Characterization of theories by hierarchies of logical formulas

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1 Introduction and Preliminaries

At first, we introduce some basic concepts in model theory.

Definition 1. A language L consists of the following:

- constant symbols,
- n-ary function symbols ($n > 0$),
- n-ary predicate symbols ($n > 0$).

Example 2. The language L_{ORing} of ordered rings is $\{c_0, c_1, f_+, f_-, f_\times, P_<\}$, where

- c_0 and c_1 are constant symbols;
- f_+ and $f_\times$ are binary function symbols;
- f_- is a unary function symbols;
- $P_<$ is a binary function symbols.

Let L be a language. We use $x, y, z, x_1, x_2, \ldots, y_1, y_2, \ldots$ as variables.

Definition 3. An L-structure M is a set with interpretations s^M for each $s \in L$, where

- If $c \in L$ is a constant symbol, then $c^M \in M$;
- If $f \in L$ is a n-ary function symbol, then $f^M : M^n \to M$;
- If $P \in L$ is an n-ary predicate symbol, then $P^M \subseteq M^n$.

Example 4. $\mathbb{R} = (\mathbb{R}; 0, 1, +, -, \cdot, <)$ is an L_{ORing}-structure.

Definition 5. An L-term is defined as follows.

- Every variable is an L-term.
- Every constant symbol of L is an L-term.
- If $f \in L$ is an n-ary function symbol and $t_1, \ldots, t_n$ are L-terms, then $f(t_1, \ldots, t_n)$ is an L-term.

Example 6. $f_+(f_\times(c_0, x), f_-(c_1))$ is an L_{ORing}-term.

Let M and N be L-structures. For each L-term $t(\bar{x})$ and $\bar{a} \in M$, the interpretation $t^M(\bar{a}) \in M$ is naturally defined.

Example 7. $f_\times(f_+(x, c_1), y)^{\mathbb{R}}(2, 3) = f_\times^{\mathbb{R}}(f_+^{\mathbb{R}}(2, c_1^{\mathbb{R}}), 3) = (2 + 1) \times 3 = 9$.

Definition 8. An atomic L-formula is defined as follows.

- If t_1 and t_2 are L-terms, then $t_1 = t_2$ is an atomic L-formula.
- If $P \in L$ is an n-ary predicate symbol and $t_1, \dots, t_n$ are L-terms, then $P(t_1, \dots, t_n)$ is an atomic L-formula.

Example 9. $f_\times(x, y) = c_0$ and $P_<(f_+(c_1, c_1), f_-(x))$ are atomic L_{ORing}-formulas.

Definition 10. An L-formula is defined as follows.

- Every atomic L-formula ia an L-formula.
- If φ is an L-formula, then $\neg\varphi$ is an L-formula.
- If φ and ψ are L-formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$ are L-formulas.
- If φ is an L-formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are L-formulas.

Example 11. $\exists y\big((\neg(x = 0)) \to P_<(f_\times(x, y), 1)\big)$ is an L_{ORing}-formula.

Definition 12. For each L-formula $\varphi(\bar{x})$ and $\bar{a} \in M$, the satisfication relation $M \models \varphi(\bar{a})$ is defined as follows.

- If $t_1(\bar{x})$ and $t_2(\bar{x})$ are L-terms, then $M \models (t_1 = t_2)(\bar{a}) \Leftrightarrow t_1^M(\bar{a}) = t_2^M(\bar{a})$.
- If $P \in L$ is an n-ary predicate symbol and $t_1, \dots, t_n$ are L-terms, then $M \models (P(t_1, \dots, t_n))(\bar{a}) \Leftrightarrow \big(t_1^M(\bar{a}), \dots, t_n^M(\bar{a})\big) \in P^M$.
- If $\psi(\bar{x})$ is an L-formula, then $M \models (\neg\psi)(\bar{a}) \Leftrightarrow M \not\models \psi(\bar{a})$.

