Asymptotic maps as a generalization of Colombeau type generalized functions

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Introduction

Generalized functions, including Schwartz distributions, make it possible to differentiate functions whose derivatives do not exist in the classical sense (e.g. locally integrable functions). They are widely used in the theory of partial differential equations, where it may be easier to establish the existence of weak solutions than classical ones, or appropriate classical solutions may not exist. In a recent paper [8], I introduced the notion of asymptotic map between arbitrary diffeological spaces. The category consisting of diffeological spaces and asymptotic maps is enriched over the category of diffeological spaces, and inherits completeness and cocompleteness. In particular, the set of asymptotic functions on a Euclidean open set include Schwartz distributions and form a Colombeau type smooth differential algebra over Robinson's field of asymptotic numbers.

In this survey, we provide a summary of the main results of [8] together with the background information needed to understand them.

Preliminaries on generalized functions

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and U be an open subset of $\mathbb{R}^d$. Denote by $C^\infty(U)$ the set of all smooth functions $U \to \mathbb{F}$. For any open or closed subset $A \subset U$, we associate an L^∞-norm

$$\|f\|_{L^\infty(A)} = \sup_{x \in A} |f(x)|$$

Then $C^\infty(U)$ is a topological vector space such that the following holds:

> Let (f_n) be a sequence in $C^\infty(U)$. Then $f_n \to f$ in $C^\infty(U)$ as $n \to \infty$ if and only if $\forall K \Subset U$ and $\forall \alpha \in \mathbb{N}^d$ such that $|\alpha| \leq k$, we have $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on K i.e. $\|\partial^\alpha f_n - \partial^\alpha f\|_{L^\infty(K)} \to 0$.

Example: For any $f(x) \in C^\infty(\mathbb{R}^n)$, let $f_n(x) = f(x/n)$. Then $f_n \to f(0)$ (constant function) as $n \to \infty$.

Test functions: A smooth function $f\colon U \to \mathbb{F}$ is called a *test function* if its support $\operatorname{supp}(f)$ is compact. Let $\mathcal{D}(U)$ be the set of test functions on U, and if $K \Subset U$ is a compact subset, let

$$\mathcal{D}(K) = \{f \in \mathcal{D}(U) \mid \operatorname{supp}(f) \subset K\}$$

Then $\mathcal{D}(U)$ is a LCTVS (locally convex topological vector space) with respect to the *canonical LF-topology*, that is, a direct limit of Fréchet spaces $\mathcal{D}(K)$ $(K \Subset U)$ with seminorms

$$\|\partial^\alpha f\|_{L^\infty(K)} = \sup_{x\in K} \|\partial^\alpha f(x)\| \quad (\forall \alpha \in \mathbb{N}^d)$$

As a locally convex vector space $\mathcal{D}(U)$ is complete, barreled, bornological, and a Montel space (i.e. reflexive, and is such that the Heine-Borel theorem is valid). We say that a sequence of test functions $\{f_n\}$ converges to f in $\mathcal{D}(U)$ if there exists $K \Subset U$ such that $\operatorname{supp}(f)$, $\operatorname{supp}(f_n) \subset K$ $(n \in \mathbb{N})$ and $f_n \to f$ in the Fréchet space $\mathcal{D}(K)$. The notion of "Cauchy sequence" is defined similarly, and we have the following.

Theorem: $\mathcal{D}(U)$ is sequentially complete.

Schwartz distributions: The set of *distributions* on U, denoted $\mathcal{D}'(U)$, is a strong dual of the topological vector space $\mathcal{D}(U)$, that is, the set of <u>continuous</u> linear functionals $\mathcal{D}(U) \to \mathbb{F}$. We denote $\langle T \mid u\rangle = T(u)$ for any linear functional T on $\mathcal{D}(U)$ and $u \in \mathcal{D}(U)$.

