Some remarks on generically large cardinals

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Abstract

In this note, we write down several observations concerning generic supercompactness and Laver-generic supercompactness.

In the following, we write down several observations concerning generic supercompactness and Laver-generic supercompactness. Most of the assertions presented here are either trivial, simple application of well-known ideals, or folklore. Their details are written just to clarify the situation.

This article is still in a state of a work in progress, and there may be some additional topics in new sections, as well as improvements and extension of the material presented here in the extended version of the article uploaded at the URL mentioned in the footnote below.

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1 First-order definability of generic supercompactness

For a class $\mathcal{P}$ of posets, a cardinal κ is said to be *generically supercompact by* $\mathcal{P}$, if, for any $\lambda \geq \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that

(1.1) $j : \mathsf{V} \overset{\preceq}{\rightarrow} M \subseteq \mathsf{V}[\mathbb{G}]$;[1)]

(1.2) $crit(j) = \kappa$, $j(\kappa) > \lambda$; and

(1.3) $j''\lambda \in M$.

We call the class mapping j as above a λ*-generically supercompact embedding for* κ.

If M is obtained as an inner model of V by ultraproduct construction with a $<\omega_1$-complete ultrafilter in V, the condition (1.3) implies ${}^{\lambda}M \subseteq M$ (see Proposition 22.4 in [5]).

In the context of generic supercompactness, the condition (1.3) still implies a certain kind of closedness of M. This can be seen in the following Lemma:

Lemma 1.1 (Lemma 2.5 in [2]) *Suppose that* $\mathbb{G}$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter for a poset* $\mathbb{P} \in \mathsf{V}$, *and* $j : \mathsf{V} \overset{\preceq}{\rightarrow} M \subseteq \mathsf{V}[\mathbb{G}]$ *is such that, for cardinals* κ, λ *in* V *with* $\kappa \leq \lambda$, $crit(j) = \kappa$ *and* $j''\lambda \in M$. *Then, we have the following:*

(1) *For any set* $A \in \mathsf{V}$ *with* $\mathsf{V} \models |A| \leq \lambda$, *we have* $j''A \in M$.

(2) $j \restriction \lambda$, $j \restriction \lambda^2 \in M$.

(3) *For any* $A \in \mathsf{V}$ *with* $A \subseteq \lambda$ *or* $A \subseteq \lambda^2$ *we have* $A \in M$.

(4) $(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$, *Thus, if* $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, *then* $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.

(5) $\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$.

(6) $j \restriction A \in M$ *for all* $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$. ❑

It is consistent (modulo a supercompact cardinal) that a successor cardinal of a regular uncountable cardinal is generically supercompact. In the following, we use Kanamori's notation of collapsing posets (see §10 of [5]).

Fact 1.2 *Suppose that* κ *is a (really) supercompact cardinal,* $\mu < \kappa$ *a regular uncountable cardinal, and* $\mathbb{P}_0 = \mathrm{Col}(\mu, \kappa)$. *Then, for a* $(\mathsf{V}, \mathbb{P}_0)$*-generic* $\mathbb{G}_0$,

$$\mathsf{V}[\mathbb{G}_0] \models \text{“}\mu^+ \textit{ is a generically supercompact cardinal by } {<}\mu\textit{-closed posets”}.$$

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1) When we write $j : \mathsf{V} \overset{\preceq}{\rightarrow} M \subseteq \mathsf{V}[\mathbb{G}]$, we always assume that M is transitive in $\mathsf{V}[\mathbb{G}]$.

Proof. Note that $\mathsf{V}[\mathbb{G}_0] \models$ "$\mu^+ = \kappa$".

For $\lambda \geq \kappa$, let $j : \mathsf{V} \overset{\preceq}{\to} M$ be a λ-supercompact embedding for κ. Then we have

$$j(\mathbb{P}_0) \underbrace{=}_{\text{by elementarity}} \mathrm{Col}(\underbrace{j(\mu)}_{=\mu}, j(\kappa))^M \overbrace{=}^{\text{by closedness of } M} \mathrm{Col}(\mu, j(\kappa))^{\mathsf{V}}.$$

For a $(\mathsf{V}[\mathbb{G}_0], \mathrm{Col}(\mu, j(\kappa) \setminus \kappa))$-generic filter $\mathbb{G}$, the lifting

$$\tilde{j} : \mathsf{V}[\mathbb{G}_0] \overset{\preceq}{\to} \underbrace{M[\mathbb{G}_0][\mathbb{G}]}_{\subseteq \mathsf{V}[\mathbb{G}_0][\mathbb{G}]};\ \underset{\sim}{a}^{\mathbb{G}_0} \mapsto j(\underset{\sim}{a})^{\mathbb{G}_0 * \mathbb{G}}$$

witnesses the generic λ-supercompactness of $\underbrace{\kappa}_{= (\mu^+)^{\mathsf{V}[\mathbb{G}_0]}}$ by μ-closed posets in $\mathsf{V}[\mathbb{G}_0]$.

□ (Fact 1.2)

For a class $\mathcal{P}$ of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω-sequence of ground model sets, the generic supercompactness by $\mathcal{P}$ is first-order definable. This is seen in the following Proposition. The Proposition is proved by imitating the proof of the characterization of supercompactness by Solovay and Reinhardt in terms of the existence of normal ultrafilters (see e.g. Theorem 22.7 in [5]).