If $\psi_1(\bar{x})$ and $\psi_2(\bar{x})$ are L-formulas, then

- $M \models (\psi_1 \wedge \psi_2)(\bar{a}) \Leftrightarrow M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a})$;
- $M \models (\psi_1 \vee \psi_2)(\bar{a}) \Leftrightarrow M \models \psi_1(\bar{a})$ or $M \models \psi_2(\bar{a})$;
- $M \models (\psi_1 \to \psi_2)(\bar{a}) \Leftrightarrow M \models \psi_1(\bar{a})$ implies $M \models \psi_2(\bar{a})$;
- $M \models (\psi_1 \leftrightarrow \psi_2)(\bar{a}) \Leftrightarrow M \models \psi_1(\bar{a})$ is equivalent to $M \models \psi_2(\bar{a})$.

If $\psi(\bar{x}, y)$ is an L-formula, then

- $M \models (\forall y\psi)(\bar{a}) \Leftrightarrow M \models \psi(\bar{a}, b)$ for all $b \in M$;
- $M \models (\exists y\psi)(\bar{a}) \Leftrightarrow M \models \psi(\bar{a}, b)$ for some $b \in M$.

Example 13. Let $\varphi(x)$ be $\exists y\,(f_+(x, x) = f_\times(y, y))$. Then $\mathbb{R} \models \varphi(3)$ because $\mathbb{R} \models f_+(3, 3) = f_\times(\sqrt{6}, \sqrt{6})$.

$\forall$ and $\exists$ are called quantifiers.

Definition 14. Let φ be an L-formula and x be a variable which appears in φ. Then x is said to be free in φ if x does not appear in the scope of any quantifier in φ. An L-formula φ is said to be an L-sentence if φ do not have any free variable.

Example 15. Let φ be $\exists y\big((\neg(x = 0)) \to f_\times(x, y) = 1\big)$. Then x is free in φ, so φ is not an L_{ORing}-sentence. Let ψ be $\forall x\exists y\big((\neg(x = 0)) \to f_\times(x, y) = 1\big)$. Then ψ is an L_{ORing}-sentence.

A set of L-sentences is called an L-theory. Let T, T_1 and T_2 be L-theories.

Definition 16. M is said to be a model of T ($M \models T$) if $M \models \varphi$ for all $\varphi \in T$.

Definition 17. T_2 is said to follow from T_1 ($T_1 \models T_2$) if $M \models T_2$ for all $M \models T_1$.

Definition 18. T_1 is said to be equivalent to T_2 if $T_1 \models T_2$ and $T_2 \models T_1$.

We introduce the hierarchy of L-formulas.

Definition 19. L-formulas are classfied as follows.

- A Δ_0 formula is a quantifier-free L-formula.
- A Π_1 formula is an L-formula of the form $\forall x_1 \forall x_2 \dots \forall x_n \psi$ where ψ is a Δ_0 formula and $n \geq 0$.
- A Σ_1 formula is an L-formula of the form $\exists x_1 \exists x_2 \dots \exists x_n \psi$ where ψ is a Δ_0 formula and $n \geq 0$.
- A Π_2 formula is an L-formula of the form $\forall x_1 \forall x_2 \dots \forall x_n \exists y_1 \exists y_2 \dots \exists y_m \psi$ where ψ is a Δ_0 formula and $n, m \geq 0$.
- A Σ_2 formula is an L-formula of the form $\exists x_1 \exists x_2 \dots \exists x_n \forall y_1 \forall y_2 \dots \forall y_m \psi$ where ψ is a Δ_0 formula and $n, m \geq 0$.
- $\cdots$.

Example 20. $\forall x \exists y \big((\neg(x = 0)) \to f_\times(x, y) = 1 \big)$ is a Π_2 L_{ORing}-sentence.

Definition 21. M is said to be a substructure of N ($M \subseteq N$) if M is a subset of N and the following holds:

- If $c \in L$ is a constant symbol, then $c^M = c^N$;
- If $f \in L$ is an n-ary funciton symbol, then $f^M = f^N|_{M^n}$;
- If $P \in L$ is an n-ary function symbol, then $P^M = P^N \cap M^n$.

Example 22. $\mathbb{Z}$ is a substructure of $\mathbb{R}$ as L_{ORing}-structures.

Definition 23. M is said to be elementarily equivalent to N ($M \equiv N$) if $M \models \varphi \Leftrightarrow N \models \varphi$ for all L-sentences φ.

We introduce one of the most important theorems in model theory.

Fact 24 (Compactness theorem)**.** T has a model if and only if every finite subset of T has a model.

The following facts follow from the compactness theorem.

