A new family of weighted operator means including the weighted Heron, logarithmic and Heinz means

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Abstract

As generalizations of the weighted arithmetic, geometric and harmonic means of two positive numbers or operators, the weighted power and Heron means are well known. On the weighted logarithmic mean, some researchers gave the definitions in their own way. Among them, we focus on the definition by Pal, Singh, Moslehian and Aujla based on the Hermite-Hadamard inequality for convex functions.

In this report, firstly we propose the notion of a transpose symmetric path of weighted $\mathfrak{M}$-means for a symmetric operator mean $\mathfrak{M}$. Next, we give an example of the transpose symmetric path of weighted $\mathfrak{M}$-means including the weighted logarithmic mean by Pal et al., which newly produces the weighted Heinz mean. From this argument, we get some relations among the weighted Heron, logarithmic and Heinz means.

1 Introduction

This report is based on [8]. The arithmetic, geometric and harmonic means are defined by $\frac{a+b}{2}$, $\sqrt{ab}$ and $\frac{2ab}{a+b}$ for two positive real numbers a and b, respectively. It is also well known that these means have their weighted versions as follows: For $t \in [0,1]$,

$$
\begin{aligned}
A_t(a,b) &= (1-t)a + tb \quad \text{(arithmetic mean)},\\
G_t(a,b) &= a^{1-t}b^t \quad \text{(geometric mean)},\\
H_t(a,b) &= \{(1-t)a^{-1} + tb^{-1}\}^{-1} \quad \text{(harmonic mean)}.
\end{aligned}
$$

If the weight parameter t is equal to $\frac{1}{2}$, then the weighted means coincide with the original (non-weighted) ones, and then we abbreviate the weight t as $A(a,b) = A_{\frac{1}{2}}(a,b)$. As one-parameter generalizations including the weighted arithmetic, geometric and harmonic means, power and Heron means are well known as follows: For $t \in [0,1]$ and $q \in \mathbb{R}$,

$$
P_{t,[q]}(a,b) = \begin{cases} \{(1-t)a^q + tb^q\}^{\frac{1}{q}} & \text{if } q \neq 0,\\ a^{1-t}b^t & \text{if } q = 0, \end{cases} \quad \text{(power mean)},
$$

$$
K_{t,[q]}(a,b) = (1-q)a^{1-t}b^t + q\{(1-t)a + tb\} \quad \text{(Heron mean)}.
$$

In this paper, we denote the power mean by $P_{t,[q]}(a,b)$ to distinguish the parameter q determining the mean from the weight parameter t, and also we use the notation for the non-weighted power mean that $P_{[q]}(a,b) = P_{\frac{1}{2},[q]}(a,b)$. We remark that the other means with parameters also obey these rules. These means have the following relations and properties.

- $A_t(a,b) = P_{t,[1]}(a,b) = K_{t,[1]}(a,b)$, $G_t(a,b) = P_{t,[0]}(a,b) = K_{t,[0]}(a,b)$ and $H_t(a,b) = P_{t,[-1]}(a,b)$.
- $H_t(a,b) \le G_t(a,b) \le A_t(a,b)$ (A-G-H mean inequality).
- $P_{t,[q]}(a,b)$ and $K_{t,[q]}(a,b)$ are monotone increasing on $q \in \mathbb{R}$.

Moreover, we know many kinds of non-weighted means besides the power and Heron means, for example power difference and Heinz means as follows: For $q \in \mathbb{R}$,

$$LM(a,b) = \frac{a-b}{\log a - \log b} \quad \text{(logarithmic mean)},$$

$$J_q(a,b) = \begin{cases} \dfrac{q}{q+1}\dfrac{a^{q+1}-b^{q+1}}{a^q - b^q} & \text{if } q \neq 0, -1, \\ \dfrac{a-b}{\log a - \log b} & \text{if } q = 0, \\ \dfrac{ab(\log a - \log b)}{a-b} & \text{if } q = -1, \end{cases} \qquad \text{(power difference mean)},$$

$$HZ_{[q]}(a,b) = \frac{a^q b^{1-q} + a^{1-q} b^q}{2} \quad \text{(Heinz mean)}.$$

These means have the following relations and properties. Related arguments are in [6, 7, 10, 17, 19, 20] and so on.

- $A(a,b) = J_{[1]}(a,b) = HZ_{[0]}(a,b) = HZ_{[1]}(a,b)$, $LM(a,b) = J_{[0]}(a,b)$, $G(a,b) = J_{[\frac{-1}{2}]}(a,b) = HZ_{[\frac{1}{2}]}(a,b)$ and $H(a,b) = J_{[-2]}(a,b)$.
- $H(a,b) \le G(a,b) \le LM(a,b) \le A(a,b)$.
- $J_{[q]}(a,b)$ is monotone increasing on $q \in \mathbb{R}$.
- $HZ_{[q]}(a,b)$ is increasing for $q \ge \frac{1}{2}$ and decreasing for $q \le \frac{1}{2}$.

