COMPACT VECTORIAL TOEPLITZ OPERATORS ON THE SEGAL-BARGMANN SPACE

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ABSTRACT. The compactness of a Toeplitz operator acting on the Segal-Bargmann space of vector-valued functions is discussed. A sufficient condition written in terms of an associated operator-valued kernel is presented.

1. INTRODUCTION

Given be a positive measure ν on $\mathscr{B}(\mathbb{C}^n)$, the σ-algebra of Borel sets in $\mathbb{C}^n$, we denote by $L^2(\nu)$ the space of all square-summable (with respect to ν) complex-valued Borel functions on $\mathbb{C}^n$ and by $\nu \otimes \nu$ we denote the product measure on $\mathscr{B}(\mathbb{C}^n \times \mathbb{C}^n) = \mathscr{B}(\mathbb{C}^{2n})$.

Suppose

$$\mu_{\mathbf{G}}(\sigma) = \frac{1}{\pi^n} \int_\sigma \exp\big(-|z|^2\big)\mathrm{d}V(z), \quad \sigma \in \mathscr{B}(\mathbb{C}^n),$$

is the n-dimensional Gaussian measure on $\mathbb{C}^n$ (here V denotes the Lebesgue measure on $\mathscr{B}(\mathbb{C}^n)$). A closed subspace of $L^2(\mu_{\mathbf{G}})$ consisting of all analytic functions belonging to $L^2(\mu_{\mathbf{G}})$ is called the *Segal-Bargmann space* and is denoted by $\mathcal{B}$. $\mathcal{B}$ turns out to be reproducing kernel Hilbert space (RKHS) with the kernel $k(z,w) = \exp\langle z, w\rangle$, $z, w \in \mathbb{C}^n$.

Let $\mathcal{H}$ be a separable Hilbert space. Then the space of all analytic functions $F\colon \mathbb{C}^n \to \mathcal{H}$ such that $\int_{\mathbb{C}^n} \|F(z)\|^2 \mathrm{d}\mu_{\mathbf{G}}(z) < \infty$ can be identified with he Hilbert tensor product of $\mathcal{B}$ and $\mathcal{H}$, denoted as usual by $\mathcal{B} \otimes \mathcal{H}$. We call it a *(vectorial) Segal-Bargmann* space. For $f \in \mathcal{B}$ and $h \in \mathcal{H}$, $f \otimes h$ stands for the function defined as $(f \otimes h)(z) = f(z)h$, $z \in \mathbb{C}^n$. The Segal-Bargmann space has the following reproducing property:

$$\langle F, k_z \otimes h\rangle = \langle F(z), h\rangle, \quad z \in \mathbb{C}^n, h \in \mathcal{H}, F \in \mathcal{B} \otimes \mathcal{H}. \tag{1.1}$$

Now, suppose $\Phi\colon \mathbb{C}^n \to \mathbf{B}(\mathcal{H})$ is a Borel function. We define the *(vectorial) Toeplitz operator*

$$\mathsf{T}_\Phi \colon \mathcal{B} \otimes \mathcal{H} \supseteq \mathscr{D}(\mathsf{T}_\Phi) \to \mathcal{B} \otimes \mathcal{H}$$

by the formula

$$\begin{aligned} \mathscr{D}(\mathsf{T}_\Phi) &= \big\{F \in \mathcal{B} \otimes \mathcal{H} \colon \Phi F \in L^2(\mu_{\mathbf{G}}) \otimes \mathcal{H}\big\}, \\ \mathsf{T}_\Phi F &= \mathsf{P}(\Phi F), \quad F \in \mathscr{D}(\mathsf{T}_\Phi), \end{aligned}$$

where P denotes the orthogonal projection from $L^2(\mu_{\mathbf{G}}) \otimes \mathcal{H}$ onto $\mathcal{B} \otimes \mathcal{H}$. Clearly, due to the reproducing property (1.1) we have

(1.2)
$$\langle (\mathsf{T}_\Phi F)(z), h\rangle = \int_{\mathbb{C}^n} \langle \Phi(w)F(w), h\rangle \overline{k_z(w)} \mathrm{d}\mu_{\mathbf{G}}(w), \quad z \in \mathbb{C}^n, F \in \mathscr{D}(\mathsf{T}_\Phi), h \in \mathcal{H},$$

where $k_\lambda(z) = \exp\langle z, \lambda\rangle$, $z \in \mathbb{C}^n$. Throughout what follows the linear span of $\{k_\lambda \otimes g\colon \lambda \in \mathbb{C}^n, g \in \mathcal{H}\}$ is denoted by $\mathscr{K} \otimes \mathcal{H}$.

Toeplitz operators on Segal-Bargmann spaces appear in the quantization of classical mechanics and are related to pseudodifferential operators (see [1, 12, 13, 14, 18, 19, 20]). They have been studied since the work of Berezin (see [5, 6]) both in the classical context (see, e.g., [2, 3, 10, 11, 15, 16, 17]) and also in the more general setting (see, e.g., [8, 9]).

