On lattice isomorphisms between lattice of topologies on linear spaces

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Abstract

We consider a lattice isomorphism between the lattice of topologies on linear spaces whose dimensions are at least two over Hausdorff topological fields. By using a result of J. Hartmanis (1958), we show that if the image of the set of linear topologies on the domain space by the lattice isomorphism coincides with the set of linear topologies on the target space, then the coefficient fields are isomorphic and the linear spaces are semi-linear isomorphic. Moreover, such a lattice isomorphism or a composition of a involution with the lattice isomorphism is induced by a composition of a linear translation with the semi-linear isomorphism.

1 Introduction

1.1 Back ground and motivation

For a fixed set X, the lattice of topologies $\Sigma(X) = (\Sigma(X), \subset)$ is a partially ordered set consisting of the set $\Sigma(X)$ of all topologies that can be defined on X endowed with the set inclusion $\subset$. It is known that this partially ordered set has supremum and infimum for any subsets of $\Sigma(X)$. In particular, we can define two binary operators on $\Sigma(X)$ by taking supermum and infimum of two elements of $\Sigma(X)$, which allows us to study $\Sigma(X)$ in algebraic manner [2]. Thus, although the lattice of topologies is one of the areas in the general topology, but also the structure of the lattice of topologies itself is an interesting object.

In this paper, we consider when X is a linear space over a Hausdorff topological filed K. Among all topologies on X, a natural topology for the linear

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space is a topology, which we call *linear topology*, such that the two linear operators

$$\text{the addition} : X \times X \to X,$$
$$\text{the scalar multiplication} : K \times X \to X$$

are continuous. We denote the subset of $\Sigma(X)$ consisting of all linear topologies by $\tau_K(X)$. It is known that the partially ordered set $\tau_K(X) = (\tau_K(X), \subset)$ also has supremum and infimum for any subsets of $\tau_K(X)$. In particular, the partially ordered set $\tau_K(X)$ has the maximum element, denoted by $T^{\max}(X)$.
The question we deal with is to what extent the structure of the linear spaces are determined by the distribution of $\tau_K(X)$ in $\Sigma(X)$. More precisely, for two linear spaces X_1 and X_2 over two Hausdorff topological fields K_1 and K_2, respectively, if there exists a lattice isomorphism Θ between $\Sigma(X_1)$ and $\Sigma(X_2)$ such that the image of $\tau_{K_1}(X_1)$ by Θ coincides with $\tau_{K_2}(X_2)$, then our question is whether K_1 and K_2 are isomorphic and X_1 and X_2 are isomorphic.

1.2 A related work

Let us explain a related work of Juris Hartmanis [3, Theorem 4], which is on the automorphism group of the lattice of topologies on a set X. We consider automorphisms of $\Sigma(X)$. Let $f : X \to X$ be a bijection map. Then f induces a lattice isomorphism f_*, called *induced map* by f, defined by

$$f_*(T) \coloneqq \{V \subset X \mid f^{-1}(V) \in T\}, \tag{1}$$

where $T \in \Sigma(X)$ is a topology on X. The inverse map of f_* is clearly, $(f^{-1})_*$ and also f^*, defined by

$$f^*(T) \coloneqq \{f^{-1}(V) \mid V \in T\}. \tag{2}$$

Another candidate for an automorphism of $\Sigma(X)$ is a map C, defined by

$$C(T) \coloneqq \{X \setminus U \mid U \in T\}.$$

J. Hartmanis showed, in [3], that C is a lattice automorphism that does not come from a bijective map when the cardinality of X is finite and at least three and that C is not a lattice automorphism when the cardinality of X is infinite. The following result by J. Hartmanis states the converse: every automorphism of $\Sigma(X)$ comes from f_* and C.

