EQUIVALENCE OF STRENGTHENINGS OF RELATIVE K-STABILITY FOR POLARIZED TORIC MANIFOLDS

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1. Introduction

The purpose of this article is to give a report of our results in [20] based on a talk given at the meeting "RIMS meeting 2021 New developments of transformation groups". This is a joint work with Shunsuke Saito.

The problem of finding canonical Kähler metrics is one of the central problems in Kähler geometry. Let (X, L) be an n-dimensional polarized manifold, that is, X is an n-dimensional compact complex manifold and L is an ample line bundle over X. Let $\mathcal{H}(X, L)$ denote the set of all Kähler metrics of X representing $c_1(L)$. Note that $\mathcal{H}(X, L) \neq \emptyset$ since L is ample. In [4], Calabi introduced the notion of an *extremal Kähler metric* as a critical point of the Calabi functional

$$\mathcal{H}(X, L) \ni \omega \mapsto \int_X (s(\omega) - \bar{s})^2 \frac{\omega^n}{n!} \in \mathbf{R}.$$

Here $s(\omega)$ is the scalar curvature of ω and

$$\bar{s} = \fint_X s(\omega) \frac{\omega^n}{n!} = -\frac{n(K_X \cdot L^{n-1})}{(L^n)}.$$

In particular, the definition of $\bar{s}$ is independent of choice of $\omega \in \mathcal{H}(X, L)$. Calabi proved in [4] that this condition is equivalent to that $\mathrm{grad}_\omega(s(\omega) - \bar{s})$ is a real holomorphic vector field on X. Hence Kähler-Einstein metrics and constant scalar curvature Kähler metrics are extremal Kähler metrics. Well-known Yau-Tian-Donaldson conjecture predicts that the existence of extremal Kähler metrics is equivalent to the stability of (X, L) in the sense of geometric invariant theory. To state this, we fix a maximal algebraic torus T in $\mathrm{Aut}(X, L)$ with maximal compact subgroup S.

Conjecture 1.1 (Yau-Tian-Donaldson). *A polarized manifold (X, L) admits an S-invariant extremal Kähler metric if and only if it is K-polystable relative to T.*

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In [18] and [23], it was shown that existence of S-invariant extremal Kähler metrics implies relatively K-polystability. Hence the problem is the opposite implication. However, an example found by Apostolov-Calderbank-Gauduchon-Tønnesen-Friedman [1] suggests that relative K-polystability might not be sufficient to ensure the existence of extremal Kähler metrics, and leads to an expectation that we need to strengthen the notion of relative K-polystability. Relative K-polystability requires the positivity of the *non-Archimedean relative K-energy* $M_V^{NA}(\mathcal{X},\mathcal{L})$ for any T-equivariant *test configuration* $(\mathcal{X},\mathcal{L})$ which is not product. To strengthen this condition, typically the following two approaches are proposed.

(a) The first approach is to introduce a kind of norm $\|(\mathcal{X},\mathcal{L})\|$ of a test configuration and requiring that there exists a $\delta > 0$ such that

$$M_V^{NA}(\mathcal{X},\mathcal{L}) \geq \delta\|(\mathcal{X},\mathcal{L})\|$$

for any T-equivariant test configuration $(\mathcal{X},\mathcal{L})$ ([3], [9], [24]). As a candidate for the norm, the *non-Archimedean reduced J-functional* $J_T^{NA}(\mathcal{X},\mathcal{L})$ is well-studied ([15]).

(b) The second approach is to find more general testing objects than test configurations and requiring $M_V^{NA}(\varphi) > 0$ for any nontrivial object φ. As a candidate for testing objects, sequences of test configurations ([19]) and filtrations of the section ring of (X, L) ([26]) are known.

Our interest is the comparison of several stability conditions which appear as described above. The main results are as follows.