Theorem 1.3 *Suppose that $\mathcal{P}$ is a class of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω-sequence of ground model sets, and $\mathcal{P}$ is closed with respect to restriction (i.e, if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$, then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$).*

An uncountable cardinal κ is generically supercompact by $\mathcal{P}$ if and only if, for any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that

$$\Vdash_{\mathbb{P}} \text{"there is a } \mathsf{V}\text{-normal ultrafilter on } \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})\text{"}.$$

Here, the notion of V-normal ultrafilter is defined as follows: Suppose that we are living in a universe W and V is an inner model. Let λ be an ordinal in V, $\mathcal{I} \in \mathsf{V}$, $\mathcal{I} \subseteq \mathcal{P}^{\mathsf{V}}(\lambda)$ a σ-ideal with $\{\xi\} \in \tau$ for all $\xi < \lambda$, and $\mathcal{B} \in \mathsf{V}$ the sub-Boolean algebra $\mathcal{B} = \mathcal{P}^{\mathsf{V}}(\mathcal{I})$ of $\mathcal{P}^{\mathsf{W}}(\mathcal{I})$.

In W, $U \subseteq \mathcal{B}$ is a V-*normal ultrafilter* if

(1.4) U is a ultrafilter on the Boolean algebra $\mathcal{B}$. I.e.,

- (i) $\emptyset \notin U$;
- (ii) $A \cap A' \in U$ for any A, $A' \in U$;
- (iii) if $A \in U$, $A \subseteq A' \in \mathcal{B}$, then $A' \in U$; and
- (iv) for any $A \in \mathcal{B}$, either $A \in U$ or $\mathcal{I} \setminus A \in U$;

(1.5) For any $x_0 \in \mathcal{I}$, we have $\{x \in \mathcal{I} : x_0 \subseteq x\} \in U$;

(1.6) For any $\langle A_\xi : \xi \in \lambda \rangle \in \mathsf{V}$, if $\{A_\xi : \xi < \lambda\} \subseteq U$, we have $\triangle_{\xi\in\lambda} A_\xi \in U$. Here, $\triangle_{\xi\in\lambda} A_\xi$ is the diagonal intersection of A_ξ's defined by

(1.7) $\triangle_{\xi\in\lambda} A_\xi := \{x \in \mathcal{I} : x \in A_\xi \text{ for all } \xi \in x\}$.

Lemma 1.4 *Suppose that $U \subseteq \mathcal{B}$ is a* V*-normal ultrafilter.*

(1) *For $\delta < \lambda$ such that $\delta \in \mathcal{I}$, and $\langle A_\xi : \xi \in \delta \rangle \in \mathsf{V}$ with $A_\xi \in U$ for all $\xi \in \delta$, we have $\bigcap_{\xi\in\delta} A_\xi \in U$.*

(2) (Pressing Down Lemma) *For any $f \in \mathsf{V}$ with $f : \mathcal{I} \to \mathsf{V}$, if $\{x \in \mathcal{I} : f(x) \in x\} \in U$, then there is $\xi < \lambda$ such that $\{x \in \mathcal{I} : f(x) = \xi\} \in U$.*

Proof. (1): Let $A_\xi := \mathcal{I}$ for all $\xi \in \lambda \setminus \delta$. Then

$$\underbrace{U \ni}_{\text{by (1.4), (ii)}} \overbrace{\triangle_{\xi\in\lambda} A_\xi}^{\in U \text{ by (1.6)}} \cap \underbrace{\{x \in \mathcal{I} : \delta \subseteq x\}}_{\in U \text{ by (1.5)}} \subseteq \bigcap_{\xi\in\delta} A_\xi.$$

Hence, $\bigcap_{\xi\in\delta} A_\xi \in U$ by (1.4), (iii).

(2): Suppose that f is a counter-example to the assertion. That is,

(1.8) $A := \{x \in \mathcal{I} : f(x) \in x\} \in U$, but

(1.9) $A_\xi := \{x \in \mathcal{I} : f(x) \neq \xi\} \in U$ for all $\xi \in \lambda$.

Then $\triangle_{\xi<\lambda} A_\xi \cap A \in U$ by (1.6) and (1.4), (ii). By (1.4), (i), there is an element x^* of this set. $f(x^*) \in x^*$ by (1.8) but $f(x^*) \neq \xi$ for all $\xi \in x^*$ by (1.9) and the definition (1.7) of diagonal intersection. This is a contradiction. □ (Lemma 1.4)

Proof of Theorem 1.3: ($\Rightarrow$): Let $\lambda \geq \kappa$ and let $\mathbb{P}$ be a $<\mu$-closed poset with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \xrightarrow{\preceq} M$ is a λ-generically supercompact embedding for κ. In particular, we have $j''\lambda \in M$. Note that

(1.10) $M \models j''\lambda \in \mathcal{P}_{j(\kappa)}(j(\lambda)) = j(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$.