Fact 1 ([3, Theorem 4])**.** Let X be a non-empty set. When the cardinality of X is one, two or infinity, the group $\mathrm{Aut}(\Sigma(X))$ of automorphisms of the lattice of topologies $\Sigma(X)$ on X is isomorphic to the symmetric group $\mathrm{Sym}(X)$ on X. When the cardinality of X is finite and at least three, the group $\mathrm{Aut}(\Sigma(X))$ is isomorphic to the direct product of $\mathrm{Sym}(X)$ and the two element group.

Although the above statement is on the structure of the automorphism group, the proof shows that elements of $\mathrm{Aut}(\Sigma(X))$ are of form f_* or $C \circ f_*$, where f is a bijection map from X to itself. Hence, by a minor change of the proof in [3], the following statement holds:

Fact 2. Let X_1 and X_2 be two non-empty sets such that their lattice of all topologies $\Sigma(X_1), \Sigma(X_2)$ are isomorphic. Let $\Theta : \Sigma(X_1) \to \Sigma(X_2)$ be a lattice isomorphism. When the cardinality of X_1 is one, two or infinity, there exists a unique bijection $\theta : X_1 \to X_2$ such that the induced map θ_* coincides with Θ. When the cardinality of X_1 is finite and at least three, there exists a unique bijection $\theta : X_1 \to X_2$ such that the induced map θ_* coincides with Θ or $C \circ \Theta$.

As a corollary of Fact 2, if the lattice of topologies of two sets are isomorphic, then the cardinality of the two sets coincides.

1.3 The main result

When X_1 and X_2 in Fact 2 are linear spaces over Hausdorff topological fields K_1, K_2, respectively, our main result is a similar statement of Fact 2. We further assume that the image of $\tau_{K_1}(X_1)$ by $\Theta : \Sigma(X_1) \to \Sigma(X_2)$ coincides with $\tau_{K_2}(X_2)$, which implies that the restriction of Θ to $\tau_{K_1}(X_1)$ is a lattice isomorphism between $\tau_{K_1}(X_1)$ and $\tau_{K_2}(X_2)$. Then it will be shown that there exists a topological field isomorphism $\phi : K_1 \to K_2$ and $\theta : X_1 \to X_2$ in Fact 2 is a composition of a ϕ-semi-linear isomorphism with a linear translation. Here, a ϕ-semi-linear isomorphism $\Phi : X_1 \to X_2$ with a field isomorphism $\phi : K_1 \to K_2$ is a bijection such that for all $x, y \in X_1$ and $\alpha \in K_1$, the map Φ satisfies

$$\Phi(x+y) = \Phi(x) + \Phi(y),$$
$$\Phi(\alpha \cdot x) = \phi(\alpha) \cdot \Phi(x).$$

More specific statement of the result is the following:

Theorem 1. [1] Let K_1, K_2 be two Hausdorff topological fields and X_1, X_2 be two linear spaces, whose dimension is at least two over K_1 and K_2,

[1] Assumptions on the topological fields and on dimensions of linear spaces are dropped from the author's research presentation at the work shop "New developments of transformation groups", held at RIMS in May 2021.

respectively. Let $\Theta : \Sigma(X_1) \to \Sigma(X_2)$ be a lattice isomorphism such that $\Theta(\tau_{K_1}(X_1)) = \tau_{K_2}(X_2)$ holds. Then there exists a unique triple (ϕ, Φ, x_0) such that when the cardinality of X_1 is infinite, the induced map by $\Phi + x_0 : X_1 \to X_2$ coincides with Θ and when the cardinality of X_1 is finite, the induced map by $\Phi + x_0$ coincides with Θ or $C \circ \Theta$, where $\phi : K_1 \to K_2$ is a isomorphism between the topological fields, $\Phi : X_1 \to X_2$ is a ϕ-semi-linear isomorphism and x_0 is a point of X_2.