Theorem 1.2 ([20]. See Theorem 3.3 for precise statement)**.** *These two approaches for strengthening relative K-polystability above are equivalent for polarized toric manifolds.*

As a biproduct of the main result, we obtain the following

Corollary 1.3 ([20])**.** *A polarized toric manifold (X, L) admits an S-invariant extremal Kähler metric if and only if it is uniformly relatively K-polystable.*

See Section 3.1 (Definition 3.2) for the definition of uniform relative K-polystability. In Section 4, we discuss on a polytopal criterion (sufficient condition) of uniform relative K-polystability for polarized toric manifolds.

2. Preliminaries

2.1. **Polarized manifolds.** Let (X, L) be an n-dimensional polarized manifold. We fix a maximal algebraic torus T in $\mathrm{Aut}(X, L)$, and set $N :=\mathrm{Hom}(\mathbf{G}_m, T)$ and $M := \mathrm{Hom}(N, \mathbf{G}_m)$. Also, let $N_k := N \otimes_{\mathbf{Z}} k$, $M_k := M \otimes_{\mathbf{Z}} k$ for each field k. Let $S := \mathrm{Hom}(\mathbf{G}_m, T) \otimes_{\mathbf{Z}} \mathbf{R}/\mathbf{Z}$ be the maximal compact subgroup of T. Note that

$$\mathrm{Lie}(S) = 2\pi\sqrt{-1}N_{\mathbf{R}}, \quad \mathrm{Lie}(S)^\vee = M_{\mathbf{R}}, \quad \mathrm{Lie}(T) = N_{\mathbf{R}} \oplus 2\pi\sqrt{-1}N_{\mathbf{R}}.$$

Let $\omega \in \mathcal{H}(X,L)^S$. Then the S-action on (X,ω) is Hamiltonian and there exists a moment map $\mu_\omega\colon X \to M_{\mathbf{R}} = \mathrm{Lie}(S)^\vee$ satisfying

$$\int_X \langle \mu_\omega, \xi \rangle \omega^n = 0 \quad (\xi \in \mathrm{Lie}(S)). \tag{2.1}$$

Here $\langle \cdot, \cdot \rangle$ is the natural pairing of $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$. Associated to the moment map, we define

$$P := \mu_\omega(X), \quad \mathrm{DH}_S := \mu_{\omega\#}\left(\frac{\omega^n}{(L^n)}\right).$$

Then P is a rational polytope in $M_{\mathbf{R}}$ and independent of choice of $\omega \in \mathcal{H}(X,L)^S$. We call P the *moment polytope* of (X,L). Also, DH_S is a Borel probability measure on $M_{\mathbf{R}}$ supported on P and independent of $\omega \in \mathcal{H}(X,L)^S$. We call DH_S the *Duistermaat-Heckman measure* of (X,L).

Definition 2.1. *The extremal Kähler vector field of* (X,L) *with respect to* T *is a rational affine function* V *on* $M_{\mathbf{R}}$ *which satisfies*

$$\int_P V(x)\,\mathrm{DH}_S(dx) = 0, \tag{2.2}$$

$$F(\xi) = \int_P \xi(x) V(x)\,\mathrm{DH}_S(dx) \quad (\xi \in N_{\mathbf{Q}}). \tag{2.3}$$

Here F *denotes the Futaki invariant of* (X,L) *(*[2], [5], [11], [12]*).*

The right-hand side of (2.3) is known as the *Futaki-Mabuchi bilinear form* ([13]). The extremal Kähler vector field is determined uniquely for each T.

2.2. **Relative K-stability.** In this subsection, we give a brief review of relative K-stability. At first, we introduce the notion of test configurations. In this article, we consider only T-*equivariant* test configurations.

