Uniformly locally o-minimal structures of the second kind and their tame topology

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概要

This paper is a brief survey on a uniformly locally o-minimal structure of the second kind for non-specialists of model theory. An o-minimal structure enjoys tame topological properties such as monotonicity theorem and definable cell decomposition theorem. A uniformly locally o-minimal structure of the second kind is a new variant of o-minimal structure. A uniformly locally o-minimal structure of the second kind enjoys the local versions of tame topological properties possessed by an o-minimal structure. It enables to develop a tame dimension theory for definable sets.

1 Introduction

This paper is a survey on a uniformly locally o-minimal structure of the second kind for non-specialists of model theory. The definition of a structure given here is slightly different from the original definition in model theory. A reader who has interest in model theory should consult textbooks such as [1, 17, 19].

The notation $\mathbb{N}$ denotes the set of positive integers. In this paper, a *structure* is a pair $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n\in\mathbb{N}})$ of a set M and the collection $\mathfrak{S}$ of families $\mathfrak{S}_n$ of subsets of M^n satisfying the following conditions:

(i) The empty set and M^n are members of $\mathfrak{S}_n$ for all $n \in \mathbb{N}$. The set $\{(x, y) \in$

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$M^2 \mid x = y\}$ is also a member of $\mathfrak{S}_2$.

(ii) The families $\mathfrak{S}_n$ are closed under the boolean algebra for all $n \in \mathbb{N}$.

(iii) The Cartesian product $S_1 \times S_2$ belongs to $\mathfrak{S}_{m+n}$ if S_1 and S_2 are members of $\mathfrak{S}_m$ and $\mathfrak{S}_n$, respectively.

(iv) Let $\pi : M^n \to M^m$ be a coordinate projection and let X be a member of $\mathfrak{S}_n$. Then, the projection image $\pi(X)$ belongs to $\mathfrak{S}_m$.

(v) Let σ be a permutation of $\{1, \ldots, n\}$. We define the map $\overline{\sigma} : M^n \to M^n$ by $\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. We have $\overline{\sigma}(X) \in \mathfrak{S}_n$ if $X \in \mathfrak{S}_n$.

When a structure $\mathcal{M}$ is given, the set M is called the *universe* or the *underlying set* of the structure $\mathcal{M}$. Members in $\mathfrak{S}_n$ are called *definable sets*. Let X and Y be definable sets. A map $f : X \to Y$ is called *definable* if its graph is a definable set.

We sometimes need to consider the family of structures such that some sets other than those given in (i) are definable. When M is a densely linearly ordered set with the order $<$, a structure $\mathcal{M} = (M, \mathfrak{S})$ is called an *expansion* of a dense linear order if the set $\{(x, y) \mid x < y\}$ is definable. When $(M, \cdot)$ is a group, a structure $\mathcal{M}$ with the universe M is called an *expansion* of a group if the set $\{(x, y, z) \in M^3 \mid x \cdot y = z\}$ is definable. We define an expansion of an ordered group, an expansion of an ordered field and so on in the same manner.

An *o-minimal* structure $\mathcal{M} = (M, \mathfrak{S})$ is an expansion of a dense linear order without endpoints such that

(vi) any definable subset of M is a finite union of points and open intervals.

An open interval is a subset of M of the form $\{x \in M \mid a < x < b\}$, where $a \in M \cup \{-\infty\}$ and $b \in M \cup \{+\infty\}$. Definable sets and definable maps in an o-minimal structures are well-behaved. For instance, for a unary definable function $f : M \to M$, the domain of definition M is decomposed into finite points and open intervals such that the restriction of f to the open intervals are monotone and continuous. It is called the monotonicity theorem. The definable cell decomposition theorem for o-minimal structures guarantees that any definable set is a finite union of good-shaped definable sets called 'cells.' Readers who are interested in o-minimal structures should consult van den Dries's book [4] and Coste's book [2]. The paper [5] is also recommended.

Many structures relaxing the condition (vi) are proposed and investigated such as weakly o-minimal structures [16], locally o-minimal structures [21] and structures having (locally) o-minimal open cores [3, 6]. A locally o-minimal structure is defined by localizing the condition (vi). A *locally o-minimal structure* is an expansion of a dense linear order without endpoints satisfying the following condition:

(vi)' Let X be a definable subset of M. For any $x \in M$, there exists an open interval I containing the point x such that $X \cap I$ is a finite union of points and open intervals.