Lemma 1. *Let $T\colon \mathcal{D}(U) \to \mathbb{F}$ be a linear functional. Then the following are equivalent.*

(1) *T is a distribution.*

(2) *$T(u_n) \to T(u) \in \mathbb{F}$ holds for every $u_n \to u \in \mathcal{D}(U)$.*

(3) *For all $K \Subset U$, there exist $C > 0$ and $m \in \mathbb{N}$ such that we have*

$$|\langle T \mid u\rangle| \le C \sum_{|\alpha|\le m} \|\partial^\alpha u\|, \qquad \forall u \in \mathcal{D}(K)$$

Any locally integrable function f gives rise to a distribution T_f defined by

$$\langle T_f \mid u\rangle = \int_U f(x)u(x)\,dx \quad (u \in \mathcal{D}(U))$$

For any $T \in \mathcal{D}'(U)$ and multi-index α we define $D^\alpha T$ by

$$\langle D^\alpha T \mid u\rangle = (-1)^{|\alpha|}\langle T \mid D^\alpha u\rangle \quad (u \in \mathcal{D}(U))$$

This extends D^α on $C^\infty(U)$ as we have $D^\alpha(T_f) = T_{D^\alpha f}$ for all $f \in C^\infty(U)$.

Important notes:

- $\mathcal{D}'(U)$ is obviously a vector space, but is not an algebra over $\mathbb{F}$. In fact, the *Schwartz impossibility result* says the following:

  > There is no associative algebra over $\mathbb{R}$ containing $\mathcal{D}'(\mathbb{R})$ as a vector subspace and the constant function 1 as unity element, having a differential operator acting like the differential operator on $\mathcal{D}'(\mathbb{R})$ and the algebra multiplication of continuous functions is like their pointwise multiplication.

- $\mathcal{D}'(U)$ is functorial with respect to submersions of U but is not fully functorial; hence there is no supercategory $\mathcal{C}$ of the category **Man** of smooth manifolds and smooth maps satisfying $\hom_{\mathcal{C}}(M, \mathbb{R}) = \mathcal{D}'(M)$.

Colombeau algebra: Schwartz' impossibility result says that there cannot be a differential algebra containing the space of distributions and preserving the product of continuous functions. However, J. F. Colombeau found that if one only wants to preserve the product of smooth functions, instead of continuous ones, such a construction of differential algebra becomes possible.

Henceforth, J denotes the interval $(0, 1]$, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{N}$ the set of non-negative integers. If X is a topological or diffeological space then its underlying set is denoted by $|X|$. Given an open set U in $\mathbb{R}^n$, let

$$\mathcal{E}_{\mathbb{F}}(U) = C^\infty(U, \mathbb{F})^J$$

be the set of J-nets in the locally convex space of smooth functions $U \to \mathbb{F}$. Then the simplified Colombeau algebra on U is defined to be the quotient

$$\mathcal{G}^s(U) = \mathcal{M}(\mathcal{E}_{\mathbb{F}}(U))/\mathcal{N}(\mathcal{E}_{\mathbb{F}}(U))$$

of the subalgebra of *moderate nets* over that of *negligible nets*. Here, we say that $(f_\epsilon)_{\epsilon \in J} \in \mathcal{E}_{\mathbb{F}}(U)$ is *moderate* if it satisfies the condition

$$\forall K \Subset U,\ \forall \alpha \in \mathbb{N}^n,\ \exists m \in \mathbb{N},\ \max_{x \in K} |D^\alpha f_\epsilon(x)| = O(1/\epsilon^m)$$

and is *negligible* if

$$\forall K \Subset U,\ \forall \alpha \in \mathbb{N}^n,\ \forall p \in \mathbb{N},\ \max_{x \in K} |D^\alpha f_\epsilon(x)| = O(\epsilon^p)$$

where the notation $K \Subset U$ indicates that K is a compact subset of U.

Partial derivatives: For every $\alpha \in \mathbb{N}^d$ $(d = \dim U)$, the partial derivative $\partial^\alpha : \mathcal{G}^s(U) \to \mathcal{G}^s(U)$ is induced by the map $(f_\epsilon) \mapsto (\partial^\alpha f_\epsilon)$.

Embedding of smooth maps: There is a natural embedding of algebras $C^\infty(U) \to \mathcal{G}^s(U)$ induced by the map $C^\infty(U) \to C^\infty(U)^J,\ f \mapsto (f)$.

