Equivariant index of a generalized Bott manifold

Yuki Sugiyama

Department of Mathematics, Graduate School of Science and Engineering, Chuo University

1 Introduction

This article is an overview of the paper [7]. A *Bott tower* of height n is a sequence:

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = \{\text{a point}\}$$

of complex manifolds $M_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j)$, where $\underline{\mathbb{C}}$ is the trivial line bundle over M_{j-1}, E_j is a holomorphic line bundle over M_{j-1}, $\mathbb{P}(\cdot)$ denotes the projectivization, and $\pi_j : M_j \to M_{j-1}$ is the projection of the $\mathbb{C}P^1$-bundle. We call M_j a *j-stage Bott manifold*. The notion of a Bott tower was introduced by Grossberg and Karshon ([3]).

A *generalized Bott tower* is a generalization of a Bott tower. A generalized Bott tower of height m is a sequence:

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

of complex manifolds $B_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, $E_j^{(k)}$ is a holomorphic line bundle over B_{j-1} for $k = 1, \ldots, n_j$. We call B_j a *j-stage generalized Bott manifold*. A generalized Bott tower has been studied from various points of view (see, e.g., [1, 2, 4]). A generalized Bott manifold is a certain class of toric manifold, so it is interesting to investigate the properties of generalized Bott towers.

In [3], Grossberg and Karshon showed the multiplicity function of the *equivariant index* for a holomorphic line bundle over a Bott manifold is given by the density function of a *twisted cube*, which is determined by the structure of the Bott manifold and the line bundle over it. From this, they derived a Demazure-type character formula.

The purpose of the paper [7] is to generalize the results in [3] to *generalized Bott manifolds*. We generalize the twisted cube, and we call it the *generalized twisted cube*. It is a special case of twisted polytope introduced by Karshon and Tolman [6] for the presymplectic toric manifold, and it is a special case of multi-polytope introduced by Hattori and Masuda [5] for the torus manifold. We show the multiplicity function of the equivariant index for the holomorphic line bundle over the generalized Bott manifold is given by the density function of the generalized twisted cube. From this, we derive a Demazure-type character formula.

2 Preliminaries

2.1 Generalized Bott manifolds

Definition 2.1 A *generalized Bott tower* of height m is a sequence :

$$B_m \stackrel{\pi_m}{\to} B_{m-1} \stackrel{\pi_{m-1}}{\to} \cdots \stackrel{\pi_2}{\to} B_1 \stackrel{\pi_1}{\to} B_0 = \{\text{a point}\},$$

of manifolds $B_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, $E_j^{(k)}$ is a holomorphic line bundle over B_{j-1} for $k = 1, \ldots n_j$, and $\mathbb{P}(\cdot)$ denotes the projectivization. We call B_j a *j-stage generalized Bott manifold.*

The construction of the generalized Bott tower is as follows. A 1-step generalized Bott tower can be written as $B_1 = \mathbb{C}P^{n_1} = (\mathbb{C}^{n_1+1})^\times/\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts diagonally. We construct a line bundle over B_1 by $E_2^{(k)} = (\mathbb{C}^{n_1+1})^\times \times_{\mathbb{C}^\times} \mathbb{C}$ for $k = 1, \ldots, n_2$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by $a : v \mapsto a^{-c_k} v$ for some integer c_k. In $E_2^{(k)}$ we have $[z_{1,0}, \ldots, z_{1,n_1}, v] = [z_{1,0}a, \ldots, z_{1,n_1}a, a^{c_k}v]$ for all $a \in \mathbb{C}^\times$. A 2-step generalized Bott tower $B_2 = \mathbb{P}(\underline{\mathbb{C}} \oplus E_2^{(1)} \oplus \cdots \oplus E_2^{(n_2)})$ can be written as $B_2 = ((\mathbb{C}^{n_1+1})^\times \times (\mathbb{C}^{n_2+1})^\times)/G$, where the right action of $G = (\mathbb{C}^\times)^2$ is given by

$$(\mathbf{z}_1, \mathbf{z}_2) \cdot (a_1, a_2) = (z_{1,0}a_1, z_{1,1}a_1, \ldots, z_{1,n_1}a_1, z_{2,0}a_2, a_1^{c_1} z_{2,1}a_2, \ldots, a_1^{c_{n_2}} z_{2,n_2}a_2),$$

where $\mathbf{z}_j = (z_{j,0}, z_{j,1}, \ldots, z_{j,n_j})$ for $j = 1, 2$.