It seems that there are not familiar weighted means except the power and Heron means. Some researchers gave the definitions of the weighted logarithmic mean in their own way in [11, 12, 15, 16]. Among them, we focus on the definition by Pal, Singh, Moslehian and Aujla [12] based on the Hermite-Hadamard inequality for convex functions as follows: For $t \in [0,1]$,

$$LM_t(a,b) = \frac{1}{\log a - \log b}\left\{\frac{1-t}{t}a^{1-t}(a^t - b^t) + \frac{t}{1-t}b^t(a^{1-t} - b^{1-t})\right\}.$$

They showed that the inequalities

$$H_t(a,b) \le G_t(a,b) \le LM_t(a,b) \le K_{t,[\frac{1}{2}]}(a,b) \le A_t(a,b) \tag{1.1}$$

always hold, which is a refinement of the above A-G-H mean inequality.

We can extend above discussion for bounded linear operators on a Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T \ge 0$) if $(Tx, x) \ge 0$ for all $x \in \mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. We denote the set of positive operators by $B^+(\mathcal{H})$. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if $A \le B$ implies $f(A) \le f(B)$ for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \le B$ means $B - A \ge 0$.

Kubo and Ando [9] constructed the general theory of operator means. We state it in the next section. As concrete examples, the following weighted operator means are known for two strictly positive operators A and B. For $t \in [0,1]$,

$$\mathfrak{A}_t(A,B) = (1-t)A + tB \quad \text{(arithmetic mean)},$$

$$\mathfrak{G}_t(A,B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} \quad \text{(geometric mean)},$$

$$\mathfrak{H}_t(A,B) = \{(1-t)A^{-1} + tB^{-1}\}^{-1} \quad \text{(harmonic mean)},$$

$$\mathfrak{P}_{t,[q]}(A,B) = \begin{cases} A^{\frac{1}{2}}\{(1-t)I + t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q\}^{\frac{1}{q}} A^{\frac{1}{2}} & \text{if } 0 < |q| \le 1, \\ A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} & \text{if } q = 0, \end{cases} \quad \text{(power mean)}.$$

Notations $A\nabla_t B$, $A\sharp_t B$, $A!_t B$ and $A\sharp_{t,q} B$ are often used instead of $\mathfrak{A}_t(A,B)$, $\mathfrak{G}_t(A,B)$, $\mathfrak{H}_t(A,B)$ and $\mathfrak{P}_{t,[q]}(A,B)$, respectively. We remark that the operator versions of other numerical means can be introduced by Kubo-Ando theory.

In this report, firstly we discuss the definition of the weighted mean and we propose the notion of a transpose symmetric path of weighted $\mathfrak{M}$-means for a symmetric operator mean $\mathfrak{M}$. Next, we give an example of the transpose symmetric path of weighted $\mathfrak{M}$-means including the weighted logarithmic mean by Pal et al., which newly produces the weighted Heinz mean. Moreover this argument leads refinements of (1.1).

2 A transpose symmetric path of weighted $\mathfrak{M}$-means

Kubo and Ando [9] defined the operator mean axiomatically as follows: A binary operation $\mathfrak{M} : \mathcal{B}^+(\mathcal{H}) \times \mathcal{B}^+(\mathcal{H}) \to \mathcal{B}^+(\mathcal{H})$ is called an operator mean if the following conditions are satisfied:

(i) $A \le C$ and $B \le D$ imply $\mathfrak{M}(A,B) \le \mathfrak{M}(C,D)$ (monotonicity),

(ii) $T^*\mathfrak{M}(A,B)T \le \mathfrak{M}(T^*AT, T^*BT)$ for every operator T (transformer inequality),

(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $\mathfrak{M}(A_n, B_n) \downarrow \mathfrak{M}(A, B)$ (upper semicontinuity),

(iv) $\mathfrak{M}(I, I) = I$ (normalization),

where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ in the strong operator topology as $n \to \infty$. We remark that (ii) leads $T^*\mathfrak{M}(A, B)T = \mathfrak{M}(T^*AT, T^*BT)$ (transformer equality) if T is invertible, and the transformer equality ensures that $\mathfrak{M}(\alpha A, \alpha B) = \alpha\mathfrak{M}(A, B)$ (homogeneity) holds for all $\alpha > 0$. The numerical mean M is defined similarly to the operator mean $\mathfrak{M}$ by monotonicity, homogeneity, continuity and normalization.

