On the nonexistence of the hierachy structure: lower rationality = higher ruledness, and very general hypersurfaces as examples

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Abstract

A short introduction to the author's study of the rationality prblem, which centers the hierarchies of the form: lower rationality = higher ruledness. Examples are given for the cases of very general hypersurfaces and complete intersections, building upon the works of Totaro, Chatzistamatiou-Levine, and Schreieder.

1 Introduction

Rationality of algebraic varieties is an authentic important concept in algebraic geometry. In fact, its mot primitive form is even taught in highschool mathematics:

$$\begin{array}{ccccccc} \{(x,y) \mid x^2+y^2=1\} & \xleftarrow{\text{birational}} & \{(x,y) \mid x^2+y^2=1\} \setminus \{(-1,0)\} & \xleftarrow{\simeq} & \mathbb{A}^1 & \xrightarrow{\text{birational}} & \mathbb{P}^1 \\ & & \left(\dfrac{1-t^2}{1+t^2}, \dfrac{2t}{1+t^2}\right) & \leftarrow\!\shortmid & t & \mapsto & (t:1) \end{array}$$

A relevant authentic important concept in algebraic geometry is ruledness. Actually, we can easily interpolate these authentic concepts of algebraic geometry canonically, in the framework of

$$\textit{Lower rationality} = \textit{Higher ruledness} \ : \tag{1}$$

Definition 1.1. *For a projective n-dimensional variety X, and $0 \le i \le n$, let us say:*

$$X \text{ is } \underline{(-i)\textbf{-rational}} \text{ (lower rationality) or } \underline{(n-i)\textbf{-ruled}} \text{ (higher ruledness)}$$

if there exist a i-dimensional Z^i and a birational map

$$\mathbb{P}^{n-i} \times Z^i - - > X.$$

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Recently, I obtained a sufficient criterion (see [M19] for a survey) for the existence of the "uni-"analogue of the above hierarchical strucutre, generalizing (actually based upon) the famous uniruledness criterion of Mori, Miyaoka-Mori, Boucksom-Demailly-Păun-Peternell.

More recently, I have embarked upon a systematic study of the nonexistence results of such a hierarchical structures. Here, let us recall various (non-existence) results of rationality have been stated with respect to the following hierachy:

$$\begin{aligned} &\text{rational} \Longrightarrow \text{stable rational} \Longrightarrow \text{ratract rational} \\ \Longrightarrow &\text{separably unirational} \Longrightarrow \text{separably rationally connected} \end{aligned} \tag{2}$$

Then I shall look after nonexistence results of the hierachical structure analogaous to (1), applied to various hierachies in (2).

Now there are two purposes of this paper. First, I shall state my first theorem Theorem **??**, which presents some practically applicable conclusions out of rectract lower-rationality conditions.

Second, I shall state retract lower irrationality theorems of very general hypersurfaces, upgrading the theorems of Totaro [T16], Chatzistamatiou-Levine [**?**] and Schreieder [S19].

I hope this would give a good flabour of hierachical phenomena.

2 Definitions of the hierachies of hierachies

So, we wish to find necessary conditions for the existence of the following hierachical structures, whose definitions are very natural in view of (1) and (2):

Definition 2.1. *For a projective n-dimensional variety X, let us say:*

(i) X *is* $\underline{\textbf{stable } (-i)\textbf{-rational}}$ *or* $\underline{\textbf{stable } (n-i)\textbf{-ruled}}$ $\quad (0 \leq i \leq n)$

if there exist an i-dimensional variety Z^i. $j \in \mathbb{Z}_{\geq 0}$ and a birational map

$$\mathbb{P}^j \times \mathbb{P}^{n-i} \times Z^i --> \mathbb{P}^j \times X.$$

(ii) X *is* $\underline{\textbf{retract } (-i)\textbf{-rational}}$ *or* $\underline{\textbf{retract } (n-i)\textbf{-ruled}}$ $\quad (0 \leq i \leq n)$

if there exist an i-dimensional variety Z^i. $N \in \mathbb{Z}_{\geq n}$ and rational maps

$$f : X --> \mathbb{P}^{N-i} \times Z^i, \quad g : \mathbb{P}^{N-i} \times Z^i --> X$$

such that the composition

$$g \circ f : X --> X$$

is defined, yielding an identity on a dense open subset of X.