As a corollary, if the lattice of topologies of two linear spaces are isomorphic with a map such that the restriction is also an isomorphism between the lattice of linear topologies, then the coefficient fields are isomorphic as topological fields and the two linear spaces are semi-isomorphic.
We consider the case when $K_1 = K_2$ as topological fields and $X_1 = X_2$ in the situation of Theorem 1. Another corollary of the theorem is on the subgroup $G(X_1)$ of $\mathrm{Aut}(\Sigma(X_1))$, defined by

$$G(X_1) \coloneqq \{\Theta \in \mathrm{Aut}(\Sigma(X_1)) \mid \Theta(\tau_{K_1}(X_1)) = \tau_{K_1}(X_1)\}.$$

By Theorem 1, the group $G(X_1)$ is isomorphic to the group of semi-direct product of the abelian group X_1 and the group $\Gamma\mathrm{GL}(X_1)$ of semi-linear automorphisms. Here, a permutation $\Phi : X_1 \to X_1$ that preserves the addition of X_1 belongs to $\Gamma\mathrm{GL}(X_1)$ if and only if there exists an automorphism $\phi : K_1 \to K_1$ of the topological field such that Φ is a ϕ-semi-linear isomorphism. The multiplication of the semi-direct group is defined by

$$(x_1, \Phi_1) \cdot (x_2, \Phi_2) \coloneqq (x_1 + \Phi_1(x_2), \Phi_1 \circ \Phi_2)$$

for $(x_1, \Phi_1), (x_2, \Phi_2) \in X_1 \times \Gamma\mathrm{GL}(X_1)$.

2 A proof of the main result

This section is devoted to give a proof of Theorem 1.

The key idea of our proof is to consider two maps N_X, T_X between linear topologies on X and linear subspaces of X, defined by the following definitions.

Definition 1. Let T be a linear topology on a linear space X. We define a subspace $N_X(T)$ of X by

$$N_X(T) \coloneqq \bigcap_{0 \in U \in T} U.$$

Definition 2. Let S be a linear subspace of X, we define a linear topology $T_X(S)$ on X by

$$T_X(S) \coloneqq {\pi_S}^* \circ \pi_{S*}(T^{\max}(X)),$$

where π_S is a quotient map from X to X/S and $\pi_{S*}, {\pi_S}^*$ are defined in the same way with (1) and (2), respectively .

Remark 1. It is easy to show that $N_X(T)$ is actually a linear subspace. Also, it is not difficult to prove that for a linear map $f : X \to Y$ between linear spaces X, Y, the map f^* sends linear topologies on Y to linear topologies on X and if f is surjective, f_* sends linear topologies on X to linear topologies on Y. Hence $T_X(S)$ is actually a linear topology on X. As for the relation between N_X and T_X, the composition $T_X \circ N_X$ is not always the identity map on the set of linear topologies because there may be more than two Hausdorff linear topologies on X. On the other hand, since we use the maximum element $T^{\max}(X)$ for the definition of $T_X(S)$, the other composition $N_X \circ T_X$ is the identity map on the set of all subspaces of X.

Let us come back to the situation of Theorem 1. From Fact 2, there exists a bijection $\theta : X_1 \to X_2$ such that the induced map θ_* coincides with Θ or with $C \circ \Theta$. We first consider the case θ_* coincides with Θ and show that there exists a triple (ϕ, Φ, x_0) satisfying $\theta = \Phi + x_0$.
We define an element $x_0 := \theta(0)$ and a bijective map $\Phi : X_1 \to X_2$ by $\Phi + x_0 = \theta$ so that $\Phi(0) = 0$ holds. Then because a linear translation map $X_2 \ni x \mapsto x + a \in X_2$ for a fixed point $a \in X_2$ induces a lattice automorphism of $\Sigma(X_2)$ such that the image of $\tau_{K_2}(X_2)$ is $\tau_{K_2}(X_2)$ itself, the induced map $\Phi_* : \Sigma(X_1) \to \Sigma(X_2)$ is a lattice isomorphism such that the image $\Phi(\tau_{K_1}(X_1))$ is $\tau_{K_2}(X_2)$.
Let S be a subspace of X_1. By definition and an easy argument, we obtain the linear subspace $N_{X_2}(\Phi_*(T_{X_1}(S)))$ equals to $\Phi(N_{X_1}(T_{X_1}(S))$. Moreover, since the map $N_{X_1} \circ T_{X_1}$ is identity, the subspace $N_{X_2}(\Phi_*(T_{X_1}(S)))$ coincides with $\Phi(S)$. Thus, the image of the subspace S by the map Φ is a subspace of X_2. Furthermore, because taking the image of subspaces by Φ preserves the set inclusion $\subset$ and Φ is a bijection, the dimension of $\Phi(S)$ is one if and only if that of S is one. As for affine subspaces, we claim the following:

Lemma 1. Let S be a linear subspace of X_1. Then the image $\Phi(a + S)$ coincides with $\Phi(a) + \Phi(S)$ for any $a \in X_1$.

Proof. Assume that there exists a point $q \in \Phi(a + S) \setminus (\Phi(a) + \Phi(S))$. Let $p \in X_1$ be a point such that $\Phi(p) = q$. Since $\Phi(S) = N_{X_2}(\Phi_*(T_{X_1}(S)))$ holds, the affine subspace $\Phi(a) + \Phi(S)$ is represented as $\bigcap_{\Phi(a) \in V \in \Phi_*(T_{X_1}(S))} V = \bigcap_{a \in U \in T_{X_1}(S)} \Phi(U)$. Thus we have an open neighborhood U of the point a with respect to the topology $T_{X_1}(S)$ that is not an open neighborhood of p. On the other hand, because p belongs to $a + S = \bigcap_{a \in U \in T_{X_1}(S)} U$, this is a contradiction. Therefore, we obtain $\Phi(a + S) \subset \Phi(a) + \Phi(S)$. The same argument for the inverse map Φ^{-1} shows that $a + S = \Phi^{-1}(\Phi(a + S)) \subset \Phi^{-1}(\Phi(a) + \Phi(S)) \subset \Phi^{-1}(\Phi(a)) + \Phi^{-1}(\Phi(S)) = a + S$, which implies the claim of this lemma. $\square$

Now, we show that $\Phi : X_1 \to X_2$ preserve the additions. We denote by $\mathrm{Span}(x)$, the one-dimensional subspace of a linear space X generated by nonzero element $x \in X$. Let x, y be nonzero elements of X_1 that are linearly independent. Note that $\Phi(x)$ and $\Phi(y)$ are also linearly independent since Φ is bijective. By Lemma 1, the image $\Phi(\mathrm{Span}(x) + y)$ is equal to $\Phi(\mathrm{Span}(x)) + \Phi(y)$ and this coincides with $\mathrm{Span}(\Phi(x)) + \Phi(y)$ since $\Phi(x)$ belongs to one-dimensional subspace $\Phi(\mathrm{Span}(x))$. The same argument shows that $\Phi(\mathrm{Span}(y) + x) = \mathrm{Span}(\Phi(y)) + \Phi(x)$. Thus, the intersection $\Phi(\mathrm{Span}(x) + y) \cap \Phi(\mathrm{Span}(y) + x)$, to which $\Phi(x + y)$ belong is a one point set $\{\Phi(x) + \Phi(y)\}$. Therefore, we obtain $\Phi(x + y) = \Phi(x) + \Phi(y)$. By using this linear independent case, similar arguments show that $\Phi(-x) = -\Phi(x)$ for nonzero x and that $\Phi(x + y) = \Phi(x) + \Phi(y)$ when x, y are not linearly independent. Here, we use the assumption of Theorem 1 on the dimension of X_1 to take an element z such that x, z are linearly independent.
Next, we define a field isomorphism $\phi : K_1 \to K_2$. Let x_0 be a non-zero element of X_1. For every $\alpha \in K_1$, since $\Phi(\alpha \cdot x_0)$ belongs to one-dimensional subspace $\mathrm{Span}(\Phi(x_0))$, there exists a unique $\beta = \beta(x_0, \alpha) \in K_2$ such that

