Automorphisms on graphs

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Abstract

We report several known results on automorphism groups of structures related to Ramsey properties.

1 Introduction

A Hausdorff topological group G is said to be extremely amenable if for every continuous action of G on a compact Hausdorff space has a fixed point. Only a few examples[2][7][8] were known before the KPT-correspondence[4] was found by Kechris, Pestov and Todorcevic. The KPT-correspondence shows that, roughly speaking, G is extremely amenable if and only if G is an automorphism group of a structure that has Ramsey-type property, and such structures are always constructed as a Fraïssé limit of a class of finitely generated structures by the amalgamation method.

In this article, we give a brief explanation of KTP-correspondence and related topics, especially about the infinite structural Ramsey property which has not been investigated so much. Throughout, ω is the set of natural numbers and the set $\{0, \cdots, n-1\}$ is denoted by $n \in \omega$. Also, groups are considered as topological groups and automorphism groups have point-wise convergence topology unless otherwise noted.

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2 The KPT-correspondence

Let G be a Hausdorff topological group.

Definition 1. Let X be a topological space.

1. An action of G on X is a continuous map $G \times X \ni (g, x) \mapsto gx \in X$ such that $ex = x$ and $g(h(x)) = (gh)x$ for every $g, h \in G$ and $x \in X$.
2. G is said to be extremely amenable if every action of G on a compact Hausdorff space X has a fixed point $x_0 \in X$, i.e., $gx_0 = x_0$ for every $g \in G$.

Remark 2. If $G \neq \{e\}$ is compact, then G cannot be extremely amenable. Indeed, the natural action of G on G itself has no fixed point unless G is trivial. Moreover, Veech [10] showed that if G is locally compact, then G is not extremely amenable.

Probably, the first concrete example of an extremely amenable group was found by Gromov and Milman in 1983. However, not so many examples were found before, in 2005, Kechris, Pestov and Todorcevic published their famous paper [4] and prove that many examples can be obtained through Ramsey structures.

Example 3. Some remarkable examples of extremely amenable groups.

1. (Gromov and Milman[2]) The group of the unitary operators on the Hilbert space l^2 with strong operator topology is extremely amenable.
2. (Pestov [7]) The automorphism group $\mathrm{Aut}(\mathbb{Q}, <)$ of the ordered set of rationals is extremely amenable.
3. (Pestov [8]) The group of isometries of the Ulysohn space is extremely amenable.

In order to explain the KPT-correspondence, we need some definitions on structural Ramsey property. We suppose the readers are familiar with the basic notions of structures in mathematical logic. (However, if you don't know about structures, then just replace structures by hyper digraphs and finitely generated substructures by finite induced subgraphs.)

Definition 4. Let L be a language and M be an (infinite) countable L-structures.

- $Age(M)$ is the set of finitely generated substructures of M.
- For $A, B \in Age(M)$, $\binom{B}{A}$ is the set of substructures A' of B such that $A' \cong A$.
- For $n \in \omega$ and $A, B, C \in Age(M)$, $C \to (B)^A_n$ means the following condition: for any coloring map $f : \binom{C}{A} \to n$, there is $B' \in \binom{C}{B}$ such that $f|\binom{B'}{A}$ is a constant map.
- M is said to be ultrahomogeneous if for every $A, A' \in Age(M)$ and an isomorphism $\sigma : A \to A'$ there is an automorphism $\bar{\sigma} \in \mathrm{Aut}(M)$ such that $\sigma \subset \bar{\sigma}$.
- M is said to be a Ramsey structure if it is ultrahomogenous and satisfies the following: For every $n \in \omega$ and $A, B \in Age(M)$ there is $C \in Age(M)$ such that $C \to (B)^A_n$.

The most basic example of Ramsey structure is the dense linear order $(\mathbb{Q}, <)$.

Example 5. One can check that $M = (\mathbb{Q}, <)$ is a Ramsey structure as follows.