(iii) X *is* $\underline{\textbf{separably } (-i)\textbf{-unirational}}$ *or* $\underline{\textbf{separably } (n-i)\textbf{-ruled}}$ $\quad (0 \leq i \leq n)$

if there exist an i-dimensional variety Z^i. $N \in \mathbb{Z}_{\geq n}$ and a separably dominant rational map

$$g : \mathbb{P}^{N-i} \times Z^i --> X$$

(iv) *when* X *is further smooth,*

$$X \text{ is } \underline{\textbf{separably } (-i)\textbf{-rationally connected}} \qquad (0 \le i \le n)$$

if there exist a morphism $f : \mathbb{P}^1 \to X$ *such that*

$$f^*T_X \;\cong\; \oplus_{1 \le j \le n} \mathcal{O}(a_j),$$
$$\text{with } a_1 \;\ge \cdots \ge\; a_{n-i} \;\ge\; \max(1, a_{n-i-1}) \;\ge\; a_{n-i-1} \;\ge \cdots \ge a_{n-1} \;\ge\; a_n \;\ge\; 0.$$

(v) *when* X *is further smooth,*

$$X \text{ is } \underline{(-i)\textbf{-rationally connected}} \qquad (0 \le i \le n)$$

if, for the maximal rationally chain connected fibration $\pi : X^n - - > Z$ *[C92][KMM92]* [1)]*,*

$$\dim \; Z \;\le\; i.$$

3 Some necessary criteria for the existence of hierarchical structures

My first main theorem states the hierarchy of retratc rationality imposes restrictions on the $\mathbb{P}^1$-invariant Nisnevich sheaves with transfers (actually, those $\mathbb{P}^1$-rigid presheaves with transfers separated with respect to Zariski topology suffice), which we now recall:

Definition 3.1. (i) [VSF00] [MVW06, Definition 1.1, Definition 1.5] *Let* $\mathbf{Cor}_F$ *be the category whose objects are the smooth separated schemes of finite type over* F*, and whose morphism from* X *to* Y *is an* **elementary correspondence** *from* X *to* Y*, i.e. an irreducible closed subset* W *of* $X \times Y$ *whose associated integral subscheme is finite and surjective over* X*.*
(ii) [VSF00] [MVW06, Definition 2.1] *A* **presheaf with transfers** *is a contravariant additive functor* $F : \mathbf{Cor}_F \to Ab$*. We will write* $PreSh\left(\mathbf{Cor}_F\right)$*, or* $\mathbf{PST}(F)$ *or even simply* $\mathbf{PST}$*, for the functor category whose objects are presheaves with transfers and whose morphisms are natural transformations.*
(iii) [KSY16, Theorem 8] [KOY21, Definition 3.1] $G \in \mathbf{PST}$ *is called* $\underline{\mathbb{P}^1\textbf{-invariant}}$*, if the structure morphism* $\sigma_{\mathbb{P}^1} : \mathbb{P}^1 \to \operatorname{Spec} F$ *induces an isomorphism* $G(U) \xrightarrow{\cong} G(U \times \mathbb{P}^1)$ *for any smooth* F*-scheme* U*.*

Denote by $\mathbf{PI}$ *(resp.* $\mathbf{PI}_{Nis}$*) the full subcategory of* $\mathbf{PST}$ *consisting of all* $\mathbb{P}^1$*-invariant presheaves (resp. Nisnevich sheaves) with transfers.*
(iv) [KSY16, Definition 6.1.3] [KOY21, Definition 3.6] $F \in \mathbf{PST}$ *is called* $\underline{\mathbb{P}^1\textbf{-rigid}}$*, if the two induced maps*

$$i_0^*, i_1^* : F(U \times \mathbb{P}^1) \to F(U)$$

are equal for any $U \in \mathrm{Sm}$*.*

Denote by $\mathbf{PRig}$ *(resp.* $\mathbf{PRig}_{Nis}$*) the full subcategory of* $\mathbf{PST}$ *consisting of all* $\mathbb{P}^1$*-rigid presheaves (resp. Nisnevich sheaves) with transfers.*

1) While the results in [C92][KMM92] are stated only for the case characteristic zero, their constructions are equally valid for the characteristic positive case as is presented in [**?**, IV, Theorem 5.2, Complement 5.2.1] (see also [**?**, p.128, Theorem 5.13]).