Fact 25. Suppose that $T_1 \not\models \varphi$ or $T_2 \not\models \neg\varphi$ for all Σ_n (Π_n) sentences φ. Then there exist $M_1 \models T_1$ and $M_2 \models T_2$ such that $M_1 \not\models \varphi$ or $M_2 \not\models \neg\varphi$ for all Σ_n (Π_n) sentences φ.

Fact 26. Suppose that $M \models \varphi \Rightarrow N \models \varphi$ for all Σ_n sentences φ. Then there exists an L-structure N' such that $M \subseteq N' \equiv N$ and $M \models \varphi(\bar{a}) \Rightarrow N' \models \varphi(\bar{a})$ for all Σ_n formulas $\varphi(\bar{x})$ and $\bar{a} \in M$.

2 Hereditary theories and Π_1 theories

Remark 27. Let M be a substructure of N. Then $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a})$ for all Δ_0 formulas $\varphi(\bar{x})$ and $\bar{a} \in M$. Thus $N \models \varphi \Rightarrow M \models \varphi$ for all Π_1 sentences φ.

We introduce hereditary theories and Π_1 theories.

Definition 28. T is said to be hereditary if the following holds: If M is a substructure of N and $N \models T$, then $M \models T$.

Definition 29. T is said to be a Π_1 theory if T is equivalent to an L-theory consisting of Π_1 sentences.

Remark 30. By Remark 27, T is hereditary if T is a Π_1 theory.

Example 31. Let $L_1 = \{\cdot\}$, where $\cdot$ is a binary function symbol. Let T_1 be a set of the following L_1-sentences:

- $\forall x \forall y \forall z\, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$,
- $\exists y \forall x\, (x \cdot y = y \cdot x = x)$,
- $\forall x \exists y\, (x \cdot y = y \cdot x = e)$.

Then T_1 is not hereditary, so T_1 is not a Π_1 theory. To make T_1 hereditary, we have to add some constant symbols and function symbols to L_1. Let $L_2 = L_1 \cup \{e, {}^{-1}\}$, where e is a constant symbol and ${}^{-1}$ is a unary function symbol. Let T_2 be a set of the following L_2-sentences:

- $\forall x \forall y \forall z\, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$,
- $\forall x\, (x \cdot e = e \cdot x = x)$,
- $\forall x\, (x \cdot x^{-1} = x^{-1} \cdot x = e)$.

Then T_2 is a Π_1 theory, so T_2 is hereditary.

The converse of Remark 30 also holds.

Theorem 32. Suppose that T is hereditary. Then T is a Π_1 theory.

Proof. Let $T^* = \{\psi : \Pi_1 \text{ sentence} \mid T \models \psi\}$. We prove $T^* \models T$. Let $\varphi \in T$. It is sufficient to show that there exists a Π_1 sentence ψ such that $T \models \psi$ and $\neg\varphi \models \neg\psi$.

Suppose that $T \not\models \psi$ or $\neg\varphi \not\models \neg\psi$ for all Π_1 sentences ψ. By Fact 25, there exist $M_1 \models T$ and $M_2 \models \neg\varphi$ such that $M_1 \not\models \psi$ or $M_2 \not\models \neg\psi$ for all Π_1 sentences ψ. Thus $M_2 \models \psi \Rightarrow M_1 \models \psi$ for all Σ_1 sentences ψ because the negation of Σ_1 formulas are equivalent to Π_1 formulas. By Fact 26, there exists M_1' such that $M_2 \subseteq M_1' \equiv M_1$. Then $M_1' \models T$. Since T is hereditary, we have $M_2 \models T$. Especially we obtain $M_2 \models \varphi$, which is a contradiction. □

Therefore, T is hereditary $\Leftrightarrow$ T is a Π_1 theory.

3 Inductive theories and Π_2 theories

Let $\omega = \{0, 1, 2, \dots\}$.

Remark 33. Let $(M_i)_{i\in\omega}$ be a chain of L-structures and $N := \bigcup_{i\in\omega} M_i$, that is,

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_i \subseteq M_{i+1} \subseteq \cdots \subseteq N \ \ (\forall i \in \omega).$$

Let φ be a Π_2 sentence. Suppose that $M_i \models \varphi$ for all $i \in \omega$. Then $N \models \varphi$.