It is also obtained in [9] that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$ and an operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$ via $f(x)I = \mathfrak{M}(I, xI)$ as follows:

$$\mathfrak{M}(A, B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{2.1}$$

if $A > 0$ and $B \geq 0$. We remark that f is called the representing function of $\mathfrak{M}$, and also it is permitted to consider binary operations given by (2.1) even if f is a general real-valued function. The representing functions of $\mathfrak{A}_t(A, B)$, $\mathfrak{G}_t(A, B)$ are $A_t(1, x)$, $G_t(1, x)$ (denoted by $A_t(x)$, $G_t(x)$) and so on. Similarly, we can introduce the operator mean $\mathfrak{M}$ corresponding to the representing function $M(1, x)$ (denoted by $M(x)$) by the numerical mean M if $M(1, x)$ is operator monotone. Refer to [14] for more details on operator means. Here we also remark that the power difference mean $\mathfrak{J}_{[q]}(A, B)$ is an operator mean if $-2 \leq q \leq 1$ (see [3, 4, 5]).

For two operator means $\mathfrak{M}$ and $\mathfrak{N}$, $\mathfrak{M} \leq \mathfrak{N}$ (resp. $\mathfrak{M} = \mathfrak{N}$) means that $\mathfrak{M}(A, B) \leq \mathfrak{N}(A, B)$ (resp. $\mathfrak{M}(A, B) = \mathfrak{N}(A, B)$) for all $A, B > 0$. An operator mean $\mathfrak{M}$ is said to be symmetric if $\mathfrak{M}(A, B) = \mathfrak{M}(B, A)$ (symmetry) holds. The weighted means in Section 1 are not symmetric except the case $t = \frac{1}{2}$, but they have the property that $\mathfrak{M}_t(A, B) = \mathfrak{M}_{1-t}(B, A)$ (transpose symmetry) holds for all $t \in [0, 1]$ instead of symmetry. For an operator mean $\mathfrak{M}$ and its representing function f, the operator mean whose representing function is $xf(x^{-1})$ is called transpose of $\mathfrak{M}$, and we denote it by $\mathfrak{M}^\circ$. We easily obtain that $\mathfrak{M}^\circ(A, B) = \mathfrak{M}(B, A)$ for $A, B > 0$, and also an operator mean $\mathfrak{M}$ is symmetric if and only if $\mathfrak{M} = \mathfrak{M}^\circ$ if and only if $f(x) = xf(x^{-1})$ for all $x > 0$.

From this argument, we can discuss numerical means and operator means simultaneously via the representing function, so we write definitions and theorems in terms of operator means from now on. We remark that we have to pay attention to operator monotonicity when we treat operator means.

Next, we discuss the definition of weighted means. We can consider plural weighted means from one symmetric mean. In fact, the weighted logarithmic mean is defined by several ways in [11, 12, 15, 16]. Moreover, in [1, 2, 13, 18], they discussed the algorithms to make weighted operator means from a given operator mean. Here, we introduce the notion of a transpose symmetric path of weighted $\mathfrak{M}$-means.

Definition 1. *Let $\mathfrak{M}$ be a symmetric operator mean and $A, B > 0$. If the following conditions hold, then $\mathfrak{M}_t$ is said to be a weighted $\mathfrak{M}$-mean, and a one-parameter family $\{\mathfrak{M}_t\}_{t\in[0,1]}$ is said to be a transpose symmetric path of weighted $\mathfrak{M}$-means.*

(i) *$\mathfrak{M}_t$ is an operator mean for all fixed $t \in [0, 1]$.*

(ii) *$\mathfrak{M}_0(A, B) = A$, $\mathfrak{M}_{\frac{1}{2}}(A, B) = \mathfrak{M}(A, B)$ and $\mathfrak{M}_1(A, B) = B$.*

(iii) *$\mathfrak{M}_t(A, B) = \mathfrak{M}_{1-t}(B, A)$ for all $t \in [0, 1]$ (transpose symmetry).*

(iv) *$\mathfrak{M}_t$ is t-weighted for all fixed $t \in [0, 1]$, that is, $f_t'(1) = t$ for the representing function f_t of $\mathfrak{M}_t$.*

In Definition 1, we consider a path of t-weighted means $\mathfrak{M}_t(A, B)$ for $t \in [0, 1]$ from A (0-weighted mean) to B (1-weighted mean) such that $\mathfrak{M}_{\frac{1}{2}}(A, B) = \mathfrak{M}(A, B)$. The families of weighted means introduced in Section 1, $\{\mathfrak{A}_t\}_{t\in[0,1]}$, $\{\mathfrak{P}_{t,[q]}\}_{t\in[0,1]}$ and so on, are all transpose symmetric paths of weighted means. We remark that the notion of t-weighted is introduced in [18] (see also [13]), and also Definition 1 is a refinement of the notion of the weighted M-mean for numerical case by Raïssouli and Sándor [16].