In $\mathsf{V}[\mathbb{G}]$, let

$$U_j := \{A \in \mathsf{V} : A \subseteq \mathcal{P}_\kappa(\lambda)^{\mathsf{V}},\ j''\lambda \in j(A)\}.$$

The following is easy to check:

Claim 1.4.1 *U_j is a* V*-normal ultrafilter on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$.* ⊣

($\Leftarrow$): Let $\lambda \geq \kappa$ and let $\mathbb{P}$ be a $<\mu$-closed poset with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and V-normal ultrafilter $U \in \mathsf{V}[\mathbb{G}]$ on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$.

Let

(1.11) $\quad \mathcal{W} := \{f \in \mathsf{V} \,:\, f : \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \to \mathsf{V}\}$

(1.12) $\quad$ For $f, g \in \mathcal{W}$, $f \sim_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \,:\, f(x) = g(x)\} \in U$;
$$f \in_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \,:\, f(x) \in g(x)\} \in U.$$

$\sim_U$ is a congruence relation to $\in_U$.
We write $\quad f/{\sim_U} \ \in_U \ g/{\sim_U} :\Leftrightarrow f \in_U g$.[2)]

Let $i_U : \mathsf{V} \to \mathcal{W}/{\sim_U}$ be defined by

(1.13) $\quad i_U(a) = const_a/{\sim_U}$

for $a \in \mathsf{V}$ where $const_a$ denote the function on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$ whose value is constantly a. Łoś's Theorem holds:

Claim 1.4.2 *For any formula $\varphi = \varphi(x_0, ..., x_{n-1})$ in $\mathcal{L}_\in$ (the language of* ZF*), and $f_0, ..., f_{n-1} \in \mathcal{W}$, we have $\langle \mathcal{W}/{\sim_U}, \in_U\rangle \models \varphi(f_0/{\sim_U}, ..., f_{n-1}/{\sim_U})$, if and only if $\{x \in \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \,:\, \mathsf{V} \models \varphi(f_0(x), ..., f_{n-1}(x))\} \in U$.*

$\vdash$ By induction on φ. $\dashv$ (Claim 1.4.2)

By Claim 1.4.2, the class mapping i_U above is an elementary embedding of V into $\langle \mathcal{W}/{\sim_U}, \in_U\rangle$.

Claim 1.4.3 *$\in_U$ is* (i) *an extensional,* (ii) *well-founded and* (iii) *set-like relation on $\mathcal{W}/{\sim_U}$.*

$\vdash$ (i): The extensionality of $\in_U$ follows from the elementarity of i_U.

(ii): Assume, toward a contradiction, that there is a sequence $\langle f_n \,:\, n \in \omega\rangle$ in $\mathcal{W}$ such that $f_{n+1} \in_U f_n$ for all $n \in \omega$. By the definition of $\in_U$, this means that $A_n = \{x \in \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \,:\, f_{n+1}(x) \in f_n(x)\} \in U$ for all $n \in \omega$. Since $\mathbb{P}$ does not add any new ω-sequence, $\langle f_n \,:\, n \in \omega\rangle \in \mathsf{V}$. Thus, we also have $\langle A_n \,:\, n \in \omega\rangle \in \mathsf{V}$. By Lemma 1.4, (1), it follows that $\bigcap_{n\in\omega} A_n \in U$. For an element x of this intersection, we have

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[2)] We apply here "Scott's trick" and define the equivalence class $f/{\sim_U}$ by

$$f/{\sim_U} := \{g \in \mathcal{W} \,:\, g \sim_U f \text{ and } g \text{ is of minimal } \in\text{-rank among elements of } \mathcal{W} \text{ with this property}\}$$

to make the equivalence class $f/{\sim_U}$ a set.

$$f_0(x) \ni f_1(x) \ni f_2(x) \ni f_3(x) \ni \cdots$$

by definition of A_n's. This is a contradiction.

(iii): Let $f \in \mathcal{W}$ be arbitrary, and let $S = \bigcup_{x\in\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})} f(x)$. Then, by Łos's Theorem, we have

$$\{g/{\sim_U} \;:\; g/{\sim_U} \in_U f/{\sim_U}\} \subseteq \{g/{\sim_U} \;:\; g : \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \to S\}$$

The right side of the inclusion is clearly a set. $\dashv$ (Claim 1.4.3)

Let $\mu_U : \langle \mathcal{W}/{\sim_U}, \in_U\rangle \to \langle M, \in\rangle$ be the Mostowski-collapse, and let $[\cdot]_U : \mathcal{W} \to M$; $f \mapsto [f]_U := \mu_U(f/{\sim_U})$.

Łós's Theorem (Claim 1.4.2) translates to the following:

Claim 1.4.4 *For any formula $\varphi = \varphi(x_0, ..., x_{n-1})$ in $\mathcal{L}_\in$ (the language of* ZF*), and $f_0, ..., f_{n-1} \in \mathcal{W}$, we have $M \models \varphi([f_0]_U, ..., [f_{n-1}]_U)$, if and only if $\{x \in \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \;:\; \mathsf{V} \models \varphi(f_0(x), ..., f_{n-1}(x))\} \in U$.* $\dashv$

Let

$$j_U : \mathsf{V} \overset{\preceq}{\to} M;\; a \mapsto [a]_U := \mu_U(i_U(a)) = [const_a]_U.$$

We show that $j_U : \mathsf{V} \overset{\preceq}{\to} M$ is a λ-generically supercompact embedding for κ.