Embedding of distributions: The correspondence $U \mapsto \mathcal{G}^s(U)$ is a fine and supple sheaf of differential algebras, and there is a sheaf embedding of vector spaces $\iota_U : \mathcal{D}'(U) \to \mathcal{G}^s(U)$ preserving partial derivatives and extending the inclusion $C^\infty(U, \mathbb{F}) \to \mathcal{G}^s(U)$ described above (see [3, Theorem 1.1]). Note, however, that the embedding ι_U is not canonical because it depends on the choice of mollifier. When $U = \mathbb{R}^n$, $\iota_{\mathbb{R}^n} : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{G}^s(\mathbb{R}^n)$ can be defined as follows.

Fix a mollifier $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (i.e. $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$), and put $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$ for $\epsilon \in J$. Then (φ_ϵ) represents Dirac's δ and we put

$$\iota_{\mathbb{R}^n}(T) = [\varphi_\epsilon * T] \qquad (T \in \mathcal{D}'(\mathbb{R}^n))$$

where $\varphi_\epsilon * T$ denotes the *convolution* of φ_ϵ with T. Recall that if $T = T_f$ for a locally integrable function f then

$$\varphi_\epsilon * T_f(x) = \int_{\mathbb{R}^n} \varphi_\epsilon(t - x) f(t)\, dt$$

Important notes:

- The scalar of Colombeau's theory is not a field but a ring with (meaningless) zero-divisors.
- Unlike $\mathcal{D}'$ or other "full" version of generalized functions, $\mathcal{G}^s(U)$ is functorial with respect to arbitrary smooth maps. As we shall see in due course, this functoriality is a huge benefit. But, of course, there is a trade-off between functoriality and fullness (i.e. canonicality of the embedding of the Schwartz distributions).

Main results

In order to describe the main results of [8], we require the notion of diffeological spaces. A set X together with a family $\mathcal{D} \subset \bigcup_U \mathbf{Set}(U, X)$ of maps from Euclidean open sets into X is called a *diffeological space* if it satisfies:

(1) Any constant map $\mathbb{R}^n \to X$ is contained in $\mathcal{D}$.

(2) $\sigma : U \to X \in \mathcal{D}$ iff ${}^\forall x \in U,\ x \in {}^\exists V \subset U,\ \sigma|V : V \to X \in \mathcal{D}$.

(3) If $\sigma : U \to X \in \mathcal{D}$ then $\sigma \circ \phi : V \to X \in \mathcal{D}$ for all smooth $\phi : V \to U$.

We call $\mathcal{D}$ a *diffeology*, and each member of $\mathcal{D}$ a *plot*. A *smooth map* from $(X, \mathcal{D})$ to $(Y, \mathcal{E})$ is a map $f : X \to Y$ such that $f \circ \sigma \in \mathcal{E}$ whenever $\sigma \in \mathcal{D}$. Diffeological spaces and smooth maps form a category $\mathbf{Diff}$.

Among noteworthy properties of $\mathbf{Diff}$, it is *convenient* in the sense that it is closed under limits, colimits, and is *cartesian closed*; in fact, its set of smooth maps $C^\infty(X, Y)$ is equipped with *functional diffeology* embracing a natural diffeomorphism

$$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z))$$

We are now ready to state the main theorem of [8].

Theorem 2. *There is a supercategory* $\widehat{\mathbf{Diff}}$ *of* $\mathbf{Diff}$ *enjoying the properties below. Let* $\mathbb{F}$ *be either* $\mathbb{R}$ *or* $\mathbb{C}$*.*

(1) $\widehat{\mathbf{Diff}}$ *has the same objects as* $\mathbf{Diff}$ *and is enriched over* $\mathbf{Diff}$*.*

(2) *There is an enriched functor* $\mathbf{Diff} \to \widehat{\mathbf{Diff}}$ *which embeds* $C^\infty(X, Y)$ *into the space of morphisms* $\widehat{C}^\infty(X, Y)$ *in* $\widehat{\mathbf{Diff}}$*.*

(3) $\widehat{\mathbf{Diff}}$ *has all small limits and colimits, and the inclusion* $\mathbf{Diff} \to \widehat{\mathbf{Diff}}$ *preserves limits and colimits.*

(4) *There is a diffeomorphism* $\widehat{C}^\infty(X \times Y, Z) \cong C^\infty(X, \widehat{C}^\infty(Y, Z))$ *which extends the exponential law* $C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z))$*.*