Proposition 3.2. [KSY16, Proposition 6.1.4] [KOY21, Lemma 3.7] *If $G \in \mathbf{PST}$ is $\mathbb{P}^1$-invariant, then it is $\mathbb{P}^1$-rigid. The converse holds if G is separated for Zariski topology.*

Then we can easily deduce the following inclusing relations (c.f. [KOY21, Lemma 3.8.(3)]):

Corollary 3.3. $\mathbf{HI}_{Nis} \subset \mathbf{PI}_{Nis} \subset \mathbf{PRig}_{Nis} \subset \mathbf{PST}$

Now, my first main theorem can be stated as follows:

Theorem 3.4. *Let X be any retract $(-i)$-rational with Z^i in Definition 2.1(ii) taken to be smooth projective. Then, for any $G \in \mathbf{PRig}_{Nis}$, $G(X)$ is a direct summand of $G(Z^i)$.*

When this conclusion holds, let us state X has $G - \dim \leq i$ for $G \in \mathbf{PRig}_{Nis}$. □

Although I can not give a complete proof here, it is much simpler comparing with "non-hierarical" predecessors [ABBvB21] [BRS20] [KOY21].

Basic idea of my proof of Theorem 3.4 is, as Merkurjev's suggestion in [CTP16, Remarque 1.6], to make use of motivic technique of Rost [KM13, Appendix RC] and [KS16]. More precisely, I work with the category of rational correspondences $\mathbf{Cor}^{\mathrm{O}}_{\mathrm{rat}}(F, A)$ of smooth projective F varieties with coefficients in a commutative ring A, studied by Rost [KM13, Appendix RC] and Kahn-Sujatha [KS16] (see [KS16, Proposition 2.3.4, Definition 2.3.5] for the definition of $\mathbf{Cor}^{\mathrm{O}}_{\mathrm{rat}}(F, A)$). Then the integral version of following concept is a core in my proof of Theorem 3.4:

Definition 3.5. *For a smooth projective F-variety X, we say X is* **integrally** *(resp.* **rationally** *)* **of birational dimension** $\leq i$ *, if for some smooth projective F-variety Z of dimension $\leq i$, $[X]$ is a direct summand of $[Z]$ in $\mathbf{Cor}^{O}_{rat}(F, \mathbb{Z})$ (resp. $\mathbf{Cor}^{O}_{rat}(F, \mathbb{Q})$).*

In fact, the rational version of this concept can be used to dereive practicable applicable conclusions out of the lower rationally conncectedness, defined in Definition 2.1(v):

Theorem 3.6. *Suppose char $F = 0$ and let X be any $(-i)$-rational connected smooth projective F variety. Then, there exists a smooth projective Z^i of dimension i, such that, for any $G \in \mathbf{PRig}_{Nis}$, $G(X) \otimes \mathbb{Q}$ is a direct summand of $G(Z^i) \otimes \mathbb{Q}$.*

When this conclusion holds, let us state X has $G_{\mathbb{Q}} - \dim \leq i$ for $G \in \mathbf{PRig}_{Nis}$. □

