Nonisomorphic Smith equivalent modules over solvable groups

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1 Introduction

Paul Smith posted the following question:

Question 1 ([17]) *Is it true that for any smooth action of a finite group G on a homotopy sphere with exactly two fixed points, the real G-modules determined on the tangent spaces at the two fixed points are isomorphic?*

If the action is semi-free, then the answer is yes [1]. Sanchez [16] showed that the answer is yes if the group G is of odd prime power order. However, Cappell and Shaneson [2] showed that the answer is no if G is a cyclic group of order $4n$ for $n \geq 2$. Many researchers studied this corresponding problem (eg. [15, 3, 7, 12, 10, 14]).

We say U and V are *Smith equivalent*, denoted by $U \asymp_S V$, if there exists a smooth G-action on a homotopy sphere S such that $S^G = \{x, y\}$ and the tangential G-modules $T_x(S)$ and $T_y(S)$ are isomorphic to U and V respectively. We put

$$\mathrm{Sm}(G) := \{[U] - [V] \mid U \asymp_S V\} \subset \mathrm{RO}(G)$$

which is called the *Smith set* of G. There are natural questions that for a group G, what the Smith set $\mathrm{Sm}(G)$ is and in particular when the Smith set is not trivial.

Let $\mathcal{S}(G)$ be the set of all subgroups of G. For sets $\mathcal{L}$, $\mathcal{P}$ of subgroups of G and a subset $\mathcal{A}$ of $\mathrm{RO}(G)$, we put

$$\begin{aligned}
\mathcal{A}_{\mathcal{P}} &= \bigcap_{P \in \mathcal{P}} \ker(\mathrm{Res}^G_P\colon\ \mathrm{RO}(G) \to \mathrm{RO}(P)) \cap \mathcal{A},\\
\mathcal{A}^{\mathcal{L}} &= \bigcap_{L \in \mathcal{L}} \ker(\mathrm{Fix}^G_L\colon\ \mathrm{RO}(G) \to \mathrm{RO}(N_G L/L)) \cap \mathcal{A},\\
\mathcal{A}^{\mathcal{L}}_{\mathcal{P}} &= (\mathcal{A}_{\mathcal{P}})^{\mathcal{L}}.
\end{aligned}$$

Let $\mathcal{P}_{\mathrm{cyc}}(G)$ denote the set of cyclic subgroups of G of odd prime power order and let $\mathcal{P}_{\mathrm{odd}}(G)$ denote the set of subgroups of G of odd prime power order. Clearly $\mathcal{P}_{\mathrm{cyc}}(G) \subset \mathcal{P}_{\mathrm{odd}}(G)$. By character theory, we have

$$\mathrm{RO}(G)_{\mathcal{P}_{\mathrm{odd}}(G)} = \mathrm{RO}(G)_{\mathcal{P}_{\mathrm{cyc}}(G)}.$$

Then by Sanchez's result, we have an implementation

$$\mathrm{Sm}(G) \subset \mathrm{RO}(G)_{\mathcal{P}_{\mathrm{odd}}(G)}.$$

It is easy to see that for the set $\mathcal{Q}(G)$ of all subgroups of G of order $1, 2, 4$, $\mathrm{Sm}(G)$ is a subset of $\mathrm{RO}(G)_{\mathcal{Q}(G)}$ and thus

$$\mathrm{Sm}(G) \subset \mathrm{RO}(G)_{\mathcal{P}_{\mathrm{odd}}(G) \cup \mathcal{Q}(G)}.$$

However, $\mathrm{Sm}(C_{4n})$ is not trivial for $n \geq 2$. Here C_{4n} denotes a cyclic group of order $4n$. In particular it is not satisfied that $\mathrm{Sm}(G)$ is a subset of $\mathrm{RO}(G)_{\mathcal{P}(G)}$.

We define a subset $\mathrm{LSm}(G)$ called the Laitinen-Smith set of G, which is a subset of $\mathrm{Sm}_{\mathcal{P}(G)}(G)$ as follows. We say U and V are *Laitinen-Smith equivalent*, denoted by $U \asymp_{LS} V$, if there exists a smooth G-action on a homotopy sphere S such that $S^G = \{x, y\}$, the tangential G-modules $T_x(S)$ and $T_y(S)$ are isomorphic to U and V respectively, and S^h is connected for any element h of G of 2-power order ≥ 8. We put

$$\mathrm{LSm}(G) := \{[U] - [V] \mid U \asymp_{LS} V\}.$$

The set $\mathrm{LSm}(G)$ is a subset of $\mathrm{Sm}(G)_{\mathcal{P}(G)}$, where $\mathcal{P}(G)$ denotes the set of subgroups of G of prime power order.