(5) *Let* $\widehat{\mathbb{F}} = \widehat{C}^\infty(\mathbb{R}^0, \mathbb{F})$*. Then* $\widehat{\mathbb{R}}$ *is a non-archimedean real closed field and* $\widehat{\mathbb{C}}$ *is an algebraically closed field of the form* $\widehat{\mathbb{C}} = \widehat{\mathbb{R}} + \sqrt{-1}\,\widehat{\mathbb{R}}$*.*

(6) *For any* $X \in \mathbf{Diff}$*,* $\widehat{C}^\infty(X, \mathbb{F})$ *is a smooth* $\widehat{\mathbb{F}}$*-algebra containing* $C^\infty(X, \mathbb{F})$ *as a smooth subalgebra over* $\mathbb{F}$*.*

(7) *If* U *is a Euclidean open set then* $\widehat{C}^\infty(U, \mathbb{F})$ *is a smooth differential algebra admitting a continuous linear embedding* $\mathcal{D}'(U) \to \widehat{C}^\infty(U, \mathbb{F})$ *which preserves partial derivatives and restricts to the inclusion of* $C^\infty(U, \mathbb{R})$*.*

A member of $\widehat{C}^\infty(X, Y)$ is called an *asymptotic map* from X to Y, and is called an *asymptotic function* if Y is $\mathbb{F}$. Beware that $\widehat{\mathbf{Diff}}$ is no longer cartesian closed because $\widehat{C}^\infty(\mathbb{R}^0, \mathbb{F})$ is not isomorphic to $\mathbb{F}$ by (5) above. Note also that (4) implies $\widehat{C}^\infty(X, \mathbb{F}) \cong C^\infty(X, \widehat{\mathbb{F}})$, meaning that $\mathbb{F}$*-valued asymptotic functions* can be viewed as $\widehat{\mathbb{F}}$*-valued smooth functions*.

Among possible applications of asymptotic maps, we are interested in the following: The fact that $\widehat{\mathbb{F}}$ is a field enables us to construct a graded differential algebra of "asymptotic differential forms" on arbitrary diffeological space. In particular, if the base space is a smooth manifold then its algebra of asymptotic differential forms includes the space of *de Rham currents*. Since $\mathbf{Diff}$ contains in a natural way a variety of singular spaces such as orbifolds, rectifiable sets and varifolds, we may expect that asymptotic maps and forms serve as an efficient tool in geometric measure theory and geometric analysis. In a subsequent paper, we shall describe the construction of asymptotic differential forms and some of their applications.

Outline of the construction of $\widehat{C}^\infty(X, Y)$

We first construct the smooth algebra of asymptotic functions $\widehat{C}^\infty(U, \mathbb{F})$ by suitably modifying the definition of simplified Colombeau algebra, and then extend it to $\widehat{C}^\infty(X, Y)$ for arbitrary X and Y.

Colombeau algebra as a diffeological space: Giordano and Wu introduced in [3, §5] a diffeological space structure for $\mathcal{G}^s(U)$ defined as follows. Instead of the locally convex topology, let us endow $C^\infty(U,\mathbb{F})$ with the functional diffeology, so that a parameterization $\sigma\colon V \to C^\infty(U,\mathbb{F})$ is a plot for $C^\infty(U,\mathbb{F})$ if and only if the composition below is smooth.

$$V \times U \xrightarrow{\sigma\times 1} C^\infty(U,\mathbb{F}) \times U \xrightarrow{\mathrm{ev}} \mathbb{F}, \quad (x,y) \mapsto \sigma(x)(y)$$

Let $\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U) = C^\infty(U,\mathbb{F})^J$, where J is regarded as a discrete diffeological space. Then moderate nets and negligible nets form diffeological subspaces $\mathcal{M}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ and $\mathcal{N}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)))$, respectively, and hence a diffeological quotient

$$\boldsymbol{\mathcal{G}}^s(U) = \mathcal{M}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))/\mathcal{N}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)).$$

Theorem 3 (Giordano-Wu)**.** *The space $\boldsymbol{\mathcal{G}}^s(U)$ is a smooth differential algebra, and there is a linear embedding $\iota_U\colon \mathcal{D}'(U) \to \boldsymbol{\mathcal{G}}^s(U)$ enjoying the following properties:*

