Integral algebra

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Abstract

A differential ring is a ring equipped with a derivation. In this paper, we introduce integral rings. An integral ring is a ring equipped with a derivation and an integration. We study the basic properties of integral rings.

1 Introduction

Throughout this paper, all rings will mean commutative rings of characteristic zero with identity. In this paper we will introduce integral rings. A differential ring is a ring equipped with a derivation. An integral ring is a ring equipped with a derivation and an integration. Differential rings and differential fields were studied in [1], [2] and [5]. However an integral ring is not studied. So we will study the basic properties of an integral ring.

2 Differential rings

In this section we introduce differential rings. We follow the notation and definition of differential rings from [4], [3] and [6].

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Definition 1. A *derivation* on a ring R is an additive homomorphism $\delta : R \to R$ such that $\delta(xy) = x\delta(y)+y\delta(x)$. A *differential ring* is a ring equipped with a derivation.

By the definition of δ, for all $x \in R$ and $n \in \mathbb{N}$, we have $\delta(x^n) = nx^{n-1}\delta(x)$.

Definition 2. We let C_R denote the kernel of the derivation δ. We call C_R the *constants* of R.

C_R is a subring of R. In particular, if R is a field, then so is C_R.

Definition 3. Given a differential ring R, the *differential polynomial ring* $R\{X\}$ in one differential indeterminate X is the ring $R\{X\}$ $:= R[X, \delta(X), \delta^2(X), \ldots]$. By extending δ to all of $R\{X\}$ in the obvious manner, we have that $R\{X\}$ is a differential ring. For nonzero $f \in R\{X\}$ we define $\mathrm{ord}(f)$, the *order* of f, to be the largest n, if any, such that $\delta^n(X)$ appears in f with non-zero coefficient.

Definition 4. We say that an ideal $I \subseteq R$ is a *differential ideal* if for all $f \in I$, we have $\delta(f) \in I$.

Definition 5. We say that a field K is a *differentially closed field* if K is a differential field such that if $f, g \in K\{X\} \setminus \{0\}$ and $\mathrm{ord}(f) > \mathrm{ord}(g)$, then there exists $x \in K$ such that $f(x) = 0$ and $g(x) \neq 0$.

Any differentially closed field is algebraically closed.

Fact 6 ([4, Lemma 2.2])**.** *Every differential field k has an extension K which is differentially closed.*

3 Integral rings

Definition 7. A *integration* on a differential ring R is a function $\rho : R \to R$ such that the followings hold.

1. For any $x \in R$, we have $\delta(\rho(x)) = x$.

2. For any $x \in R$, there exists $\alpha \in R$ such that $\rho(\delta(x)) = x + \alpha$ and $\delta(\alpha) = 0$.

A *integral ring* is a differential ring equipped with an integration.

By the definition of an integration ρ, it follows that δ is surjective and ρ is injective. Since $\delta(x^n) = nx^{n-1}\delta(x)$ for any $x \in R$ and $n \in \mathbb{N}$, there exists $\alpha \in C_R$ such that $\rho(x^n\delta(x)) = \frac{1}{n+1}x^{n+1} + \alpha$.

In this paper we let integration be a function, but we may let integration be a relation.

Remark 8. *There exists a differential ring which is not an integral ring. For example, let $\mathbb{R}\left\{\frac{1}{X}\right\}$ be a differential ring, where $\delta\left(\frac{1}{X}\right) = -\frac{1}{X^2}$. Then, $\mathbb{R}\left\{\frac{1}{X}\right\}$ is not an integral ring. By Fact 6, there exists a differentially closed field $K \supseteq \mathbb{R}\left\{\frac{1}{X}\right\}$. Then, there exists $x \in K$ such that $\delta(x) = \frac{1}{X}$.*

Lemma 9. *Let $x, y \in R$. The followings hold.*

1. *There exists $\alpha \in C_R$ such that $\rho(x + y) = \rho(x) + \rho(y) + \alpha$.*
2. *For any $\alpha \in C_R$, there exists $\beta \in C_R$ such that $\rho(\alpha x) = \alpha\rho(x) + \beta$.*
3. *There exists $\alpha \in C_R$ such that $\rho(\delta(x)y) = xy - \rho(x\delta(y)) + \alpha$.*

Definition 10. Suppose that R is an integral ring. The *strong constant ring* of R is the ring $C'_R := \{x \in R : \exists n \in \mathbb{N}, \delta^n(x) = 0\}$

The constant ring C_R of R is not an integral ring.

Lemma 11. *The strong constant ring C'_R of R is an integral ring.*

Definition 12. We say that a differential ideal $I \subseteq R$ is a *integral ideal* if for all $f \in I$, we have $\rho(f) \in I$.

If $I \subseteq R$ is a differential ideal, then naturally R/I is a differential quotient ring of R. However, if $I \subseteq R$ be a integral ideal, then R/I is not an integral quotient ring of R.

Lemma 13. *Let $I \subsetneq R$ be an integral ideal. Then there exists a maximal integral ideal $J \supseteq I$.*

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