Claim 1.4.5 (1) *$j_U(\xi) = \xi$ for all $\xi \in \kappa$.*

(2) *$j_U(\kappa) > \kappa$.*

(3) *$j_U{}''\lambda \in M$.*

$\vdash$ (1): Note that $j_U(\xi) = \mu_U(i_U(\xi)) = [const_\xi]_U$. Thus, for $\xi < \kappa$ and $f \in \mathcal{W}$,

$$\begin{aligned}
[f]_U \in j_U(\xi) \quad &\Leftrightarrow \quad [f]_U \in [const_\xi]_U \\
&\underbrace{\Leftrightarrow}_{\text{Claim 1.4.4}} \{x \in \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \;:\; f(x) \in \underbrace{\xi}_{= const_\xi(x)}\} \in U \\
&\underbrace{\Leftrightarrow}_{\text{by Lemma 1.4, (2) and (1.5)}} \{x \in \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \;:\; f(x) = \underbrace{\eta^*}_{= const_{\eta^*}(x)}\} \in U \text{ for some } \eta^* \in \xi \\
&\underbrace{\Leftrightarrow}_{\text{Claim 1.4.4}} [f]_U = j_U(\eta^*) \text{ for some } \eta^* \in \xi.
\end{aligned}$$

Thus, by induction on $\xi < \kappa$, we obtain $j_U(\xi) = \xi$ for all $\xi < \kappa$.

(2): Let $\iota : \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) \to \mathsf{V}$; $x \mapsto \sup(x \cap \kappa)$

For all $\xi < \kappa$, we have

$$\underbrace{\xi = }_{(1)} j_U(\xi) = [const_\xi]_U \underbrace{<}_{\text{Claim 1.4.4 and (1.5)}} [\iota]_U \overbrace{<}^{\text{Claim 1.4.4 and (1.5)}} [const_\kappa]_U = j_U(\kappa).$$

Thus $\kappa \leq [\iota]_U < j(\kappa)$.

(3): We show that $[id_{\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}}]_U = j_U{}''\lambda$.

For an arbitrary $f \in \mathcal{W}$

$$[f]_U \in [id_{\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}}]_U \underbrace{\Leftrightarrow}_{\text{by Claim 1.4.4}} \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} : f(x) \in \underbrace{x}_{= id_{\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}}(x)} \} \in U$$

$$\underbrace{\Leftrightarrow}_{\text{by Lemma 1.4, (2)}} \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} : f(x) = \underbrace{\xi^*}_{= const_{\xi^*}(x)} \} \in U \text{ for some } \xi^* < \lambda$$

$$\underbrace{\Leftrightarrow}_{\text{by Claim 1.4.4}} [f]_U = j_U(\xi^*) \text{ for some } \xi^* < \lambda.$$

⊣ (Claim 1.4.5)

It follows that there is $\mathbb{p} \in \mathbb{G}$ such that

$$\mathbb{p} \Vdash_{\mathbb{P}} \text{“there is a } \mathsf{V}\text{-normal ultrafilter on } \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})\text{”}.$$

Since $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ by the assumption on $\mathcal{P}$, we obtain the desired condition for λ by replacing $\mathbb{P}$ with $\mathbb{P} \restriction \mathbb{p}$. □ (Theorem 1.3)

Note that the proof of Claim 1.4.3 relies on the property of $\mathcal{P}$ that no $\mathbb{P}$ adds any new ω-sequence ground model sets. Note also that the argument using the fact that the well-foundedness of a relation is Δ_1 is irrelevant here since the relation $\in_U$ is not in the ground model.

Thus, the proof of Theorem 1.3 cannot simply be applied to the generic supercompactness by a class of posets $\mathcal{P}$ whose elements might add new ω-sequences of ground model sets.

By Theorem 1.3 we obtain another characterization of generic supercompactness by a $\mathcal{P}$ as in Theorem 1.3:

Corollary 1.5 *Suppose that $\mathcal{P}$ is a class of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω-sequence of ground model sets, and $\mathcal{P}$ is closed with respect to restriction. Then, the following are equivalent:*

(a) *κ is generically supercompact by $\mathcal{P}$.*

(b) *For any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that*

$$\Vdash_{\mathbb{P}} \text{“there is a } \mathsf{V}\text{-normal ultrafilter on } \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})\text{”}.$$

(c) *For any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \xrightarrow{\preceq} M$; $crit(j) = \kappa$; $j(\kappa) > \lambda$ and $j''\lambda \in M$.*

□

2 Rado Conjectures of height $>\omega_1$

For an infinite cardinal μ, a tree $T = \langle T, \leq_T \rangle$ is said to be *μ-special* if T is the union of μ-many antichains (i.e. subsets whose elements are pairwise incomparable). Note that

(2.1) Any tree of height $< \mu^+$ is μ-special, and any tree of height $> \mu^+$ is not μ-special.