For an element g of G, the real conjugacy class of g, denoted by $(g)^{\pm}$ is the union of the conjugacy class of g and that of g^{-1}. A finite group G is called an Oliver group if there does not exist a sequence

$$P \triangleleft H \triangleleft G$$

such that P and G/H is of prime power order and H/P is cyclic.

Theorem 2 ([11, 5]) *The followings are equivalent.*

(1) *G is an Oliver group.*

(2) *There exists a fixed-point-free G-action of a disk.*

(3) *There exists a one fixed point G-action of a sphere.*

Question 3 ([6]) *Let G be a finite Oliver group. Is it true that* $\mathrm{LSm}(G)$ *is not trivial if G has at least two real conjugacy classes of elements of G not of prime power order?*

Morimoto [8] pointed out that the answer is no and moreover showed that $\mathrm{Sm}(G)$ is a subset of $\mathrm{RO}(G)^{\cap_2(G)}$, where $\cap_2(G)$ is the set of subgroups of G with index 1 or 2. Furthermore Morimoto and his students [4] showed that if a Sylow 2-subgroup of G is a normal subgroup of G then $\mathrm{Sm}(G)$ is a subset of $\mathrm{RO}(G)^{\cap(G)}$, where $\cap(G)$ is the set of normal subgroups of G with index 1 or prime.

In this note, we recall and study Oliver groups G satisfying that $\mathrm{LSm}(G)^{\mathcal{L}(G)}$ is not trivial.

2 Some groups of which Smith sets are nontrivial

Let a_G denote the number of real conjugacy classes of elements not of prime power order. If $\mathrm{LSm}(G)$ is not trivial then $a_G \geq 2$ holds. For $G = \mathrm{Aut}(A_6)$, $a_G = 2$ and $\mathrm{Sm}(G)$ is trivial. Let $b_{G,2}$ be the number of real conjugacy classes of $G/\cap_{L \in \cap_2(G)} L$ which is the image of the real conjugacy classes of elements of G not of prime power order.

Proposition 4 *If* $a_G = b_{G,2}$ *then* $\mathrm{LSm}(G)$ *is trivial.*

Equivalently if $\mathrm{LSm}(G)$ is not trivial then $a_G > b_{G,2}$ holds.

For a prime p, the Dress subgroup $O^p(G)$ is defined as the intersection of all subgroups of G with p-power index. Put

$$G^{\mathrm{nil}} = \bigcap_p O^p(G).$$

The factor group G/G^{nil} is nilpotent. Let $\mathcal{L}(G)$ be the set of large subgroups, that is the set of subgroups of G which includes a Dress subgroup $O^p(G)$ for some prime p.

For a subset $\mathcal{S}$ of $\mathcal{S}(G)$, we say that K is an $\mathcal{S}$-gap subgroup of G if there exists an $(\mathcal{S} \cap K)$-free K-module V such that

$$\dim V^P > 2 \dim V^H$$

for any $P < H \leq K$ and $P \in \mathcal{P}(K)$, where

$$\mathcal{S} \cap K = \{S \cap K \mid S \in \mathcal{S}\}.$$

A finite group G is called a gap group if G is an $\mathcal{L}(G)$-free gap subgroup of G. We say that a G-module V satisfies the *weak gap condition* if

$$\dim V^P \geq 2 \dim V^H$$

for any $P < H \leq G$ and $P \in \mathcal{P}(G)$. Also, we say that G satisfies the G^{nil}-*coset condition*[1] if there exists a G^{nil}-coset of G containing two elements x and y not of prime power order that are not real conjugate in G and, in addition,

(1) the elements x and y are both in some $\mathcal{L}(G)$-gap subgroup K of G, or

(2) the orders of x and y are even and the elements of order 2 of $\langle x \rangle$ and $\langle y \rangle$ are conjugate in $G \smallsetminus O^2(G)$, where $\langle x \rangle$ and $\langle y \rangle$ are the cyclic groups generated by x and y, respectively.

Remark 5 (cf. [18, Theorem B]) *If an Oliver group G has an element x not of prime power order such that $\langle x \rangle \cap O^2(G)$ is a group of ever order then $\langle x \rangle O^2(G)$ is an $\mathcal{L}(G)$-gap subgroup of G.*

Theorem 6 (cf. [14, Theorem 5.6]) *If G satisfies* (2), *then there exists an element of* $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ *can be described as the difference of nonisomorphic G-modules satisfying the weak gap condition and it belongs to* $\mathrm{LSm}(G)$.