We can construct higher generalized Bott tower in a similar way. In this way we get an m-step generalized Bott manifold $B_m = \mathbb{P}(\underline{\mathbb{C}} \oplus E_m^{(1)} \oplus \cdots \oplus E_m^{(n_m)})$ from any collection of integers $\{c_{i,j}^{(k)}\}$:

$$B_m = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times)/G,$$

where the right action of $G = (\mathbb{C}^\times)^m$ is given by

$$(\mathbf{z}_1, \ldots, \mathbf{z}_m) \cdot \mathbf{a} = (\mathbf{z}'_1, \mathbf{z}'_2, \ldots, \mathbf{z}'_m),$$

where $\mathbf{z}_i = (z_{i,0}, \ldots, z_{i,n_i})$ for $i = 1, \ldots, m$, $\mathbf{a} = (a_1, \ldots, a_m) \in (\mathbb{C}^\times)^m$,
$\mathbf{z}'_1 = (z_{1,0}a_1, z_{1,1}a_1, \ldots, z_{1,n_1}a_1)$ and $\mathbf{z}'_j = (z_{j,0}a_j, a_1^{c_{1,j}^{(1)}} \cdots a_{j-1}^{c_{j-1,j}^{(1)}} z_{j,1}a_j, \ldots, a_1^{c_{1,j}^{(n_j)}} \cdots a_{j-1}^{c_{j-1,j}^{(n_j)}} z_{j,n_j}a_j)$ for $j = 2, \ldots, m$. We can construct a line bundle over B_m from the integers $(\ell_1, \ldots, \ell_m)$ by

$$\mathbf{L} = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times) \times_G \mathbb{C},$$

where $G = (\mathbb{C}^\times)^m$ acts by

$$((\mathbf{z}_1, \ldots, \mathbf{z}_m), v) \cdot \mathbf{a} = (\mathbf{z}'_1, \mathbf{z}'_2, \ldots, \mathbf{z}'_m, a_1^{\ell_1} \cdots a_m^{\ell_m} v). \tag{2.1}$$

2.2 Torus action on generalized Bott towers

Let $N = \sum_{j=1}^{m} n_j$ and let $T^N = S^1 \times \cdots \times S^1$. We consider the action of T^N on B_m as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m] = [\mathbf{t}_1 \cdot \mathbf{z}_1, \ldots, \mathbf{t}_m \cdot \mathbf{z}_m],$$

where $\mathbf{t}_i = (t_{i,1}, \ldots, t_{i,n_i})$ and $\mathbf{t}_i \cdot \mathbf{z}_i = (z_{i,0}, t_{i,1}z_{i,1}, \ldots, t_{i,n_i}z_{i,n_i})$ for $i = 1, \ldots, m$. Also we consider the action of $T = T^N \times S^1$ on $\mathbf{L}$ as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m, t_{m+1}) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m, v] = [\mathbf{t}_1 \cdot \mathbf{z}_1, \ldots, \mathbf{t}_m \cdot \mathbf{z}_m, t_{m+1}v]. \tag{2.2}$$

2.3 Generalized twisted cubes

Definition 2.2 A *generalized twisted cube* is defined to be the set of $x = (x_{1,1}, \ldots, x_{m,n_m}) \in \mathbb{R}^N$ which satisfies

$$\begin{aligned} &A_i(x) \leq \sum_{k=1}^{n_i} x_{i,k} \leq 0,\ \ x_{i,k} \leq 0 \ \ (1 \leq k \leq n_i) \\ &\text{or } 0 < \sum_{k=1}^{n_i} x_{i,k} < A_i(x),\ \ x_{i,k} > 0 \ \ (1 \leq k \leq n_i), \end{aligned} \tag{2.3}$$

for all $1 \leq i \leq m$, where

$$A_i(x) = \begin{cases} -\ell_m & (i = m) \\ -(\ell_i + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}) & (1 \leq i \leq m-1). \end{cases}$$

Definition 2.3 Let C be the generalized twisted cube. We define $\operatorname{sgn}(x_{i,k}) = 1$ for $x_{i,k} > 0$ and $\operatorname{sgn}(x_{i,k}) = -1$ for $x_{i,k} \leq 0$. The *density function* of the generalized twisted cube is then defined to be $\rho(x) = (-1)^N \prod_{i,k} \operatorname{sgn}(x_{i,k})$ when $x \in C$ and 0 elsewhere.