Unfortunately, even a localized version of monotonicity theorem is unavailable in a general local o-minimal structure [21, Proposition 2.11]. This is the reason why the author proposed a uniformly locally o-minimal structure of the second kind in [8]. A local monotonicity theorem holds true in a uniformly locally o-minimal structure of the second kind [8, Corollary 3.1]. A definably complete locally o-minimal structure admits local definable cell decomposition if and only if it is a uniformly locally o-minimal structure of the second kind [8, Corollary 4.1]. This paper summarizes the results on uniformly locally o-minimal structures of the second kind including the above theorems. It is a survey paper, and does not give a new insight on uniformly locally o-minimal structures of the second kind.

This paper is organized as follows. We first define a uniformly locally o-minimal structure of the second kind and related structures in Section 2. Topology of definable sets in a uniformly locally o-minimal structure of the second kind is discussed in Section 3. We develop a dimension theory for definable sets in a uniformly locally o-minimal structure of the second kind in Section 4 using the results of Section 3. We conclude this paper with remarks in Section 5.

2 Baisc Definitions

We first review the definitions given in [18, 21, 15, 8].

Definition 2.1. We consider an expansion $\mathcal{M} = (M, \mathfrak{S})$ of a dense linear order without endpoints. It is *definably complete* if every definable subset of M has both a supremum and an infimum in $M \cup \{\pm\infty\}$ [18]. A definably complete expansion of an ordered group is divisible and abelian [18, Proposition 2.2].

We review the definition of locally o-minimal structures. The structure $\mathcal{M}$ is *locally o-minimal* if, for every definable subset X of M and for every point $a \in M$, there exists an open interval I containing the point a such that $X \cap I$ is a finite union of points and open intervals. A locally o-minimal structure $\mathcal{M}$ is *strongly locally o-minimal* if, for every point $a \in M$, there exists an open interval I containing the point a such that $X \cap I$ is a finite union of points and open intervals for every definable subset X of M.

A locally o-minimal structure $\mathcal{M}$ is a *uniformly locally o-minimal structure of the first kind* if, for any positive integer n, any definable set $X \subset M^{n+1}$ and $a \in M$, there exists an open interval I containing the point a such that the definable sets $X_y \cap I$ are finite unions of points and open intervals for all $y \in M^n$. Here, X_y denotes the fiber $\{x \in M \mid (x, y) \in X\}$. A uniformly locally o-minimal structure of the first kind is called a uniformly locally o-minimal structure in [15].

A locally o-minimal structure $\mathcal{M}$ is a *uniformly locally o-minimal structure of the second kind* if, for any positive integer n, any definable set $X \subset M^{n+1}$, $a \in M$ and $b \in M^n$, there exist an open interval I containing the point a and an open box B containing b such that the definable sets $X_y \cap I$ are finite unions of points and open intervals for all $y \in B$.

We frequently consider a definably complete uniformly locally o-minimal expansion of the second kind of an ordered group. We simply call it a *DCULOAS structure* in this paper.

A locally o-minimal structure whose universe is the set of reals $\mathbb{R}$ is strongly locally o-minimal [21, Corollary 3.4]. A strongly locally o-minimal structure is always a uniformly locally o-minimal structure of the first kind. But the converse is not true in general. A definably complete uniformly locally o-minimal structure of the first kind which is not strongly o-minimal is found in [8, Example 2.4]. A uniformly locally o-minimal structure of the first kind is a uniformly locally o-minimal structure of the second kind. The converse is not true, neither. A counterexample is [8, Example 2.3]. A locally o-minimal structure is not necessarily a uniformly locally o-minimal structure of the second kind. Its counterexample is [8, Example 2.2].

The following proposition indicates that it is futile to consider a uniformly locally o-minimal expansion of the second kind of an ordered field.

Proposition 2.2 ([8, Proposition 2.1]). *A uniformly locally o-minimal expansion of the second kind of an ordered field is o-minimal.*

3 Tame topology

The following is the local monotonicity theorem for uniformly locally o-minimal structure of the second kind.

Theorem 3.1 (Local monotonicity theorem). *Consider a uniformly locally o-minimal structure of the second kind $\mathcal{M} = (M, \mathfrak{S})$. Let I be an interval and $f : I \to M$ be a definable function. For any $(a, b) \in M^2$, there exist an open interval J_1 containing the point a, an open interval J_2 containing the point b and a mutually disjoint definable partition*

$$f^{-1}(J_2) \cap J_1 = X_d \cup X_c \cup X_+ \cup X_-$$

satisfying the following conditions:

(1) the definable set X_d is discrete and closed;
(2) the definable set X_c is open and f is locally constant on X_c;
(3) the definable set X_+ is open and f is locally strictly increasing and continuous on X_+;
(4) the definable set X_- is open and f is locally strictly decreasing and continuous on X_-.