For cardinals μ, κ with $\kappa > \mu^+$, the *Rado Conjecture of height μ^+ with reflection point $<\kappa$* is the principle:

$\mathsf{RC}(\mu, <\kappa)$: For any tree T, if T is not μ-special, then there is $T' \in [T]^{<\kappa}$ such that T' is not μ-special.

The following is a straight-forward generalization of Lemma 12 in [6]:

Lemma 2.1 *If a tree T is μ-special and $\mathbb{P}$ a $<\mu^+$-closed poset, then we have $\Vdash_{\mathbb{P}}$ "T is not μ-special".*

Proof. By (2.1), we may assume that $ht(T) = \mu^+$. Suppose that $\Vdash_{\mathbb{P}}$ "T is μ-special", and let $\underset{\sim}{f}$ be a $\mathbb{P}$-name such that

(2.2) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{T} \to \check{\mu}$ and
$\underset{\sim}{f}^{-1}{}''\{\xi\}$ is an antichain in $\check{T}$ for all $\xi < \check{\mu}$".

We want to prove that T is μ-special (in V).

By induction on $\alpha < \mu$, we can take $\mathbb{p}_t \in \mathbb{P}$ and $\xi_t \in \mu$ for $t \in T_\alpha$ such that

(2.3) if $t' \leq_T t$ then $\mathbb{p}_t \leq_{\mathbb{P}} \mathbb{p}_{t'}$; and

(2.4) $\mathbb{p}_t \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\check{t}) = \check{\xi}_t$".

Note that, for each $t \in T$, if $\mathbb{p}_{t'}$, for all $t' \leq_T t$ have been defined according to (2.3) and (2.4), there is $\mathbb{p} \in \mathbb{P}$ with $\mathbb{p} \leq_T \mathbb{p}_{t'}$ for all $t' \leq_T t$ by $<\mu^+$-closedness of $\mathbb{P}$. Thus we can choose $\mathbb{p}_t \leq_{\mathbb{P}} \mathbb{p}$ such that it satisfies (2.4).

For $\xi < \mu$, let

$$A_\xi := \{t \in T : \xi_t = \xi\}.$$

Then $T = \bigcup_{\xi<\mu} A_\xi$, and each A_ξ for $\xi < \mu$ is an antichain by (2.2), (2.3), and (2.4).

□ (Lemma 2.1)

Proposition 2.2 *Suppose that $\mu^+ < \kappa$ and κ is a generically supercompact cardinal by $<\mu^+$-closed posets. Then $\mathsf{RC}(\mu', <\kappa)$ holds for all $\omega \leq \mu' \leq \mu$.*

Proof. Suppose that $\omega \leq \mu' \leq \mu$ and T is not μ'-special. Let $|T| = \lambda$. We want to show that there is a subtree T' of T of cardinality $< \kappa$ which is not μ'-special.

Without loss of generality, we may assume that the underlying set of T is λ. That is, we assume that $T = \langle \lambda, \leq_T \rangle$.

Let $\mathbb{P}$ be a $< \mu^+$-closed poset, and $\mathbb{G}$ a $(\mathsf{V}, \mathbb{P})$-generic set with j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that

(1.1) $j : \mathsf{V} \overset{\preceq}{\rightarrow} M \subseteq \mathsf{V}[\mathbb{G}]$;

(1.2) $crit(j) = \kappa$, $j(\kappa) > \lambda$; and

(1.3) $j''\lambda \in M$.

By $< \mu^+$-closedness of $\mathbb{P}$ [3] and Lemma 2.1, we have

(2.5) $\mathsf{V}[\mathbb{G}] \models$ "T is not μ'-special".

The tree $j''\lambda = \langle j''\lambda, j''{\leq_T} \rangle$ is isomorphic to T. Thus we have

$$\mathsf{V}[\mathbb{G}] \models \text{"} j''T \text{ is not } \mu'\text{-special"}.$$

Since the tree $j''T$ is an element of M by Lemma 1.1, it follows that $M \models$ "$j''T$ is not μ'-special". Thus, we have

$$M \models \text{"there is a subtree } T' \text{ of } j(T) \text{ of size} < j(\kappa) \text{ which is not } \underbrace{\mu'}_{= j(\mu')}\text{-special"}.$$

By elementarity, it follows that

$$\mathsf{V} \models \text{"there is a subtree } T' \text{ of } T \text{ of size} < \kappa \text{ which is not } \mu'\text{-special"}.$$

□ (Proposition 2.2)

3 Laver-generically supercompact cardinals

The notion of Laver-generically large cardinal was introduced in [2]. The Laver-genericity for a class $\mathcal{P}$ of posets, as we define here, is stronger than the one given in [2], and it corresponds to the definition of Laver-genericity for $(\mathcal{P}, \mathcal{P})$ in [3].

A class $\mathcal{P}$ of posets is *iterable* if

(3.1) $\mathcal{P}$ is closed with respect to forcing equivalence. That is, if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}'$ is forcing equivalent to $\mathbb{P}$, then $\mathbb{P}' \in \mathcal{P}$;

(3.2) $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$; and

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[3] Note that $< \mu'^+$-closedness of $\mathbb{P}$ follows from this.