2. Compactness

Among the many questions on Toeplitz operators (not necessarily acting on Segal-Bargmann space) that have been addressed in the literature one was the question of the compactness of these operators. A classical and well-known results due to Stroethoff (see [21, Theorem 5]) states

Theorem 2.1. *Let $\phi \in L^\infty(\mathbb{C}^n)$. Then the Toeplitz operator T_ϕ acting on the Segal-Bargmann space $\mathcal{B}$ is compact if and only if*

$$\lim_{|\lambda|\to\infty} \|\mathrm{P}(\phi \circ \tau_\lambda)\| = 0 \tag{2.1}$$

where τ_λ is the translation on $\mathbb{C}^n$ by λ and P is the orthogonal projection from $L^2(\mu_{\mathbf{G}})$ onto $\mathcal{B}$

The above is related to a characterization of compact multiplication operators in terms of Berezin symbols due to Berger and Coburn (see [7, Theorem C]). Similar results to Theorem 2.1 can be proved in other classical RKHS's (see [22] for more on this). Hence the question arises as to whether one can characterize the compactness of the Toeplitz operator acting in a vectorial Segal-Bargmann space via the oscillation at infinity type condition. An attempt to answer this has been tackled recently in [4].

Suppose $\mathscr{K} \otimes \mathcal{H} \subseteq \mathscr{D}(\mathsf{T}_\Phi)$ and $z, w \in \mathbb{C}^n$. Let $\Xi_\Phi(z,w) \in \mathbf{B}(\mathcal{H})$ be given by

$$\Xi_\Phi(z,w)g = \int_{\mathbb{C}^n} \Phi_z(u) g\, \overline{k_w(u)}\, \mathrm{d}\mu_{\mathbf{G}}(u) = \mathsf{P}(\Phi_z \otimes g)(w), \quad g \in \mathcal{H}.$$

The operator kernel above is used as a vectorial counterpart of $\mathrm{P}(\phi \circ \tau_z)$ in possible generalizations of the condition of oscillation at infinity (2.1). The first comes from [4, Proposition 3.7]

$$\lim_{|z|\to\infty} \int_{\mathbb{C}^n} \|\Xi_\Phi(z,w)g\|^2 \mathrm{d}\mu_{\mathbf{G}}(w) = 0, \quad g \in \mathcal{H}. \tag{2.2}$$

Another possible generalization of (2.1) comes from [4, Theorem 3.6]

$$\lim_{|z|\to\infty} \int_{\mathbb{C}^n} \|\Xi_\Phi(z,w)\|_{\mathrm{HS}}^2 \mathrm{d}\mu_{\mathbf{G}}(w) = 0. \tag{2.3}$$

Here, and alter on, $\|A\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm of an operator A. The conditions differ essentially and what is more none of them characterizes the compactness of vectorial Toeplitz operators. Nonetheless we have the following.

Proposition 2.2 ([4, Proposition 3.7]). *Let $\mathsf{T}_\Phi \in \mathbf{B}(\mathcal{B} \otimes \mathcal{H})$ be compact. Then* (2.2) *is satisfied.*

On the other hand we also have

Theorem 2.3 ([4, Theorem 3.6]). *Let $\Phi \in L^\infty\big(\mu_{\mathbf{G}}; \mathbf{HS}(\mathcal{H})\big)$. Suppose that* (2.3) *holds. Then T_Φ is compact.*

It is important to note that there is a non-compact vectorial Toeplitz operator which satisfies (2.2).

Example 2.4 ([4, Example 3.9]). Let $\phi\colon \mathbb{C}^n \to \mathbb{C}$ be a non-zero function such that the Toeplitz operator T_ϕ on $\mathcal{B}$ is compact. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\Phi\colon \mathbb{C}^n \to \mathbf{B}(\mathcal{H})$ be given by $\Phi(z) = \phi(z)\, I$, where I is the identity operator on $\mathcal{H}$. Then by [21, Theorem 5] we have

$$\lim_{|z|\to\infty} \int_{\mathbb{C}^n} |\mathrm{P}(\phi \circ \tau_z)(w)|^2 \mathrm{d}\mu_{\mathbf{G}}(w) = 0, \tag{2.4}$$

and so (2.2) holds. On the other hand, T_Φ is not compact as the image of $\{\chi_{\mathbb{C}^n} \otimes e_i \colon i \in \mathbb{N}\}$ via T_Φ, where $\{e_i \colon i \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$, does not contain a convergent subsequence.

Also worth noting that condition (2.3) may characterize the compactness of T_Φ only in the case when Φ is $\mathbf{HS}(\mathcal{H})$-valued which is not necessary.

Example 2.5 ([4, Example 3.10]). Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let A be a compact operator A which is not Hilbert-Schmidt. Put $\Phi(z) = \phi(z)A$, $z \in \mathbb{C}^n$, where $\phi\colon \mathbb{C}^n \to \mathbb{C}$ is a non-zero function such that T_ϕ on $\mathcal{B}$ is compact. Then T_Φ is compact, however (2.3) does not hold.

All the above means that characterizing the compactness of Toeplitz operators on vectorial Segal-Bargmann space remains to be an open problem and calls for further attention.

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