Furthermore, if the structure $\mathcal{M}$ is a DCULOAS structure, we can choose $J_1 = I$ and $J_2 = M$.

This theorem is first proved in [15, Proposition 11] only for strongly locally o-minimal structures. For any point $a \in M^n$, there exists an open box B such that the intersection of B with a definable subset of M^n in the given strongly locally o-minimal structure is a definable subset in an o-minimal structure having the same universe [15, Theorem 9]. The above theorem for strongly locally o-minimal structures is a direct corollary of it and the monotonicity theorem for o-minmal structures. Theorem 3.1 is found in [8, Corollary 3.1] and its proof is not so easy. The 'furthermore' part follows from [10, Theorem 2.11, Proposition 2.13].

We review the definitions of cells and definable cell decomposition.

Definition 3.2 (Definable cell decomposition)**.** Consider an expansion of dense linear order $\mathcal{M} = (M, \mathfrak{S})$. Let $(i_1, \dots, i_n)$ be a sequence of zeros and ones of length n. $(i_1, \dots, i_n)$-*cells* are definable subsets of M^n defined inductively as follows:

- A (0)-cell is a point in M and a (1)-cell is an open interval in M.
- An $(i_1, \dots, i_n, 0)$-cell is the graph of a continuous definable function defined on an $(i_1, \dots, i_n)$-cell. An $(i_1, \dots, i_n, 1)$-cell is a definable set of the form $\{(x, y) \in C \times M \mid f(x) < y < g(x)\}$, where C is an $(i_1, \dots, i_n)$-cell and f and g are definable continuous functions defined on C with $f < g$.

A *cell* is an $(i_1, \dots, i_n)$-cell for some sequence $(i_1, \dots, i_n)$ of zeros and ones. An *open cell* is a $(1, 1, \dots, 1)$-cell.

We inductively define a *definable cell decomposition* of an open box $B \subset M^n$. For $n = 1$, a definable cell decomposition of B is a partition $B = \bigcup_{i=1}^m C_i$ into finite cells. For $n > 1$, a definable cell decomposition of B is a partition $B = \bigcup_{i=1}^m C_i$ into finite cells such that $\pi(B) = \bigcup_{i=1}^m \pi(C_i)$ is also a definable cell decomposition of $\pi(B)$, where $\pi : M^n \to M^{n-1}$ is the projection forgetting the last coordinate. Given a finite family $\{A_\lambda\}_{\lambda \in \Lambda}$ of definable subsets of B, a *definable cell decomposition of B partitioning* $\{A_\lambda\}_{\lambda \in \Lambda}$ is a definable cell decomposition of B such that the definable sets A_λ are unions of cells for all $\lambda \in \Lambda$.

In an o-minimal structure, global definable cell decomposition is available. It means that, for any finite family of definable subsets of M^n, there exists a definable cell decomposition of M^n partitioning the given family [4, Chapter 3, Cell decomposition theorem 2.11]. In a general local o-minimal structure, even local definable cell decomposition is unavailable. The following theorem says that it is available when the structure is a definablly complete uniformly locally o-minimal structure of the second kind.

Theorem 3.3 (Local definable cell decomposition theorem)**.** *Consider a strongly locally o-minimal structure or a definably complete uniformly locally o-minimal structure of the second kind $\mathcal{M} = (M, \mathfrak{S})$.*

Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of M^n. For any point $a \in M^n$, *there exist an open box B containing the point a and a definable cell decomposition of B partitioning the finite family $\{B \cap A_\lambda \mid \lambda \in \Lambda$ and $B \cap A_\lambda \neq \emptyset\}$.*

This theorem is first proved in [15, Proposition 13] only for strongly locally o-minimal structures. It is also a direct corollary of [15, Theorem 9] and the definable cell decomposition theorem for o-minimal structures. The case in which the structure is a definably complete uniformly locally o-minimal structure of the second kind is [8, Theorem 4.2].

Definition 3.4. A locally o-minimal structure *admits local definable cell decomposition* if the assertion in Theorem 3.3 hold true for all positive integers n.