(1) *ι_U is continuous with respect to the weak dual topology on $\mathcal{D}'(U)$ and the D-topology on $\boldsymbol{\mathcal{G}}^s(U)$.*

(2) *ι_U preserves partial derivatives, i.e. $D^\alpha(\iota_U(u)) = \iota_U(D^\alpha u)$ holds for all $u \in \mathcal{D}'(U)$ and $\alpha \in \mathbb{N}^n$.*

(3) *ι_U restricts to the embedding of smooth differential algebra $C^\infty(U) \to \boldsymbol{\mathcal{G}}^s(U)$ which takes f to the class of the constant net with value f.*

Note. The embedding ι_U preserves multiplication of smooth functions, but does not preserve multiplication of continuous functions because if otherwise, it contradicts to the Schwartz impossibility result [7].

Asymptotic functions on Euclidean open sets: We first modify the definition of $\boldsymbol{\mathcal{G}}^s(U)$ so that the set of scalars (the case $U = \mathbb{R}^0$) becomes a true field. For this purpose, we introduce, as in the definition of Todorov-Vernaeve's algebra [9], a $\{0,1\}$-valued finitely additive measure on $J = (0,1]$ given by an ultrafilter $\mathcal{U}$ generated by the collection $\{(0,\epsilon) \mid 0 < \epsilon \leq 1\}$. Given a predicate $P(\epsilon)$ defined on J we write "$P(\epsilon)$ a.e." to mean that the set $\{\epsilon \in J \mid P(\epsilon)\}$ belongs to $\mathcal{U}$.

As in the previous section, let $\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U) = C^\infty(U,\mathbb{F})^J$ and put

$$\widehat{C}^\infty(U,\mathbb{F}) = \widehat{\mathcal{M}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))/\widehat{\mathcal{N}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$$

where $\widehat{\mathcal{M}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ consists of *moderate nets* $(f_\epsilon) \in \boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)$ satisfying

$$\forall K \Subset U,\ \forall \alpha \in \mathbb{N}^n,\ \exists m \in \mathbb{N},\ \max_{x\in K}|D^\alpha f_\epsilon(x)| = O(1/\epsilon^m) \text{ a.e.}$$

and $\widehat{\mathcal{N}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ consists of *negligible nets* $(f_\epsilon) \in \boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)$ satisfying

$$\forall K \Subset U,\ \forall \alpha \in \mathbb{N}^n,\ \forall p \in \mathbb{N},\ \max_{x\in K}|D^\alpha f_\epsilon(x)| = O(\epsilon^p) \text{ a.e.}$$

Clearly, $\widehat{C}^{\infty}(U,\mathbb{F})$ inherits ring operations from $C^{\infty}(U,\mathbb{F})$ and every smooth map $F\colon W \to U$ between Euclidean open sets induces a functorial homomorphism of $\mathbb{F}$-algebras

$$F^{*}\colon\ \widehat{C}^{\infty}(U,\mathbb{F}) \to \widehat{C}^{\infty}(W,\mathbb{F}), \quad F^{*}([f_\epsilon]) = [f_\epsilon \circ F]$$

Moreover, by arguing as in [3, Theorem 5.1] we see that $\widehat{C}^{\infty}(U,\mathbb{F})$ is a smooth differential algebra with respect to partial differential operators

$$D^{\alpha}\colon\ \widehat{C}^{\infty}(U,\mathbb{F}) \to \widehat{C}^{\infty}(U,\mathbb{F}), \quad D^{\alpha}([f_\epsilon]) = [D^{\alpha} f_\epsilon] \quad (\alpha \in \mathbb{N}^{n})$$

and there is a natural inclusion of smooth differential algebras

$$i_U\colon\ C^{\infty}(U,\mathbb{F}) \to \widehat{C}^{\infty}(U,\mathbb{F})$$

which takes a smooth map f to the class of the constant net with value f.

In particular, let $\widehat{\mathbb{F}} = \widehat{C}^{\infty}(\mathbb{R}^{0},\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. The following result is essentially the same as [9, Theorem 7.3].