2.4 Equivariant index

Let $\mathbf{L}$ be a holomorphic line bundle over a generalized Bott manifold B_m with the action of the torus T as in (2.2). Let $\mathcal{O}_{\mathbf{L}}$ be the sheaf of holomorphic sections. The *equivariant index* of a generalized Bott manifold is the formal sum of representation of T:

$$\operatorname{index}(B_m, \mathcal{O}_{\mathbf{L}}) = \sum (-1)^i H^i(B_m, \mathcal{O}_{\mathbf{L}})$$

The *character* of the equivariant index is the function $\chi : T \to \mathbb{C}$ which is given by $\chi = \sum (-1)^i \chi^i$ where $\chi^i(a) = \operatorname{trace}\{a : H^i(B_m, \mathcal{O}_{\mathbf{L}}) \to H^i(B_m, \mathcal{O}_{\mathbf{L}})\}$ for $a \in T$. Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^*$ be its dual space. Every μ in the integral weight lattice $\ell^* \subset i\mathfrak{t}^*$ defines a homomorphism $\lambda^\mu : T \to S^1$. We can write $\chi = \sum_{\mu \in \ell^*} m_\mu \lambda^\mu$. The coefficients are given by a function mult $: \ell^* \to \mathbb{Z}$, sending $\mu \mapsto m_\mu$, called the *multiplicity function* for the equivariant index.

3 Main results

In this section, we state the main results and give the example.

Theorem 3.1 *Fix integers* $\{c_{i,j}^{(k)}\}$ *and* $\{\ell_j\}$*. Let* $\mathbf{L} \to B_m$ *be the corresponding line bundle over a generalized Bott manifold. Let* $\rho : \mathbb{R}^N \to \{-1, 0, 1\}$ *be the density function of the generalized twisted cube which is determined by these integers. Consider the action of* $T = T^N \times S^1$*. Then the multiplicity function for* $\ell^* \cong \mathbb{Z}^N \times \mathbb{Z}$ *is given by*

$$\operatorname{mult}(x, k) = \begin{cases} \rho(x) & (k = 1) \\ 0 & (k \neq 1). \end{cases}$$

Definition 3.2 Let $\{e_{1,1}, \dots, e_{m,n_m}, e_{m+1}\}$ be the standard basis in $\mathbb{R}^{N+1}$, $x_i = (x_{i,1}, \dots, x_{i,n_i})$ and $e_i = (e_{i,1}, \dots, e_{i,n_i})$. Let $\Delta^-_{n,r} = \left\{z = (z_1, \dots, z_n) \in \mathbb{Z}^n_{\leq 0} \,\middle|\, z_1 + \cdots + z_n = -r\right\}$, and let $\Delta^+_{n,r} = \{z = (z_1, \dots, z_n) \in \mathbb{Z}^n_{>0} \,|\, z_1 + \cdots + z_n = r - 1\}$. Then the operators $D_i : \mathbb{Z}[T] \to \mathbb{Z}[T]$ are defined using $c_{i,j}^{(k)}$ and ℓ_j in the following way:

$$D_i(\lambda^\mu) = \begin{cases} \displaystyle\sum_{0 \leq r \leq k_i} \sum_{x_i \in \Delta^-_{n_i,r}} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \geq 0 \\ 0 & \text{if } -n_i \leq k_i \leq -1 \\ \displaystyle\sum_{n_i+1 \leq r \leq -k_i} \sum_{x_i \in \Delta^+_{n_i,r}} (-1)^{n_i} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \leq -n_i - 1, \end{cases}$$

where the functions k_i are defined as follows: if $\mu = e_{m+1} + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} x_{j,k} e_{j,k}$, then $k_i(\mu) = \ell_i + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}$.

From Theorem 3.1, we immediately obtain the following theorem.