When the structure is definably complete, we can get the following important corollary:

Corollary 3.5 ([8, Corollary 4.1])**.** *A definably complete locally o-minimal structure admits local definable cell decomposition if and only if it is a uniformly locally o-minimal structure of the second kind.*

The local definable cell decomposition theorem (Theorem 3.3) cannot be extended to the global one. We want to decompose a definable set into finite good-shaped definable sets, which may not be as good-shaped as cells. One candidate is a quasi-special submanifold defined as follows:

Definition 3.6 (Quasi-special submanifolds)**.** Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n \in \mathbb{N}})$. Let X be a definable subset of M^n and $\pi : M^n \to M^d$ be a coordinate projection. The definable set X is a *π-quasi-special submanifold* or simply a *quasi-special submanifold* if, $\pi(X)$ is a definable open set and, for every point $x \in \pi(X)$, there exists an open box U in M^d containing the point x satisfying the following condition: For any $y \in X \cap \pi^{-1}(x)$, there exist an open box V in M^n and a definable continuous map $\tau : U \to M^n$ such that $\pi(V) = U$, $\tau(U) = X \cap V$ and the composition $\pi \circ \tau$ is the identity map on U.

Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. A *decomposition of M^n into quasi-special submanifolds partitioning $\{X_i\}_{i=1}^m$* is a finite family of quasi-special submanifolds $\{C_i\}_{i=1}^N$ such that

- $\bigcup_{i=1}^N C_i = M^n$,
- $C_i \cap C_j = \emptyset$ when $i \neq j$ and
- either C_i has an empty intersection with X_j or is contained in X_j for any $1 \leq i \leq m$ and $1 \leq j \leq N$.

A decomposition $\{C_i\}_{i=1}^N$ of M^n into quasi-special submanifolds *satisfies the frontier condition* if the closure of any quasi-special submanifold $\overline{C_i}$ is the union of a subfamily of the decomposition.

The following theorem says that any definable set is a disjoint union of finite quasi-special submanifolds.

Theorem 3.7 ([10, Theorem 4.5])**.** *Consider a DCULOAS structure $\mathcal{M} = (M, \mathfrak{S})$. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. There exists a decomposition $\{C_i\}_{i=1}^N$ of M^n into quasi-special submanifolds partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. Furthermore, the number N of quasi-special submanifolds in the decomposition is not greater than the number uniquely determined only by m and n.*

4 Dimension theory

Assuming that the considered structure admits local definable cell decomposition, we can develop a good dimension theory. We first define the dimension of a definable set as follows.

Definition 4.1 (Dimension of a definable set)**.** Consider an expansion of dense linear order $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n \in \mathbb{N}})$. A definable set $X \subset M^n$ is of $\dim(X) \geq m$ if there exists an open box $B \subset M^m$ and a definable continuous injective map $f : B \to X$ which is homeomorphic onto its image. A definable set $X \subset M^n$ is of $\dim(X) = m$ if it is of $\dim(X) \geq m$ and it is not of $\dim(X) \geq m + 1$. The empty set is defined to be of dimension $-\infty$.

The following corollary gives equivalent definitions of dimension.

Corollary 4.2 ([8, Corollary 5.3])**.** *Consider a locally o-minimal structure $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n \in \mathbb{N}})$ which admits local definable cell decomposition. The following* *conditions are equivalent for any definable subset* $X \subset M^n$*:*

- $\dim(X) \geq m$*;*
- *the definable set* X *contains an* $(i_1, \ldots, i_n)$*-cell with* $\sum_{j=1}^n i_j \geq m$*, and*
- *there exist a coordinate projection* $\pi : M^n \to M^m$ *and a point* $a \in M^n$ *such that the definable set* $\pi(B \cap X)$ *has a nonempty interior for any open box* B *containing the point* a*.*

The dimension defined above possesses the good features which we naturally expect as in Theorem 4.3 and Theorem 4.5.

Theorem 4.3 ([8, Lemma 5.1, Corollary 5.4, Theorem 5.6])**.** *Consider a locally o-minimal structure* $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n\in\mathbb{N}})$ *which admits local definable cell decomposition. The following assertions hold true:*

(1) Let $X \subset Y$ *be definable sets. Then, the inequality* $\dim(X) \leq \dim(Y)$ *holds true.*
(2) Let σ *be a permutation of the set* $\{1, \ldots, n\}$*. The definable map* $\overline{\sigma} : M^n \to M^n$ *is defined by* $\overline{\sigma}(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$*. Then, we have* $\dim(X) = \dim(\overline{\sigma}(X))$ *for any definable subset* X *of* M^n*.*
(3) Let X *and* Y *be definable sets. We have* $\dim(X \times Y) = \dim(X) + \dim(Y)$*.*
(4) Let X *and* Y *be definable subsets of* M^n*. We have*

$$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}.$$

(5) Let X *be a definable set. The notation* ∂X *denotes the frontier of* X *defined by* $\partial X = \overline{X} \setminus X$*. We have* $\dim(\partial X) < \dim X$*.*

In the course of the proof of Theorem 4.5, we demonstrate the following strong definable Baire property, which is a definable variant of the famous Baire property.