Definition 2.2. *Let* $r \in \mathbf{Z}_{>0}$. *A* T*-equivariant test configuration for a polarized manifold* (X,L) *of exponent* r *consists of the following data:*

(1) a normal variety $\mathcal{X}$ *admitting an action of* $T \times \mathbf{G}_m$*;*
(2) a $T \times \mathbf{G}_m$*-equivariant flat projective morphism* $\pi\colon \mathcal{X} \to \mathbf{A}^1$ *with respect to the trivial action of* T *on* $\mathbf{A}^1$ *and the standard action of* $\mathbf{G}_m$ *on* $\mathbf{A}^1$ *via multiplication;*
(3) a $T \times \mathbf{G}_m$*-linearlized* π*-ample* $\mathbf{Q}$*-line bundle* $\mathcal{L} \to \mathcal{X}$*;*
(4) a $T \times \mathbf{G}_m$*-equivariant isomorphism* $(\mathcal{X}_{\mathbf{G}_m}, \mathcal{L}_{\mathbf{G}_m}) \cong (X \times \mathbf{G}_m, p_1^* L^r)$*, where* $(\mathcal{X}_{\mathbf{G}_m}, \mathcal{L}_{\mathbf{G}_m}) \to \mathbf{G}_m$ *is the base change of* $(\mathcal{X},\mathcal{L}) \to \mathbf{A}^1$ *with respect to the inclusion* $\mathbf{G}_m \hookrightarrow \mathbf{A}^1$*, and* $p_1\colon X \times \mathbf{G}_m \to X$ *is the projection to the first factor.*

A T*-equivariant test configuration* $(\mathcal{X},\mathcal{L})$ *is called a product configuration if there exists a* T*-equivariant isomorphism* $\mathcal{X} \cong X \times \mathbf{A}^1$*.*

As mentioned in the introduction, associated to a T-equivariant test configuration $(\mathcal{X},\mathcal{L})$, the *non-Archimedean relative K-energy* $M_V^{NA}(\mathcal{X},\mathcal{L})$ and *non-Archimedean reduced J-functional* $J_T^{NA}(\mathcal{X},\mathcal{L})$ is given as numerical invariants. Also, it is known that $J_T^{NA}(\mathcal{X},\mathcal{L}) \geq 0$ for any $(\mathcal{X},\mathcal{L})$ and equality holds if and only if $(\mathcal{X},\mathcal{L})$ is a product configuration ([3], [15], [16]). We will not explain precise definitions in this paper, but give explicit expressions on polarized toric manifolds in the next section. See Theorem 2.4 for details.

Definition 2.3. *Let (X, L) be a polarized manifold.*

(1) (X, L) is relatively K-semistable if $M_V^{NA}(\mathcal{X},\mathcal{L}) \geq 0$ for any T-equivariant test configuration $(\mathcal{X},\mathcal{L})$.

(2) (X, L) is relatively K-polystable if $M_V^{NA}(\mathcal{X},\mathcal{L}) > 0$ for any T-equivariant test configuration $(\mathcal{X},\mathcal{L})$ which is not a product configuration.

(3) (X, L) is relatively K-unstable if (X, L) is not relatively K-semistable.

In case $V = 0$, the modifier "relatively" will be omitted.

2.3. **Polarized toric manifolds.** In this section, we assume the polarized manifold (X, L) is *toric*, that is, $\dim_{\mathbf{C}} T = \dim_{\mathbf{C}} X = n$, and the T-action on X is effective and has an open dense orbit. Then the moment polytope P is a special type of polytope known as *Delzant polytope*. Furthermore, after a suitable translation, we can make P into an integral polytope. Conversely, we can recover (X, L) from the integral Delzant polytope. Let us explain this below (see [8] for details).