Theorem 4 (Todorov-Vernaeve)**.** *$\widehat{\mathbb{C}}$ is an algebraically closed field of the form $\widehat{\mathbb{R}} + \sqrt{-1}\,\widehat{\mathbb{R}}$.*

Corollary 5. *$\widehat{\mathbb{R}}$ is a real closed field and both $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{C}}$ are non-archimedean fields in the sense that they contain non-zero infinitesimals.*

Note. The constant ρ is called the *canonical infinitesimal* in $\widehat{\mathbb{R}}$. If we regard ρ as a positive infinitesimal in the nonstandard reals ${}^{*}\mathbb{R} = \mathbb{R}^{J}/\sim_{\mathcal{U}}$ then $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{R}}$ are isomorphic to Robinson's fields of ρ-asymptotic numbers ${}^{\rho}\mathbb{C}$ and ${}^{\rho}\mathbb{R}$, respectively. For more details, see Todorov-Vernaeve [9, Section 7].

It is clear that $\mathcal{M}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ and $\mathcal{N}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ are subalgebras of $\widehat{\mathcal{M}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ and $\widehat{\mathcal{N}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$, respectively. Hence we have

Proposition 6. *The natural map $\boldsymbol{\mathcal{G}}^{s}(U) \to \widehat{C}^{\infty}(U,\mathbb{F})$ induced by the inclusions $\mathcal{M}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)) \to \widehat{\mathcal{M}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ and $\mathcal{N}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)) \to \widehat{\mathcal{N}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U))$ is a homomorphism of smooth differential algebras over $\mathbb{F}$.*

Let $\mathcal{I}_U$ be the composite $\mathcal{D}'(U) \to \boldsymbol{\mathcal{G}}^{s}(U) \to \widehat{C}^{\infty}(U,\mathbb{F})$. Then $\mathcal{I}_U$ is a continuous linear map by Theorem 3 and Proposition 6. We show that $\mathcal{I}_U$ is an injection. For this, consider the pairing

$$\widehat{C}^{\infty}(U,\mathbb{F}) \times \mathcal{D}(U) \to \widehat{\mathbb{F}}$$

which assigns to $(F,\tau) = ([f_\epsilon],\tau)$ an asymptotic number $\langle F \,|\, \tau\rangle = [\langle f_\epsilon \,|\, \tau\rangle]$, where $\langle f_\epsilon \,|\, \tau\rangle = \int_U f_\epsilon(x)\tau(x)\,dx$ $(\epsilon \in J)$. The next proposition shows that the pairing above is compatible with the duality pairing $\mathcal{D}'(U)\times\mathcal{D}(U) \to \mathbb{F}$ under the linear map $\mathcal{I}_U : \mathcal{D}'(U) \to \widehat{C}^{\infty}(U,\mathbb{F})$.

Proposition 7. *$\langle \mathcal{I}_U(T) \,|\, \tau\rangle = \langle T \,|\, \tau\rangle$ holds for every $T \in \mathcal{D}'(U)$ and $\tau \in \mathcal{D}(U)$.*

In particular, $\mathcal{I}_U : \mathcal{D}'(U) \to \widehat{C}^{\infty}(U,\mathbb{F})$ is injective, and hence we have

Corollary 8. *There exists a (non-canonical) continuous linear embedding $\mathcal{D}'(U) \to \widehat{C}^{\infty}(U,\mathbb{F})$ preserving partial derivatives.*

Asymptotic maps between arbitrary diffeological spaces: Let X and Y be arbitrary diffeological spaces. We say that $(f_\epsilon) \in C^\infty(X,\mathbb{F})^J$ is a *moderate net* if its pullback $(f_\epsilon \circ \sigma) \in \boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)$ is moderate for every plot $\sigma\colon U \to X$. Let $\widehat{\mathcal{M}}(X,\mathbb{F})$ be the subalgebra of moderate nets in $C^\infty(X,\mathbb{F})^J$, and define a subspace $\widehat{\mathcal{M}}(X,Y)$ of $C^\infty(X,Y)^J$ by the formula

$$\widehat{\mathcal{M}}(X,Y) = \{(f_\epsilon) \mid (u_\epsilon \circ f_\epsilon) \in \widehat{\mathcal{M}}(X,\mathbb{F}) \text{ for } \forall (u_\epsilon) \in \widehat{\mathcal{M}}(Y,\mathbb{F})\}.$$