$$\Phi(\alpha \cdot x_0) = \beta \cdot \Phi(x_0).$$

Let us show that β does not depend on the choice of x_0. Take an another nonzero element $x_1 \in X_1$. If x_0, x_1 are linearly independent, since Φ preserves the additions, we have the following equality:

$$\begin{aligned}
\beta(x_0 + x_1, \alpha) \cdot (\Phi(x_0) + \Phi(x_1)) =& \beta(x_0 + x_1, \alpha)\Phi(x_0 + x_1) \\
=& \Phi(\alpha \cdot (x_0 + x_1)) \\
=& \Phi(\alpha \cdot x_0) + \Phi(\alpha \cdot x_1) \\
=& \beta(x_0, \alpha) \cdot \Phi(x_0) + \beta(x_1, \alpha) \cdot \Phi(x_1).
\end{aligned}$$

Since $\Phi(x_0)$ and $\Phi(x_1)$ are linearly independent, the coefficients $\beta(x_0+x_1, \alpha)$ and $\beta(x_i, \alpha)$ coincides for $i = 0, 1$. If x_0, x_1 are linearly dependent, since $\dim X_1$ is at least two, we can take $z \in X_1$ so that x_0, z and x_1, z are both linearly independent. Then from the linearly independent case, we have $\beta(x_0, \alpha) = \beta(z, \alpha) = \beta(x_1, \alpha)$. Therefore, β depends only on α and we define a map $\phi : K_1 \to K_2$ by $\phi(\alpha) \coloneqq \beta(x_0, \alpha)$. Since Φ preserves the additions, $\phi : K_1 \to K_2$ also preserves the additions. The independence of the choice of nonzero x_0 for $\beta(x_0, \alpha)$ shows that ϕ preserves the multiplications. By the same argument for the inverse map Φ^{-1}, the map ϕ is also a bijective, and thus $\phi : K_1 \to K_2$ is a field isomorphism. Now, we show that $\phi : K_1 \to K_2$ is a homeomorphism. Note that because Φ_* sends the maximum element $T^{\max}(X_1)$ to the maximum element $T^{\max}(X_2)$, the map $\Phi : (X_1, T^{\max}(X_1)) \to (X_2, T^{\max}(X_2))$ is a homeomorphism. Now, fix a nonzero element $x_1 \in X_1$. Then linear isomorphisms $\psi_1 : K_1 \ni \alpha \mapsto \alpha \cdot x_1 \in$

$\mathrm{Span}(x_1)$ and $\psi_2 : K_2 \ni \beta \mapsto \beta \cdot \Phi(x_1) \in \mathrm{Span}(\Phi(x_1))$ are homeomorphisms, where $\mathrm{Span}(x_1) \subset X_1$ and $\mathrm{Span}(\Phi(x_1)) \subset X_2$ endowed with the relative topology of $T^{\max}(X_1)$ and $T^{\max}(X_2)$, respectively. Because the composition ${\psi_2}^{-1} \circ \Phi \restriction_{\mathrm{Span}(x_1)} \circ \psi_1 : K_1 \to K_2$ is homeomorphism, which is equal to ϕ, the map $\phi : K_1 \to K_2$ is an isomorphism between topological fields K_1 and K_2.

We conclude that ϕ is an isomorphism of topological fields and by definition, Φ is a ϕ-semi-linear isomorphism and the triple (ϕ, Φ, x_0) satisfies $(\Phi + x_0)_* = \Theta$.