Let $C(P) \subset M_{\mathbf{R}} \times \mathbf{R}$ denote the cone over P with respect to the origin. Then the set of all lattice points $S_P := C(P) \cap (M \times \mathbf{Z})$ is a finitely generated commutative semigroup. Let $\mathbf{C}[S_P]$ denotes the semigroup ring of S_P. This is a finitely generated $\mathbf{C}$-algebra, and has a natural grading by the coordinate of $\mathbf{Z}$. Then the resulting $(\operatorname{Proj} \mathbf{C}[S_P], \mathcal{O}_{\operatorname{Proj} \mathbf{C}[S_P]}(1))$ is a polarized toric manifold. It is known that this is a one to one correspondence between isomorphism classes of polarized toric manifolds and isomorphism classes of integral Delzant polytopes.

For polarized toric manifolds, it is natural to consider test configurations whose total space are also toric. Let f be a rational piecewise affine convex function on P, that is, f is a convex function of the form

$$f = \max\{\ell_1, \ldots, \ell_m\}$$

with each ℓ_j an affine function having rational coefficients. Choosing an integer A so that $A > \max_P f$, let

$$\mathcal{P} := \{(x, y) \in M_{\mathbf{R}} \times \mathbf{R} \mid x \in P,\ f(x) - A \leq y\}.$$

Then $\mathcal{P}$ is an $(n+1)$-dimensional rational polyhedron in $M_{\mathbf{R}} \times \mathbf{R}$. By replacing f, A, $\mathcal{P}$ by rf, rA, $r\mathcal{P}$ for suitable $r \in \mathbf{Z}_{>0}$ if necessary, we may assume that each ℓ_j has integral coefficients and $\mathcal{P}$ is an integral polyhedron. Then the similar construction described above gives us an $(n+1)$-dimensional normal toric variety $\mathcal{X}_f$ with $T \times \mathbf{G}_m$-linearized $\mathbf{Q}$-line bundle $\mathcal{L}_f$ over $\mathcal{X}_f$. Further, we have a natural $T \times \mathbf{G}_m$-equivariant morphism $\pi_f \colon \mathcal{X}_f \to \mathbf{A}^1$ and see that $(\mathcal{X}_f, \mathcal{L}_f)$ is a T-equivariant test configuration for (X, L). We call this a *toric test configuration* associated to f.

For toric test configurations, the followings are known.

Theorem 2.4. *Let $(\mathcal{X}_f, \mathcal{L}_f)$ be the toric test configuration associated to a rational piecewise affine convex function f. Then the following holds:*

$$M_V^{NA}(\mathcal{X}_f, \mathcal{L}_f) = M_V^{NA}(f) := \frac{1}{\mathrm{vol}(P)} \left(\int_{\partial P} f(\zeta)\, d\sigma - \int_P (\bar{s} + V(x)) f(x)\, dx \right),$$

$$J_T^{NA}(\mathcal{X}_f, \mathcal{L}_f) = J_T^{NA}(f) := \inf_{\xi\,:\,\text{affine}} \left(\fint_P (f(x) + \xi(x))\, dx - \inf_P (f + \xi) \right).$$

Here σ is the natural Borel measure on ∂P which appears in the coeffient of the Ehrhart polynomial of P:

$$E_P(r) := \#(rP \cap M) = \mathrm{vol}(P) r^n + \frac{\sigma(\partial P)}{2} r^{n-1} + \cdots + 1.$$

Note that $J_T^{NA}(\mathcal{X}_f, \mathcal{L}_f) = 0$ is equivalent to $(\mathcal{X}_f, \mathcal{L}_f)$ being a product configuration, but in terms of a convex function, it is also equivalent to f being affine.

3. Main results

3.1. **Equivalence of stabilities.** In this section, we present the main results in [20]. Let (X, L) be a polarized toric manifold with the moment polytope P. Firstly, we introduce several spaces of convex functions on P.

Definition 3.1. *Let P^* denote the union of interior of P and relative interiors of each facet of P. We fix an interior point x_0 of P.*

(1) $\mathcal{C}_*$ denotes the set of all continuous convex functions on P^* which are integrable on ∂P.
(2) $\mathcal{C}_{PL}$ denotes the set of all piecewise affine convex functions on P.
(3) $\mathcal{C}_{PL}^{\mathbf{Q}}$ denotes the set of all rational piecewise affine convex functions on P.
(4) $\mathcal{C}_\infty$ denotes the set of all continuous convex functions on P which are smooth in the interior.