Note that the definition of $\widehat{\mathcal{M}}(X,Y)$ does not depend on $\mathbb{F}$ because we have $\widehat{\mathcal{M}}(X,\mathbb{C}) = \widehat{\mathcal{M}}(X,\mathbb{R}) + \sqrt{-1}\,\widehat{\mathcal{M}}(X,\mathbb{R})$. Also, if we put $Y = \mathbb{F}$ in $\widehat{\mathcal{M}}(X,Y)$ then it coincides with previously defined $\widehat{\mathcal{M}}(X,\mathbb{F})$ by the chain rule.

Now, let $\widehat{\mathcal{N}}(X,\mathbb{F})$ be the subalgebra of $\widehat{\mathcal{M}}(X,\mathbb{F})$ given by

$$\widehat{\mathcal{N}}(X,\mathbb{F}) = \{(f_\epsilon) \mid (f_\epsilon \circ \sigma_\epsilon) \in \widehat{\mathcal{N}}(\boldsymbol{\mathcal{E}}_{\mathbb{F}}(U)) \text{ for } \forall (\sigma_\epsilon) \in \widehat{\mathcal{M}}(U,X)\}$$

and define the space of *asymptotic maps* from X to Y as the quotient space

$$\widehat{C}^\infty(X,Y) = \widehat{\mathcal{M}}(X,Y)/\sim$$

where $(f_\epsilon) \sim (g_\epsilon)$ in $\widehat{\mathcal{M}}(X,Y)$ if and only if $u_\epsilon \circ g_\epsilon - u_\epsilon \circ f_\epsilon \in \widehat{\mathcal{N}}(X,\mathbb{F})$ holds for all $(u_\epsilon) \in \widehat{\mathcal{M}}(Y,\mathbb{F})$. In particular, if we take $\mathbb{F}$ as Y then the space

$$\widehat{C}^\infty(X,\mathbb{F}) = \widehat{\mathcal{M}}(X,\mathbb{F})/\widehat{\mathcal{N}}(X,\mathbb{F})$$

becomes a smooth algebra over $\widehat{\mathbb{F}}$. The next lemma shows that $\widehat{C}^\infty(X,\mathbb{F})$ coincides with previously defined $\widehat{C}^\infty(U,\mathbb{F})$ if X is a Euclidean open set U.

Proposition 9. *There exists a supercategory* $\widehat{\mathbf{Diff}}$ *of* **Diff** *which has the same objects as* **Diff** *and is enriched over* **Diff** *with* $\widehat{C}^\infty(X,Y)$ *as the space of morphisms from* X *to* Y. *There is an enriched functor* $\mathbf{Diff} \to \widehat{\mathbf{Diff}}$ *which embeds each hom-set* $C^\infty(X,Y)$ *as a subspace of* $\widehat{C}^\infty(X,Y)$.

The theorem above implies the naturality of asymptotic functions with respect to asymptotic maps. More precisely, we have the following.

Proposition 10. *To any asymptotic map* $f\colon X \to Y$ *there attached an* $\widehat{\mathbb{F}}$*-algebra map* $f^*\colon\ \widehat{C}^\infty(Y,\mathbb{F}) \to \widehat{C}^\infty(X,\mathbb{F})$ *enjoying the following properties:*

(1) $(g \circ f)^* = f^* \circ g^*$ *holds for any* $f \in \widehat{C}^\infty(X,Y)$ *and* $g \in \widehat{C}^\infty(Y,Z)$, *and the identity of* X *induces the identity of* $\widehat{C}^\infty(X,\mathbb{F})$.

(2) *If* $f\colon X \to Y$ *is smooth then* $f^*\colon\ \widehat{C}^\infty(Y,\mathbb{F}) \to \widehat{C}^\infty(X,\mathbb{F})$ *restricts to an* $\mathbb{F}$*-algebra map* $C^\infty(Y,\mathbb{F}) \to C^\infty(X,\mathbb{F})$.