Actually, Theorem 3.4 and Theorem 3.6 are parts of the following bird's-eye diagram of implications of various hierarchies for a smooth projective F-variety X to satisfy (Here, I have only considered the stronger versions of the conditions in Definition **??** with Z^i (or Z) smooth projective.):

$$
\begin{tikzcd}
(-i)\text{-rational} \arrow[r] & \text{stable } (-i)\text{-rational} \arrow[dl] & \\
\text{retract } (-i)\text{-rational} \arrow[r] \arrow[dd, "\mathbf{(IV)}"'] & \text{separably } (-i)\text{-unirational} \arrow[r] \arrow[d, equal, "\text{if char} F = 0"] \arrow[d, bend right] & \text{separably } (-i)\text{-rationally connected} \arrow[d, equal, "\text{if char} F = 0"] \arrow[d, bend right] \arrow[ddddll, "\text{if } F = \overline{F}"] \\
 & (-i)\text{-unirational} \arrow[r] & (-i)\text{-rationally connected} \arrow[d] \\
\text{integrally of birational dimension} \leq i \arrow[rr] \arrow[d] & & \text{rationally of birational dimension} \leq i \arrow[d] \\
G - \dim \leq i,\ \forall G \in \mathbf{PRig}_{Nis} \arrow[rr] \arrow[d, "\text{if } F = \overline{F}"] \arrow[dd, bend right=60, "\text{if } F = \overline{F}"] & & G_{\mathbb{Q}} - \dim \leq i,\ \forall G \in \mathbf{PRig}_{Nis} \arrow[d, "\text{if } F = \overline{F}"] \\
H^0\left(X, \Omega_X^j\right) = 0\ (j > i) \arrow[rr] & & H^0\left(X, \Omega_X^j\right) = 0\ (j > i),\ \underline{\text{char } F = 0}. \\
H^j_{ur}(X, A) = 0\ (j > i). & &
\end{tikzcd}
$$

4 Hierachical versions of the theorems of Totaro, Chatzistamatiou-Levine, Schreieder

Theorems of Totaro [T16], Chatzistamatiou-Levine [CL17] and Schreieder [?] [?] for very general hypersurfaces (Totaro, Schreieder) and very general complete intersections (Chatzistamatiou-Levine) can be upgraded to statements of non retract lower-rationality statements, as follows:

Theorem 4.1. *[T16, Theorem 2.1] [CL17, Theorem 6.1]*

A very general complete intersection $X_{d_1,\cdots,d_r} \subset \mathbb{P}^{n+r}_{\mathbb{C}}$ of type $(d_1, \ldots, d_r)$ with the Fano condition $\sum_{1\leq i\leq r} d_i \leq n + r$ is not stable 2-ruled. actually, not even retract $-(n-2)$-rational, provided, for some $1 \leq i \leq r$,

$$d_i \;\geq\; 2\left\lceil \frac{n + r + 1 - \sum_{\substack{1\leq j\leq r \\ j\neq i}} d_j}{3} \right\rceil .$$

Theorem 4.2. *[?, Theorem 1.1] [S21, Theorem 7.1] For a natural number n, express it uniquely as*

$$n = l + r \quad \textit{such that} \quad 2^{l-1} - 2 \leq r \leq 2^l - 2,$$

and set

$$L_2 n := l = \min\left\{l \in \mathbb{N} \mid l + 2^l - 2 \geq n\right\} \ \left(\leq \lceil \log_2 n \rceil \right).$$

Then a very general hypersurface $X_d \subset \mathbb{P}^{n+1}_K$ defined over an uncountable field K is not retract $-(L_{2^n} - 1)$-rational under the following conditions:

$$\begin{cases} d \geq 2 + L_2 n & \textit{if} \quad \textit{char } K \neq 2 \\ d \geq 3 + L_2 n & \textit{if} \quad \textit{char } K = 2 \end{cases}$$

Unfortunately, my first main theorem Theorem 3.4 is not strong enough to prove these theorems. This is because, proofs of Totaro, Chatzistamatiou-Levine, and Schreieder make use of the specialization argument, which yield singular varieties. Actually, my proovs simply follow and upgrade the original proofs of Chatzistamatiou-Levine, and Schreieder to hierarical versions. However, for the Schreieder's version: Theorem 4.2, I have recently found a more transparent proof, in the spirit of my proof of Theorem 3.4.

The details will be put in ArXiv soon.

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