1. $M = (\mathbb{Q}, <)$ is ultrahomogeneous. Indeed, if $\sigma : A \to A'$ is an order preserving bijection with finite subsets $A, A' \subset \mathbb{Q}$, then it is easy to find an order preserving bijection $\bar{\sigma} : \mathbb{Q} \to \mathbb{Q}$ extending σ.
2. Let $A, B \in Age(M)$ and suppose that $|A| = i$ and $|B| = j$. Let $k \in \omega$ be large enough such that finite Ramsey theory with respect to n-coloring holds for (i, j, k). Then any $C \in Age(M)$ with $|C| \geq k$ satisfies $C \to (B)^A_n$.

Remark 6. One may notice that, in the above example, we don't need the ordering to prove the Ramsey property. In fact, the infinite set ω with no structure (i.e., empty language) satisfies the definition of Ramsey structure. However, to understand the KPT-correspondence, it is important to consider a kind of ordering which forces every $A \in Age(M)$ has no nontrivial automorphism.

Definition 7. Let M be an L-structure and $A \in Age(M)$. A is said to be rigid if $\mathrm{Aut}(A) = \{id_A\}$.

Example 8. The following structures M are Ramsey structures. Moreover, $Age(M)$ consists of rigid elements since they have a linear ordering.

1. (Nešetřil and Rödl [6]) The ordered random graph: $M = (V; E, <)$.
2. (Graham, Leeb, and Rothschild [1]) A countable infinite vector space V over a finite field $\mathbb{F}$: $M = (V; 0, +, \{\lambda\cdot\}_{\lambda\in\mathbb{F}})$. One can expand this structure with an anti-lexicographic order with an linear ordering over F, and it remains to be a Ramsey structure.
3. Many other structures such as metric spaces, posets, lattices, and so on, are known as Ramsey structures. See [5] for example.

Interestingly, if we omit the ordering from the firs example, then the random graph isn't a Ramsey structure anymore.

The following fact is not so hard to show.

Fact 9. Let G be a subgroup of S_∞. G is closed if and only if $G = \mathrm{Aut}(M)$ for some ultrahomogeneous structure M.

Now we have prepared to see KPT-correspondence.

Theorem 10 (KPT [4])**.** Let G be a closed subgroup of S_∞. The following are equivalent.

1. G is extremely amenable.
2. $G = \mathrm{Aut}(M)$ for some Ramsey structure M such that $Age(M)$ consists of rigid elements.

Recently, as an analogy of KPT-correspondence, some researchers investigated automorphism groups of metric structures in the context of continuous logic (replacing usual structures with first order logic), and they found the similar correspondence holds. (See [3] for example.)

3 Infinite Ramsey property for structures

In this section, we see a short discussion on the infinite Ramsey property of structures.

The following are not trivial but one can prove it by using König's Lemma.

Fact 11. Let M be an ultrahomogeneous L-structure with finite relational language L. The following are equivalent.

1. M is a Ramsey structure.

2. For any $n \in \omega$ and $A, B \in Age(M)$, $M \to (B)^A_n$ holds.

Hence, the definition below gives a stronger notion of Ramsey structures.

Definition 12. Let M be a countable L-structure and $A \in Age(M)$.

1. We say M has infinite Ramsey property with respect to A if for every $n \in \omega$, $M \to (M)^A_n$ holds.
2. We say M has infinite Ramsey property if for every $n \in \omega$ and $A \in Age(M)$, $M \to (M)^A_n$ holds.

Of course the infinite set ω with empty language satisfies the definition of infinite Ramsey property. However, $(\mathbb{Q}, <)$ does not have infinite Ramsey property.

Proposition 13. Let $M = (\mathbb{Q}, <)$.

1. M has infinite Ramsey property with respect to a singleton $A = \{*\}$.
2. M does not have infinite Ramsey property with respect to any two points set $A = \{a < b\}$.

Also, it is well known that the (ordered) random graph has infinite Ramsey property with respect to a singleton, however, it has no infinite Ramsey property with respect to an edge.

Fact 14. Let $M = (V, E, <)$ be a random graph. Then for any partition of V into n-sets, $V = \bigsqcup_{i<n} V_i$, one of V_i is isomorphic to M.

However, Sauer et. al. found that if we add a special dense linear order $<_*$ to M, they have infinite Ramsey property with respect to (at least) any two point set A (See [9]). Unfortunately, their technique is complicated for the author to find the connection of infinite Ramsey property and topological dynamics, however it may be interesting to investigate infinite Ramsey property in the point of view from topological dynamics.

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