(3.3) if $\mathbb{P} \in \mathcal{P}$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \varepsilon \mathcal{P}$", then $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

For a cardinal κ and an iterable class $\mathcal{P}$ of posets, we call κ a *Laver-generically supercompact for* $\mathcal{P}$ (or *L-g supercompact*, for short) if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name of a poset $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \varepsilon \mathcal{P}$" such that, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter $\mathbb{H}$, there are M, $j \subseteq \mathsf{V}[\mathbb{H}]$ such that

(3.4) $j : \mathsf{V} \overset{\preceq}{\rightarrow} M$,

(3.5) $crit(j) = \kappa$, $j(\kappa) > \lambda$,

(3.6) $\mathbb{P}, \mathbb{H} \in M$ and

(3.7) $j''\lambda \in M$.

We shall call j as above *a* λ *L-g supercompact embedding (with the critical point* κ*, associated with* $\mathbb{H}$ *over* V*)*.

Even in the case that the class of $\mathbb{P}$ of posets consists of $<\mu$-closed posets, the first-order formulizability of the notion of Laver-generic supercompactness is unknown: An argument like that of Proposition 1.3 cannot help because it apparently cannot create the situation with (3.6).

Thus, at least at the moment, we have to treat a Laver-generic large cardinal merely as a scheme. In each of the concrete instances we encounter, this is no problem since we know there exactly how the elementary embeddings j, and inner models M are constructed.

The situation depicted in the following theorem is archetypal for this:

Theorem 3.1 *μ^+ is L-g supercompact in the model given in Fact 1.2. More precisely, if κ is a (really) supercompact cardinal, $\mu < \kappa$ a regular uncountable cardinal, and $\mathbb{P}_0 = \mathrm{Col}(\mu, \kappa)$, then, for a $(\mathsf{V}, \mathbb{P}_0)$-generic $\mathbb{G}_0$,*

$$\mathsf{V}[\mathbb{G}_0] \models \text{“}\mu^+ \textit{ is a L-g supercompact cardinal for } {<}\,\mu\textit{-closed posets”}.$$

The theorem above follows from the corollary (Corollary 3.4) of the next theorem which is a generalization of Proposition 10.20 in Kanamori [5]:

Theorem 3.2 (see Theorem 1.5 in [2]) *Suppose that μ, and λ are regular with $\mu < \lambda$. If $\mathbb{P}$ is a separative poset such that $|\,\mathbb{P}\,| = \lambda$, $\mathbb{P}$ is μ-closed, and*

(3.8) $\Vdash_{\mathbb{P}}$ "*there is a surjection* $\check{\mu} \rightarrow \check{\lambda}$",

then $\mathrm{ro}(\mathbb{P}) \cong \mathrm{ro}(\mathrm{Col}(\mu, \{\lambda\}))$. □

The following are well-known and easy to prove:

Lemma 3.3 *Let μ be an uncountable regular cardinal. Then*

(1) *For disjoint sets S_0, S_1, we have* $\mathrm{Col}(\mu, S_0 \dot\cup S_1) \sim \mathrm{Col}(\mu, S_1) \times \mathrm{Col}(\mu, S_1)$.

(2) *If $\mathbb{P}_0$ and $\mathbb{P}_1$ are $<\mu$-closed, then $\mathbb{P}_0 \times \mathbb{P}_1$ is $<\mu$-closed.*

(3) *If $\mathbb{P}_0$ is $<\mu$-closed and $\Vdash_{\mathbb{P}_0}$ "$\underset{\sim}{\mathbb{P}}_1$ is $<\mu$-closed", then $\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}_1$ is $<\mu$-closed.*

□

Corollary 3.4 (Corollary 1.6, (2) in [2]) *For any $<\mu$-closed poset $\mathbb{P}$ and cardinals ν, λ_0, λ with $|\mathbb{P}| \leq \lambda_0 = (\lambda_0)^{<\mu} < \lambda$, and $\nu \leq \lambda_0$, we have*

$$\mathrm{Col}(\mu, \lambda \setminus \nu) \underbrace{\sim}_{①} \mathrm{Col}(\mu, \lambda) \underbrace{\sim}_{②} \mathbb{P} \times \mathrm{Col}(\mu, \lambda) \underbrace{\sim}_{③} \mathbb{P} * \mathrm{Col}(\mu, \lambda)^{\mathsf{V}^{\mathbb{P}}}.$$

Proof. ① : Since $|\mathrm{Col}(\mu, \lambda_0 + 2 \setminus \nu)| = |\mathrm{Col}(\mu, \lambda_0 + 2)| = \lambda_0$ and both of the posets add a surjection from μ to λ_0, we have

$$\mathrm{Col}(\mu, \lambda_0 + 2 \setminus \nu) \sim \mathrm{Col}(\mu, \{\lambda_0\}) \sim \mathrm{Col}(\mu, \lambda_0 + 2) \tag{3.9}$$

by Theorem 3.2. Thus

$$\begin{aligned} \mathrm{Col}(\nu, \lambda \setminus \nu) &\underbrace{\sim}_{\text{by Lemma 3.3, (1)}} \mathrm{Col}(\mu, \lambda_0 + 2 \setminus \nu) \times \mathrm{Col}(\nu, \lambda \setminus \lambda_0 + 2) \\ &\underbrace{\sim}_{\text{by (3.9)}} \mathrm{Col}(\mu, \lambda_0 + 2) \times \mathrm{Col}(\nu, \lambda \setminus \lambda_0 + 2) \underbrace{\sim}_{\text{by Lemma 3.3, (1)}} \mathrm{Col}(\mu, \lambda) \end{aligned}$$