We introduce inductive theories and Π_2 theories.

Definition 34. T is said to be inductive if the union of any chain of models of T is a model of T.

Definition 35. T is said to be a Π_2 theory if T is equivalent to an L-theory consisting of Π_2 sentences.

Remark 36. By Remark 33, T is inductive if T is a Π_2 theory.

Example 37. The theories of groups, rings, fields and dense linear orders without endpoints are Π_2 theories, so these are inductive.

Example 38. Let $L = \{<\}$, where $<$ is a binary predicate symbol. Then $T := \{\varphi : L\text{-sentence} \mid \mathbb{Z} \models \varphi\}$ is not inductive, so T is a Π_2 theory.

$\because$) Consider the following chain: $\mathbb{Z} \subset \frac{1}{2}\mathbb{Z} \subset \frac{1}{4}\mathbb{Z} \subset \cdots$.

The union of this chain is dense. However, $\mathbb{Z}$ is not dense, that is, $\mathbb{Z} \models \exists x \exists y \forall z (x < y \wedge \neg(x < z < y))$.

Definition 39. Let M be a substructure of N. Then M is an elementary substructure of N ($M \preceq N$) if $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a})$ for all L-formulas $\varphi(\bar{x})$ and $\bar{a} \in M$.

Fact 40. Let $(N_i)_{i\in\omega}$ be a chain of L-structures and $N := \bigcup_{i\in\omega} N_i$. Suppose that $N_i \preceq N_{i+1}$ for all $i \in \omega$. Then $N_i \preceq N$ for all $i \in \omega$.

The converse of Remark 36 also holds.

Theorem 41. Suppose that T is inductive. Then T is a Π_2 theory.

Proof. Let $T^* = \{\psi : \Pi_2 \text{ sentence} \mid T \models \psi\}$. We prove $T^* \models T$. Let $\varphi \in T$. It is sufficient to show that there exists a Π_2 sentence ψ such that $T \models \psi$ and $\neg\varphi \models \neg\psi$. Suppose that $T \not\models \psi$ or $\neg\varphi \not\models \neg\psi$ for all Π_2 sentences ψ. By Fact 25, there exist $M \models T$ and $N_0 \models \neg\varphi$ such that $M \not\models \psi$ or $N_0 \not\models \neg\psi$ for all Π_2 sentences ψ. Hence $N_0 \models \psi \Rightarrow M \models \psi$ for all Σ_2 sentences ψ. By Fact 26, there exists an L-structure M_0 such that $N_0 \subseteq M_0 \equiv M$ and $N_0 \models \psi(\bar{a}) \Rightarrow M_0 \models \psi(\bar{a})$ for all Σ_2 formulas $\psi(\bar{x})$ and $\bar{a} \in N_0$. Thus $M_0 \models \psi(\bar{a}) \Rightarrow N_0 \models \psi(\bar{a})$ for all Σ_1 formulas $\psi(\bar{x})$ and $\bar{a} \in N_0$. Consider in the language $L(N_0) := L \cup N_0$, where each $a \in N_0$ is a constant symbol. Then $M_0 \models \psi \Rightarrow N_0 \models \psi$ for all Σ_1 $L(N_0)$-sentences ψ. By Fact 26, there exists an $L(N_0)$-structure N_1 such that $M_0 \subseteq N_1 \equiv N_0$. Since $N_0 \equiv N_1$ as $L(N_0)$-structures, $N_0 \preceq N_1$ as L-structures.

By repeating the above discussion, we obtain the following chain:

$$N_0 \subseteq M_0 \subseteq N_1 \subseteq M_1 \subseteq N_2 \subseteq \cdots,$$

where $M_i \equiv M$ and $N_i \preceq N_{i+1}$ for all $i \in \omega$. Let $N = \bigcup_{i\in\omega} N_i = \bigcup_{i\in\omega} M_i$. By Fact 40, we have $N_i \preceq N$ for all $i \in \omega$. Hence $N \models \neg\varphi$. Since $M_i \models T$ for all $i \in \omega$ and T is inductive, we have $N \models T$. Especially we have $N \models \varphi$, which is a contradiction. □

Therefore, T is inductive $\Leftrightarrow$ T is a Π_2 theory.

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