Theorem 3.3 *Consider the action of the torus* T*. We denote the* $(N + 1)$*-th component of the standard basis in* $\mathbb{R}^{N+1}$ *by* e_{m+1}*. Then the character is given by the following element of* $\mathbb{Z}[T]$ *:*

$$\chi = D_1 \cdots D_m(\lambda^{e_{m+1}}).$$

Example 3.4 Suppose that $m = 2, n_1 = 1, n_2 = 2, \ell_1 = 1$, and $\ell_2 = 2$. We set $c_{1,2}^{(1)} = 2$ and $c_{1,2}^{(2)} = -1$. Then the generalized twisted cube is the set of $x = (x_{1,1}, x_{2,1}, x_{2,2})$ which satisfies

- $-2 \leq x_{2,1} + x_{2,2} \leq 0,\ x_{2,1}, x_{2,2} \leq 0,$
- $-1 - 2x_{2,1} + x_{2,2} \leq x_{1,1} \leq 0$ or $0 < x_{1,1} < -1 - 2x_{2,1} + x_{2,2}$.

In Figure 1, the black dots represent the lattice points of the sign $+1$ and the white dots represent the sign -1.

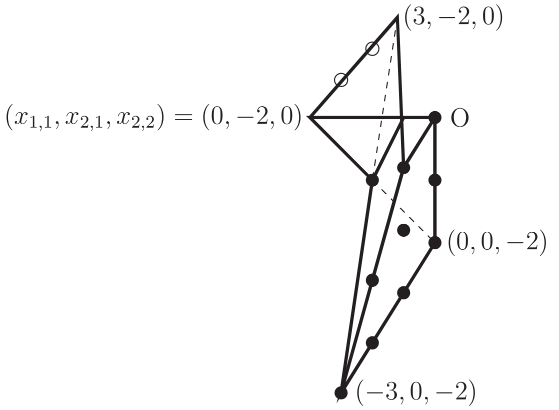

Figure 1

The corresponding character χ is given by

$$\begin{aligned}
\chi &= D_1 D_2(\lambda^{e_3}) \\
&= D_1(\lambda^{e_3} + \lambda^{e_3 - e_{2,1}} + \lambda^{e_3 - e_{2,2}} + \lambda^{e_3 - 2e_{2,1}} + \lambda^{e_3 - e_{2,1} - e_{2,2}} + \lambda^{e_3 - 2e_{2,2}}) \\
&= \lambda^{e_3} + \lambda^{e_3 - e_{1,1}} + \lambda^{e_3 - e_{2,2}} + \lambda^{e_3 - e_{2,2} - e_{1,1}} + \lambda^{e_3 - e_{2,2} - 2e_{1,1}} - \lambda^{e_3 - 2e_{2,1} + e_{1,1}} - \lambda^{e_3 - 2e_{2,1} + 2e_{1,1}} \\
&\quad + \lambda^{e_3 - e_{2,1} - e_{2,2}} + \lambda^{e_3 - 2e_{2,2}} + \lambda^{e_3 - 2e_{2,2} - e_{1,1}} + \lambda^{e_3 - 2e_{2,2} - 2e_{1,1}} + \lambda^{e_3 - 2e_{2,2} - 3e_{1,1}}.
\end{aligned}$$

References

[1] S. Choi, M. Masuda, and D. Y. Suh, *Quasitoric manifolds over a product of simplices*, Osaka J. Math., 47(1) (2010), 109–129.

[2] S. Choi, M. Masuda, and D. Y. Suh, *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc., 362(2) (2010), 1097–1112.

[3] M. Grossberg, and Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke math. J., 76(1) (1994), 23–58.

[4] T. Hwang, E. Lee, and D. Y. Suh, *The Gromov width of generalized Bott manifolds*, International Math. Research Notices, 2021(9) (2019), 7096–7131.

[5] A. Hattori, and M. Masuda, *Theory of multi-fans*, Osaka J. Math., 40 (2003), 1–68.

[6] Y. Karshon, and S. Tolman, *The moment map and line bundles over presymplectic toric manifolds*, J. Differential geometry, 38 (1993), 465–484.

[7] Y. Sugiyama, *On equivariant index of a generalized Bott manifold*, arXiv:2107.12054v1 (2021).