② : By Lemma 3.3, (2) and Theorem 3.2, we have

$$\mathbb{P} \times \mathrm{Col}(\mu, \lambda_0 + 1) \sim \mathrm{Col}(\mu, \{\lambda_0\}) \sim \mathrm{Col}(\mu, \lambda_0 + 1). \tag{3.10}$$

Thus

$$\begin{aligned} \mathbb{P} \times \mathrm{Col}(\mu, \lambda) &\underbrace{\sim}_{\text{by Lemma 3.3, (1)}} \mathbb{P} \times \mathrm{Col}(\mu, \lambda_0 + 1) \times \mathrm{Col}(\mu, \lambda \setminus \lambda_0 + 1) \\ &\underbrace{\sim}_{\text{by (3.10)}} \mathrm{Col}(\mu, \lambda_0 + 1) \times \mathrm{Col}(\mu, \lambda \setminus \lambda_0 + 1) \underbrace{\sim}_{\text{by Lemma 3.3, (1)}} \mathrm{Col}(\mu, \lambda). \end{aligned}$$

③ : follows from the $<\mu$-closedness of $\mathbb{P}$. □ (Corollary 3.4)

Proof of Theorem 3.1: Suppose that $\mathsf{V}[\mathbb{G}_0] \models \mathbb{P}$ is $<\mu$-closed.

Let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_0$-name of $\mathbb{P}$, and let $\lambda \geq \kappa$ be arbitrary. Let λ_0 be such that $|\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}|$, $\lambda \leq \lambda_0$ and $(\lambda_0)^{<\mu} = \lambda_0$. Without loss of generality, we may assume that the underlying set of $\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}$ is a cardinal $\leq \lambda_0$.

Let $j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}$ be a λ_0-supercompact embedding for κ. Note that $\lambda_0 < j(\kappa) \leq j(\lambda_0)$.[4)]

For an arbitrary $(\mathsf{V}[\mathbb{G}_0], \mathbb{P})$-generic set $\mathbb{G}$, let $\mathbb{H}_0$ be a $(\mathsf{V}[\mathbb{G}_0][\mathbb{G}], \underbrace{\mathrm{Col}(\mu, j(\lambda_0))^{\mathsf{V}[\mathbb{G}_0][\mathbb{G}]}}_{(=\mathrm{Col}(\mu, j(\lambda_0))^{\mathsf{V}})})$-generic set. In $\mathsf{V}[\mathbb{G}_0]$, Let $\mathbb{Q} = \mathbb{P} * \mathrm{Col}(\mu, j(\lambda_0))^{(\mathsf{V}[\mathbb{G}_0])^{\mathbb{P}}}$. $\mathbb{G} * \mathbb{H}_0$ is then a $(\mathsf{V}[\mathbb{G}_0], \mathbb{Q})$-generic set.

By Corollary 3.4, there is a $(\mathsf{V}, \mathrm{Col}(\mu, j(\lambda_0)))$-generic set $\mathbb{H}$ such that $j''\mathbb{G}_0 = \mathbb{G}_0 \subseteq \mathbb{H}$ and $\mathsf{V}[\mathbb{H}] = \mathsf{V}[\mathbb{G}_0][\mathbb{G}][\mathbb{H}_0]$.

Let

$$(3.11) \qquad \tilde{j} : \mathsf{V}[\mathbb{G}_0] \xrightarrow{\preceq} M[\mathbb{H}] \subseteq \mathsf{V}[\mathbb{H}]; \quad \underset{\sim}{a}^{\mathbb{G}_0} \mapsto j(\underset{\sim}{a})^{\mathbb{H}}.$$

Since $\mathbb{P}_0 \in M$ by the closedness of M (as a target model of λ_0-supercompact embedding for κ) and Lemma 1.1, we have $\mathbb{P}_0 \in M[\mathbb{G}_0]$. Hence we also have $\mathbb{G}_0 \in M[\mathbb{G}_0]$. By the closedness of $M[\mathbb{G}_0]$ ($\tilde{j}''\lambda_0 = j''\lambda_0 \in M \subseteq M[\mathbb{G}_0]$), we have $\mathbb{P} \in M[\mathbb{G}_0]$ and $\mathrm{Col}(\mu, j(\lambda_0))^{\mathsf{V}[\mathbb{G}_0][\mathbb{G}]} = \mathrm{Col}(\mu, j(\lambda_0))^{\mathsf{V}} \in M \subseteq M[\mathbb{H}]$.

Thus we have $\mathbb{G}$, $\mathbb{H}_0 \in M[\mathbb{H}]$ and $M[\mathbb{H}] = M[\mathbb{G}_0][\mathbb{G}][\mathbb{H}_0]$. It follows that $\tilde{j}$ is a λ-L-g supercompact embedding with the critical point κ, associated with $\mathbb{G} * \mathbb{H}_0$ over $\mathsf{V}[\mathbb{G}_0]$. □ (Theorem 3.1)

In [2], it is proved that a/the L-g supercompact cardinal for $<\aleph_1$-closed poset is $\aleph_2$ (if it exists). The proof can be generalized to show that a L-g supercompact cardinal for $<\aleph_n$-closed poset is $\aleph_{n+1}$ for each $n \in \omega$.