Homotopy theoretic applications of asymptotic maps

We can utilize asymptotic maps to make homotopy theory of diffeological spaces much easier and adaptable. Despite of several efforts (cf. [1], [4], [6]), it is cumbersome to fully develop homotopy theory by using only smooth maps. For example, the fact that I^n (n-cube), Δ^n (n-simplex) and B^n (n-ball) are not mutually diffeomorphic causes confusion regarding which should be used as building blocks of "smooth cell complex." Furthermore, whichever cell-type you adopt, the resulting cell complex does not enjoy *homotopy extension property* in general. But we can overcome these difficulties by introducing asymptotic maps. As we shall below, I^n, Δ^n and B^n are asymptotically isomorphic to each other, and (relative) cell complexes always enjoy homotopy extension property (and covering homotopy property, too).

Denote by I the unit interval $[0,1]$ equipped with standard diffeology.

Definition 11. A pair of diffeological spaces (X, A) is called a *smooth relative cell complex* if there is an ordinal δ and a δ-sequence $Z\colon \delta \to \mathbf{Diff}$ such that the composition $Z_0 \to \operatorname{colim} Z$ coincides with the inclusion $i\colon A \to X$ and for each successor ordinal $\beta < \delta$, there is a smooth map $\phi_\beta\colon \partial I^n \to Z_{\beta-1}$, called an *attaching map*, such that Z_β is diffeomorphic to the adjunction space $Z_{\beta-1} \cup_{\phi_\beta} I^n$. In particular, if $A = \emptyset$ then X is called a *smooth cell complex*.

Proposition 12. *Let p be a positive valued continuous function on $S^{n-1} = \partial B^n$. Then the subspace $D = \{tx \in \mathbb{R}^n \mid x \in S^{n-1},\ 0 \le t \le p(x)\}$ of $\mathbb{R}^n$ is asymptotically isomorphic to B^n.*

Evidently, this implies the following.

Corollary 13. *The cells I^n, Δ^n and B^n are asymptotically isomorphic to each other.*

The proposition above can be proved as in the following way By using a smooth partition of unity, we can embed the space of distributions on S^{n-1} into $\widehat{C}^\infty(S^{n-1}, \mathbb{R})$. In particular, there is a net of smooth functions (p_ϵ) which represents p as an asymptotic function on S^{n-1}. We may assume that $p_\epsilon(x) > 0$ holds for all $x \in S^{n-1}$ because $p(x)$ is positive and S^{n-1} is compact. For each $\epsilon \in J$, let $M_\epsilon = \max_x\{p_\epsilon(x)/p(x)\}$, $m_\epsilon = \min_x\{p_\epsilon(x)/p(x)\}$, and let

$$\bar{p}_\epsilon(x) = p_\epsilon(x)/M_\epsilon \quad (x \in S^{n-1}), \quad \bar{q}_\epsilon(y) = m_\epsilon/p_\epsilon(y/\|y\|) \quad (y \in \partial D)$$

Then we have

$$\bar{p}_\epsilon(x) \le p(x) \quad (x \in S^{n-1}), \quad \bar{q}_\epsilon(y) \le 1/p(y/\|y\|) \quad (y \in \partial D)$$

and hence smooth maps $P_\epsilon \in C^\infty(B^n, D)$, $Q_\epsilon \in C^\infty(D, B^n)$ defined by

$$P_\epsilon(tx) = t\bar{p}_\epsilon(x)x, \quad Q_\epsilon(ty) = t\bar{q}_\epsilon(y)y \quad (x \in S^{n-1},\ y \in \partial D,\ 0 \le t \le 1)$$

It is now easy to check that the nets (P_ϵ) and (Q_ϵ) represent respectively an asymptotic map $P\colon B^n \to D$ and its inverse $Q\colon D \to B^n$.

The next theorem says that every smooth relative cell complex enjoys homotopy extension property in the extended category $\widehat{\mathbf{Diff}}$.

Theorem 14 (Theorem 5.4 of [8]). *Let (X, A) be a smooth relative cell complex, and $f: X \to Y$ be an asymptotic map. Suppose there is an asymptotic homotopy $h: A \times I \to Y$ satisfying $h_0 = f|A$. Then there exists an asymptotic homotopy $H: X \times I \to Y$ satisfying $H_0 = f$ and $H|A \times I = h$.*

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