Theorem 7 ([8, 9, 14]) *Let G be a finite nonsolvable group. Suppose that there exist two or more real conjugacy classes of elements of G not of prime power order. Then G satisfies one of the following property.*

(1) *G is isomorphic to* $\mathrm{Aut}(A_6)$.

(2) *G is isomorphic to* $P\Sigma L(2, 27)$.

[1] It was defined in [14]. This version is a little bit extended so that [14, Theorem 5.6].

(3) *G satisfies the G^{nil}-coset condition.*

Furthermore, $\mathrm{LSm}(G)$ *is nontrivial for any group G satisfying (2) or (3), and if* $\mathrm{Sm}(G)$ *is trivial then (1) holds.*

Question 8 *Determine nonsolvable groups G of which Smith set is trivial. Is it true that if $a_G \leq 1$, then* $\mathrm{Sm}(G) = 0$*?*

By the above theorem and [12, Classification Theorem], such groups are finitely many. For an epimorphism $f\colon G \to F$, we easily see that $f^* \mathrm{Sm}(F)$ isomorphic to $\mathrm{Sm}(F)$, is a subset of $\mathrm{Sm}(G)$. However it is not known whether the action on a sphere giving the Smith equivalence is effective. If G is an Oliver group and there is a pair (V, W) of $\mathcal{L}(F)$-free F-modules such that $[V]-[W] \in \mathrm{RO}(F)_{\mathcal{P}(F)}$ and both V and W satisfy the weak gap condition, then we construct an effective action of a sphere giving $f^*([V]-[W]) \in \mathrm{Sm}(G)$.

Let $\mathrm{WRO}(G)$ be the set of the difference of $\mathcal{L}(G)$-free G-modules satisfying the weak gap condition and put $\mathrm{WLO}(G) = \mathrm{LO}(G) \cap \mathrm{WRO}(G)$. The following proposition is known.

Proposition 9 *Let G be an Oliver group.*

$$\mathrm{WLO}(G) \subset \mathrm{LSm}(G)^{\mathcal{L}(G)} \subset \mathrm{Sm}(G)_{\mathcal{P}(G)}.$$

Note that $\mathrm{WLO}(N) = \mathrm{LO}(N)$ for a nilpotent group N. We have $\mathrm{LO}(N) \subset \mathrm{LSm}(N)$. Then we have

Proposition 10 *If* $\mathrm{LSm}(G) = 0$ *then* $\mathrm{LSm}(G/G^{\mathrm{nil}}) = 0$ *holds.*

We remark that nilpotent groups N with $LO(N) = 0$ are known:

Proposition 11 ([13]) *Let N be a finite nilpotent group not of prime power order. If* $\mathrm{LO}(N) = 0$*, then N is isomorphic to $C_2 \times Q$, Q a group of odd prime power order, or $P \times C_3$ for a 2-group P such that x and x^{-1} are conjugate for any element $x \in P$.*

3 Case when $\mathrm{LO}(G) \neq 0$ implies $\mathrm{WLO}(G) \neq 0$

It is unknown that there exists a finite group G such that $\mathrm{LO}(G) \neq \mathrm{WLO}(G)$. There exist many groups satisfying the equality:

Proposition 12 *If G is a gap group then* $\mathrm{LO}(G) = \mathrm{WLO}(G)$.

Proposition 13 *If an Oliver group G has an element whose order is divisible by at least three distinct primes, then* $0 \neq \mathrm{WLO}(G) \subset \mathrm{LO}(G) \subset \mathrm{LSm}(G)$.

Proof By the assumption, G is a gap group and satisfies (1) and (2) of the G^{nil}-condition. ∎

Theorem 14 *Let G be a finite Oliver group with $O^p(G) \neq G$ for some odd prime p. Then for any $x \in \mathrm{LO}(G)$ there exists an $\mathcal{L}(G)$-free G-module W satisfying the weak gap condition such that $x + [W]$ is represented by a $\mathcal{L}(G)$-free G-module satisfying the weak gap condition, which implies* $\mathrm{WLO}(G) = \mathrm{LO}(G) \subset \mathrm{LSm}(G)$.