In general we have the following. Let us first see the situation with an arbitrary class $\mathcal{P}$ of posets:

Lemma 3.5 *If κ is generically supercompact by a class $\mathcal{P}$ of posets, and κ is a limit cardinal, then κ is a Mahlo cardinal.*[5)]

Proof. We prove first that κ is a regular cardinal. Suppose not. Then there is a strictly increasing sequence $\langle \alpha_x i : \xi < \delta \rangle$ of ordinals such that $\delta < \kappa$ and $\lim_{\xi<\delta} \alpha_\xi = \kappa$.

Let $\mathbb{P} \in \mathcal{P}$ be such that, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and j, $H \subseteq \mathsf{V}[\mathbb{G}]$,

$$(3.12) \qquad j : \mathsf{V} \xrightarrow{\preceq} M, \text{ and}$$

$$(3.13) \qquad \mathit{crit}(j) = \kappa.$$

4) $j(\lambda_0)$ is going to play the role of λ in Corollary 3.4.

5) Actually, for the following proof, it is enough to assume that κ is generically measurable. Here, a cardinal κ is said to be *generically mesearable* by $\mathcal{P}$, if there is a $\mathbb{P} \in \mathcal{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, j, $M \in \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}[\mathbb{G}]$; and $\mathit{crit}(j) = \kappa$.

By the elementarity (3.12) and (3.13), we have $j(\langle \alpha_x i : \xi < \delta\rangle) = \langle \alpha_x i : \xi < \delta\rangle$. Hence, again by elementarity, $\mathsf{V}[\mathbb{G}] \models j(\kappa) = \lim_{\xi<\delta} \alpha_\xi = \kappa$. This is a contradiction to (3.13).

Suppose now that $C \subseteq \kappa$ is a club. Then, for $\mathbb{P}$, $\mathbb{G}$, j, M as above, we have $M \models$ "$j(C)$ is a club in $j(\kappa)$" and $M \ni j(c) \cap \kappa = C$. It follows that $M \models \kappa \in j(C)$. Since $M \models$ "κ is regular", we have $M \models$ "there is a regular cardinal $\in j(C)$". By elementarity, it follows that $\mathsf{V} \models$ " there is a regular cardinal $\in C$". □ (Lemma 3.5)

Lemma 3.6 (1) *Suppose that κ is a generically measurable cardinal by a $< \mu$-closed poset. If κ is a successor cardinal then $\mu < \kappa$.*

(2) *Suppose that κ is a L-g supercompact cardinal for a class $\mathcal{P}$ of posets with* $\mathrm{Col}(\mu, \{\mu^+\}) \in \mathcal{P}$ *for $\mu < \kappa$. Then we have $\kappa = \mu^+$.*

(3) *Suppose that κ is a L-g supercompact cardinal for $< \mu$-closed posets. If κ is a successor cardinal, then $\kappa = \mu^+$.*

Proof. (1): Suppose that $\kappa = (\kappa_0)^+$. Toward a contradiction, assume $\mu \geq \kappa$. Let *poset* $\mathbb{P}$ be a $< \mu$-closed poset such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$, we have $j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}[\mathbb{G}]$ and $\mathit{crit}(j) = \kappa$.

Then

$$M \models (\underbrace{j(\kappa_0)}_{=\kappa_0})^+ = j(\kappa) \tag{3.14}$$

by elementarity. On the other hand, $\mathsf{V}[\mathbb{G}] \models$ "κ is a cardinal" by the $< \mu$-closedness of $\mathbb{P}$. Hence $M \models$ "κ is a cardinal" and $M \models \kappa_0 < \kappa < j(\kappa)$. This is a contradiction to (3.14).

(2): Suppose that $\kappa > \mu^+$. Let $\mathbb{P} = \mathrm{Col}(\mu, \{\mu^+\})$ and let $\mathbb{Q}$ be such that $\mathbb{P} \lessdot \mathbb{Q}$, $\mathbb{Q}$ is $< \mu$-closed, and, for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \xrightarrow{\preceq} M$, $\kappa = \mathit{crit}(j)$, and $\mathbb{P}$, $\mathbb{H} \in M$.

By elementarity, we have

$$M \models \text{“}\underbrace{j((\mu^+)^{\mathsf{V}})}_{=(\mu^+)^{\mathsf{V}}} \text{ is the successor cardinal of } \underbrace{j(\mu)}_{=\mu}\text{”}.$$

However, $\mathbb{H} \cap \mathbb{P}$ $(\in M)$ collapses $(\mu^+)^{\mathsf{V}}$ to an ordinal of cardinality μ. This is a contradiction.

(3): follows from (1) and (2). □ (Lemma 3.6)

Problem 3.7 *Is it consistent that for some regular uncountable μ, there is a limit cardinal κ which is L-g supercompact for $< \mu$-closed posets?*

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