When $\mathcal{F}$ is one of the spaces $\mathcal{C}_*$, $\mathcal{C}_{PL}$, $\mathcal{C}_{PL}^{\mathbf{Q}}$ or $\mathcal{C}_\infty$, we set

$$\widetilde{\mathcal{F}} := \{ f \in \mathcal{F} \mid \inf_P f = f(x_0) = 0 \}.$$

As mentioned in section 2.3, $\mathcal{C}_{PL}^{\mathbf{Q}}$ is nothing but the set of all toric test configurations for (X, L). Also, $\mathcal{C}_{PL}^{\mathbf{Q}}$ is a proper subset of $\mathcal{C}_*$, and $M_V^{NA}(f)$ and $J_T^{NA}(f)$ are well-defined for any $f \in \mathcal{C}_*$. Following each of two approaches (a) and (b) described in the introduction, we define the strengthenings of relative K-polystability as follows.

Definition 3.2. *Let (X, L) be a polarized toric manifold with the moment polytope P.*

(1) (X, L) *is uniformly relatively K-polystable if there exists a* $\delta > 0$ *such that*

$$M_V^{NA}(f) \geq \delta J_T^{NA}(f)$$

for any $f \in \mathcal{C}_{PL}^{\mathbf{Q}}$*.*

(2) (X, L) *is relatively* K_**-polystable if*

$$M_V^{NA}(f) > 0$$

for any $f \in \mathcal{C}_*$ *which is not affine.*

Under the above notations and conventions, our main results are stated as follows.

Theorem 3.3 ([20])**.** *Let* $\mathcal{F}$ *be one of the spaces* $\mathcal{C}_*$, $\mathcal{C}_{PL}$, $\mathcal{C}_{PL}^{\mathbf{Q}}$ *or* $\mathcal{C}_\infty$*. Then the following are equivalent.*

$(b)_{\mathcal{F}}$ *There exists a* $\delta > 0$ *such that*

$$M_V^{NA}(f) \geq \delta \int_{\partial P} f \, d\sigma$$

for any $f \in \widetilde{\mathcal{F}}$*.*

$(J)_{\mathcal{F}}$ *There exists a* $\delta > 0$ *such that*

$$M_V^{NA}(f) \geq \delta J_T^{NA}(f)$$

for any $f \in \mathcal{F}$*.*

(K_*) (X, L) *is relatively* K_**-polystable.*

In particular, (X, L) *is uniformly relatively K-polystable if and only if it is relatively* K_**-polystable.*

The keys of our proof of Theorem 3.3 are the approximation and a kind of compactness results for convex functions in $\mathcal{C}_*$, which is improvements of [7, Lemma 3.1] and [10, Propositions 5.2.6 and 5.2.8]. For the detailed proof, see [20].

3.2. **Toric Yau-Tian-Donaldson correspondence.** As a consequence of Theorem 3.3, we obtain the following

Corollary 3.4 ([20]. Toric YTD correspondence)**.** *A polarized toric manifold* (X, L) *admits an* S*-invariant extremal Kähler metric if and only if it is uniformly relatively K-polystable.*

Proof. By combining the results of [7], [14], and [28], we obtain the equivalence of the existence of S-invariant extremal Kähler metrics and the condition $(b)_{\mathcal{C}_\infty}$. On the other hand, the condition $(b)_{\mathcal{C}_\infty}$ is equivalent to uniform relative K-polystability by Theorem 3.3. □

The following corollary is a simple application of Corollary 3.4.