Proof It suffice to show that

$$\mathrm{WRO}(G)^{\{O^2(G)\}}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{O^2(G)\}}_{\mathcal{P}(G)}$$

for an Oliver group G. Since if G is a gap group then it is clear, we assume that G is not a gap group. First we assume that $[G : O^2(G)] = 2$. Then by [19, Theorem 7.7], the group G^{nil} is of odd order. By Sylow's theorem, we see the claim. In the case when G is an Oliver nongap group with $[G : O^2(G)] > 2$. By [18, Theorem A], we see the claim. ∎

Suppose that G/G^{nil} has 2-power order. Let x and y be elements of G not of prime power order such that $xO^2(G) = yO^2(G)$ and $|xO^2(G)| > 2$. Then we can construct an nontrivial element of $\mathrm{WLO}(G)$ by using elements of $\mathrm{RO}(\langle x\rangle)_{\mathcal{P}(G)}$ and $\mathrm{RO}(\langle y\rangle)_{\mathcal{P}(G)}$ by using the G^{nil}-condition. Thus we see

Theorem 15 *Let G be an Oliver group. Suppose that any subgroup K of G with $[K : O^2(G)] = 2$ is a gap group. Then* $\mathrm{LO}(G) = \mathrm{WLO}(G) \subset \mathrm{LSm}(G)$.

Therefore, the case when $[G : G^{\mathrm{nil}}] = 2$ is important:

Theorem 16 *Let G be a finite nongap Oliver group such that G/G^{nil} is cyclic.* $\mathrm{LO}(G) \otimes \mathbb{Q} \neq \mathrm{WLO}(G) \otimes \mathbb{Q}$ *if and only if there exist two elements x and y of G not of prime power order, $|x|$ and $|y|$ is even and not divisible by 4, and elements of order 2 of $\langle x\rangle$ and $\langle y\rangle$ are elements of $G \smallsetminus O^2(G)$ and are not conjugate in G.*

Remark 17 *There exist finite Oliver groups G such that*

(1) *G is not a gap group,*

(2) *$[G : G^{\mathrm{nil}}] = 2$, and*

(3) *there exist two elements of $G \smallsetminus G^{\mathrm{nil}}$ of order 2 which are not conjugate in G.*

4 Oliver solvable groups

Nonsolvable groups are Oliver groups. In this section we consider finite solvable Oliver nongap groups. Ronald Solomon pointed out the structure of a nongap group G with $[G : G^{\mathrm{nil}}] = 2$ as follows. Suppose that there exists an element of order 2 of $G \smallsetminus G^{\mathrm{nil}}$. The group G/G_2 is a solvable group for some normal subgroup G_2 of G of odd order: There exist normal subgroups G_1, G_2 of G such that

(1) $G^{\mathrm{nil}} \geq G_1 \geq G_2$,

(2) G_2 is of odd order,

(3) G_1/G_2 is a 2-group,

(4) G^{nil}/G_1 is an abelian group of odd order, and every element of $(G/G_1) \smallsetminus (G^{\mathrm{nil}}/G_1)$ is an element of order 2.

Note that $(G \setminus G^{\mathrm{nil}})/G_2$ does not have an element not of prime power order.

For $K \le G$, we put

$$c_{K,G} := \#\{(k)^{\pm} \mid k \in \mathrm{NPP}(K)\},$$

where $(k)^{\pm}$ is the real conjugacy class of k. Recall the following proposition.

Proposition 18 ([12, Second Rank Lemma and Subgroup Lemma]) $\mathrm{LSm}(G) \neq 0$ *for an Oliver group G with $c_{G^{\mathrm{nil}},G} \ge 2$.*

Thus suppose that $c_{G^{\mathrm{nil}},G} \le 1$. Under this condition, we see

Theorem 19 *Let G be a nongap Oliver solvable group with $[G : G^{\mathrm{nil}}] = 2$ and $c_{G^{\mathrm{nil}},G} \le 1$. Then $\mathrm{LSm}(G) = \mathrm{Sm}(G)_{\mathcal{P}(G)} = \mathrm{WRO}(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$ and $\mathrm{Ind}^{G}_{G^{\mathrm{nil}}} \mathrm{RO}(G^{\mathrm{nil}})^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)} = 0$.*

The proof depends on the structure of a nongap group G with $[G : G^{\mathrm{nil}}] = 2$ by Ronald Solomon. Note that if $\mathrm{LO}(G) \neq 0$ then $0 \neq \mathrm{WLO}(G) \subset \mathrm{LSm}(G)$ for any nonsolvable group G.

Finally remark that there exist a few Oliver groups G such that $\mathrm{LO}(G) = 0$ and $\mathrm{LSm}(G) \neq 0$. The author expects that this is a rare case.

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