Proposition 4.4 (Strong definable Baire property, [9, Theorem 4.3])**.** *Consider a DCULOAS structure* $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n\in\mathbb{N}})$*. Take* $c \in M$*. Let* $\{X\langle r\rangle\}_{r>c}$ *be a parameterized increasing family of definable sets of* M^n*; that is, there exists a definable subset* $\mathcal{X}$ *of* M^{n+1} *such that* $X\langle r\rangle$ *coincides with the fiber* $\mathcal{X}_r$ *for any* $r > c$ *and we have* $X\langle r\rangle \subset X\langle s\rangle$ *if* $r < s$*. Set* $X = \bigcup_{r>c} X\langle r\rangle$*. The definable set* $X\langle r\rangle$ *has a nonempty interior for some* $r > c$ *if* X *has a nonempty interior.*

Structures satisfying a weaker definable Baire property are discussed in [7, 14]

Theorem 4.5 ([9, Theorem 1.1, Corollary 1.2], [10, Theorem 3.14])**.** *Consider a DCULOAS structure* $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n \in \mathbb{N}})$. *The following assertions hold true:*

(1) Let $f : X \to M^n$ *be a definable map. We have* $\dim(f(X)) \leq \dim X$.
(2) Let $f : X \to M^n$ *be a definable map. The notation* $\mathcal{D}(f)$ *denotes the set of points at which the map* f *is discontinuous. We have* $\dim \mathcal{D}(f) < \dim X$.
(3) (Addition Property) Let $\varphi : X \to Y$ *be a definable surjective map whose fibers are equi-dimensional; that is, the dimensions of the fibers* $\varphi^{-1}(y)$ *are constant. We have* $\dim X = \dim Y + \dim \varphi^{-1}(y)$ *for all* $y \in Y$.

5 Remarks

We conclude this paper with several remarks. The most restrictive structure considered in this paper is a DCULOAS structure. However, by [10], all the assertions except the local definable cell decomposition theorem (Theorem 3.3) are satisfied in any definably complete locally o-minimal structure such that

($*$) the image of a discrete definable set under a coordinate projection is again discrete.

A DCULOAS structure satisfies the condition ($*$). Shoutens proposed a locally o-minimal structure called a *model of DCTC* [20]. A model of DCTC is a definably complete locally o-minimal structure with the property ($*$). A locally o-minimal expansion of an ordered field falls into a model of DCTC.

A natural unsolved question is as follows:

Conjecture. Does any definably complete locally o-minimal structure satisfy the property ($*$)?

The strongly locally o-minimal structure in [15, Example 12] is definably complete nor satisfies the property ($*$), neither.

Since a DCULOAS structure has tame topological properties, definable functions are also expected to have tame properties. Definable equi-continuity is defined and investigated in [11]. A variant of the Arzela-Ascoli theorem is demonstrated in the same paper.

Consider two structures $\mathcal{M} = (M, \mathfrak{S} = \{\mathfrak{S}_n\}_{n\in\mathbb{N}})$ and $\mathcal{M}' = (M, \mathfrak{S}' = \{\mathfrak{S}'_n\}_{n\in\mathbb{N}})$ having the same universe M. When $\mathfrak{S}$ is a subset of $\mathfrak{S}'$, $\mathcal{M}$ is called a *reduct* of $\mathcal{M}'$, and $\mathcal{M}'$ is called an *expansion* of $\mathcal{M}$. For a given structure $\mathcal{M}$, the reduct generated by the open sets definable in $\mathcal{M}$ is called the *open core* of $\mathcal{M}$. A sufficient condition for a structure having an o-minimal open core is discussed in [3]. Definably complete expansions of ordered fields having locally o-minimal open cores are treated in [6]. The author gave a sufficient condition for a structure having uniformly locally o-minimal open core of the first/second kind in [12].

A locally o-minimal structure whose universe is the set of reals $\mathbb{R}$ is strongly o-minimal. It enjoys more tame condition called almost o-minimality.

Definition 5.1. An expansion of densely linearly ordered set without endpoints is *almost o-minimal* if any bounded definable set in M is a finite union of points and open intervals. Here, M is the universe of the expansion.

The notion of almost o-minimality was formulated in [13].

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