Corollary 3.5. *Let* (X_1, L_1) *and* (X_2, L_2) *be polarized toric manifolds, and set* $X := X_1 \times X_2$, $L := L_1 \boxtimes L_2$*. If* (X_1, L_1) *and* (X_2, L_2) *are uniformly relatively K-polystable, then so is* (X, L)*.*

Proof. Since (X_1, L_1) and (X_2, L_2) are uniformly relatively K-polystable, they admit torus invariant extremal Kähler metrics by Corollary 3.4. Since the product metric of two extremal Kähler metrics is also an extremal Kähler metric, (X, L) is uniformly relatively K-polystable by Corollary 3.4 again. □

4. A polytopal sufficient condition for uniform relative K-polystability

In the final section, we provide a polytopal sufficient condition for unirofm relative K-polystability. Let (X, L) be an n-dimensional polarized toric manifold. Then the integral Delzant polytope P can be written by

$$P = \{x \in M_{\mathbf{R}} \mid \langle x, \lambda_j \rangle + d_j \geq 0 \ (j = 1, \ldots, r)\}, \tag{4.1}$$

where r is the number of facets of P, $\lambda_j \in N$, $d_j \in \mathbf{Z}$ and each λ_j is primitive. We fix an interior point x_0 of P, and put

$$d_{x_0,j} := \langle x_0, \lambda_j \rangle + d_j$$

for each $j = 1, \ldots, r$. Note that $d_{x_0,j} > 0$ for any j.

Theorem 4.1. *Let $d_{x_0} := \max\{d_{x_0,1}, \ldots, d_{x_0,r}\}$, and assume P satisfies either*

$$V = 0 \quad \textit{and} \quad \overline{s} < \frac{n+1}{d_{x_0}}, \tag{4.2}$$

or

$$V \neq 0 \quad \textit{and} \quad \overline{s} + \max_P V \leq \frac{n+1}{d_{x_0}}. \tag{4.3}$$

Then (X, L) is uniformly relatively K-polystable.

In [28], Zhou and Zhu showed relative K-polystability of polarized toric manifolds under the same assumption. Theorem 4.1 strengthens their result to uniform relative K-polystability.

Now suppose that $(X, L) = (X, K_X^{-1})$ is a toric Fano manifold with anticanonical polarization. Then $\overline{s} = n$, and P is a reflexive polytope. In particular, we have $d_j = 1$ for any $j = 1, \ldots, r$. Moreover, we can choose x_0 to be the origin of $M_{\mathbf{R}}$. Then the conditions (4.2) and (4.3) reduce

$$V = 0 \tag{4.4}$$

and

$$V \neq 0 \quad \text{and} \quad \max_P V \leq 1, \tag{4.5}$$

respectively. Combining this with the computations in [21] and [28] yields the following corollary.

Corollary 4.2. *All toric Del Pezzo surfaces with anticanonical polarization are uniformly relatively K-polystable.*

It is known that all toric del Pezzo surfaces admit an extremal Kähler metric in its anticanonical class by the works of [4], [6], [22], and [27]. Corollary 4.2 gives the stability counterpart of this. The computation of $\max_P V$ for toric Fano manifolds up to dimension 4 can be seen in [21].

Finally, let us consider uniform K-polystability of toric Fano manifolds. Note that the condition (4.4) is equivalent to vanishing of the Futaki invariant of (X, K_X^{-1}) ([13]). Also, it is known that the Futaki invariant vanishes if and only if the barycenter of P coincides with the origin of $M_{\mathbf{R}}$ ([17]). Hence we obtain the following.

Corollary 4.3. *Let (X, K_X^{-1}) be a toric Fano manifold. Then the following are equivalent.*

(1) (X, K_X^{-1}) is uniformly K-polystable.
(2) (X, K_X^{-1}) is K-polystable.
(3) (X, K_X^{-1}) is K-semistable.
(4) The Futaki invariant of (X, K_X^{-1}) vanishes.
(5) The barycenter of P is the origin of $M_{\mathbf{R}}$.
(6) $V = 0$.

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