Next, we consider the case when Φ_* is $C \circ \Theta$. Recall that this case occurs when the cardinality of X_1 is finite. Because the cardinality of $\Sigma(X_1)$ and that of $\Sigma(X_2)$ are the same, X_2 is also a finite set, and thus K_2 is a finite field with the discrete topology. It is known that for finite-dimensional linear space X whose coefficient field K is a discrete finite field, every linear topology $T \in \tau_K(X)$ has an open base of the form $\{S + a \mid a \in X\}$ for a subspace S (See [1] for a proof of this fact for example). Thus, every linear topology on X_2 is mapped to itself by $C : \Sigma(X_2) \to \Sigma(X_2)$. Hence, $C \circ \Theta$ also satisfies the assumption on Θ, i.e. $C \circ \Theta : \Sigma(X_1) \to \Sigma(X_2)$ is a lattice isomorphism such that the image $C \circ \Theta(\tau_{K_1}(X_1))$ coincides with $\tau_{K_2}(X_2)$. Therefore, the same argument in the case when $\theta_* = \Theta$ shows that there exists a triple (ϕ, Φ, x_0) such that the induced map $(\Phi + x_0)_*$ is $C \circ \Theta$, where $\phi : K_1 \to K_2$ is a topological field isomorphism, $\Phi : X_1 \to X_2$ is a ϕ-semi-linear isomorphism and $x_0 \in X_2$.

We end the proof by showing the uniqueness of the triple (ϕ, Φ, x_0). Let (ϕ, Φ, x_0) and $(\phi', \Phi', {x_0}')$ be two triples such that the induced maps $(\Phi + x_0)_*$ and $(\Phi' + {x_0}')_*$ is Θ or $C \circ \Theta$. Because the cardinality of X_2 is at least four, only one of Θ or $C \circ \Theta$ is induced by a bijection, which implies that $(\Phi + x_0)_* = (\Phi' + {x_0}')_*$. Because every induced map f_* by a bijection $f : X_1 \to X_2$ sends a topology of form $\{\emptyset, \{p\}, X_1\}$ to $\{\emptyset, \{f(p)\}, X_2\}$ for every point $p \in X_1$, two maps $\Phi + x_0$ and $\Phi' + {x_0}'$ are the same map, and thus we obtain $x_0 = {x_0}'$ and $\Phi = \Phi'$. By definition of semi-linear isomorphism and $\Phi = \Phi'$, two field isomorphisms ϕ and ϕ' are the same.

3 Questions

We can consider a similar problem to Theorem 1 for groups, rings or other algebraic systems. We end this article by considering the case of groups. For every group G, we denote by $\tau_{\mathrm{group}}(G)$, the set of group topologies, which consists of topologies on G such that the group operators $G \times G \ni (x, y) \to x \cdot y \in G$ and $G \ni x \to x^- \in G$ are continuous. Then an analogy of Theorem 1 is the following: Let G_1 and G_2 be two groups and assume that there exists a lattice isomorphism $\Theta : \Sigma(G_1) \to \Sigma(G_2)$ with $\Theta(\tau_{\mathrm{group}}(G_1)) = \tau_{\mathrm{group}}(G_2)$.

Then is there a group isomorphism $\Phi : G_1 \to G_2$ such that $\Phi_* = \Theta$ or $\Phi_* = C \circ \Theta$ holds? This does not hold because every finite cyclic group only admits the discrete and indiscrete topology as a group topology, and thus any bijection from a cyclic group G to itself induces an identity map on $\tau_{\mathrm{group}}(G)$. However, this examples does not imply the failure of the corollary of Theorem 1: The existence of such a lattice isomorphism implies two groups are isomorphic. Our first question is whether there exist groups G and H such that they are not isomorphic and that a lattice isomorphism Θ exists with $\Theta(\tau_{\mathrm{group}}(G)) = \tau_{\mathrm{group}}(H)$. The next question is whether a characterization of groups that satisfies the corollary of Theorem 1 exists or not. From a more general view point, another question is there exists a characterization of algebraic systems such that Theorem 1 holds.

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