Similarly to symmetry, (iii) in Definition 1 holds if and only if $\mathfrak{M}_t = \mathfrak{M}_{1-t}^{\circ}$ if and only if $f_t(x) = xf_{1-t}(x^{-1})$ for all $x > 0$, where $f_t(x)$ is the representing function of $\mathfrak{M}_t$. We can easily verify that every symmetric mean is $\frac{1}{2}$-weighted, but the converse does not hold in general. We remark that the weight parameter t in $\{\mathfrak{M}_t\}_{t\in[0,1]}$ does not always equal $f_t'(1)$ even if (i)–(iii) hold in Definition 1.

3 Results

In this section, we construct a transpose symmetric path of weighted $\mathfrak{M}$-means including some fundamental weighted means stated in Section 1, and also we get some relations among the weighted Heron, logarithmic and Heinz means.

Let $\{\mathfrak{M}_t\}_{t\in[0,1]}$ be a transpose symmetric path of weighted $\mathfrak{M}$-means and

$$\mathcal{R} = \left\{\{f_t\}_{t\in[0,1]} : f_t \text{ is the representing function of } \mathfrak{M}_t \in \{\mathfrak{M}_t\}_{t\in[0,1]}\right\}.$$

We denote $\{f_t\}_{t\in[0,1]}$ by $\{f_t\}$ briefly. Now we consider the following function.

Definition 2. *Let $\{\varphi_s\} \in \mathcal{R}$. Then we define a function $m_t[\varphi_s] : [0, \infty) \to [0, \infty)$ as*

$$m_t[\varphi_s](x) = (1-t)\varphi_{1-s}(x^t) + tx^t\varphi_s(x^{1-t}) \quad \text{for } t, s \in [0, 1].$$

In particular, if φ is the representing function of a symmetric mean, then we define

$$m_t[\varphi](x) = (1-t)\varphi(x^t) + tx^t\varphi(x^{1-t}) \quad \text{for } t \in [0, 1].$$

Then the function $m_t[\varphi_s]$ makes a transpose symmetric path of weighted $\mathfrak{M}[\varphi_s]$-means by the following Theorem 3.1. We recognize that t and s in $m_t[\varphi_s]$ express the weight parameter and the parameter determining the path, respectively. We remark that we do not have to consider a one-parameter family $\{\varphi_s\} \in \mathcal{R}$ if we choose $s = \frac{1}{2}$, the representing function φ of a symmetric mean, in Definition 2.

Theorem 3.1. *Let $\{\varphi_s\} \in \mathcal{R}$ and $m_t[\varphi_s]$ be as in Definition 2. Let $\mathfrak{M}_t[\varphi_s]$ be the binary operation whose representing function is $m_t[\varphi_s]$, and also $\mathfrak{M}[\varphi_s] = \mathfrak{M}_{\frac{1}{2}}[\varphi_s]$. Then the family $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ is a transpose symmetric path of weighted $\mathfrak{M}[\varphi_s]$-means.*

To prove Theorem 3.1, we shall verify that $\mathfrak{M}_t[\varphi_s]$ has four properties in Definition 1 by using its representing function $m_t[\varphi_s]$. We omit the detail.

For $A, B > 0$ and binary operations $\mathfrak{M}, \mathfrak{N}$ given by (2.1), we note that $(\mathfrak{M} + \mathfrak{N})(A, B) \equiv \mathfrak{M}(A, B) + \mathfrak{N}(A, B)$ and $(\alpha\mathfrak{M})(A, B) \equiv \alpha\mathfrak{M}(A, B)$ $(\alpha > 0)$. We have the following fundamental properties of the weighted operator mean $\mathfrak{M}_t[\varphi_s]$ in Theorem 3.1 by considering its representing function.

Theorem 3.2. *Let $\{\varphi_s\}, \{\psi_s\} \in \mathcal{R}$. If $\varphi_s \leq \psi_s$ for each $s \in [0, 1]$, then $\mathfrak{M}_t[\varphi_s] \leq \mathfrak{M}_t[\psi_s]$ for all $t \in [0, 1]$.*

Theorem 3.3. *Let $\{\psi_s^{(j)}\} \in \mathcal{R}$ and $\alpha_j > 0$ for $j = 1, \ldots, n$ with $\alpha_1 + \cdots + \alpha_n = 1$. If $\varphi_s = \alpha_1\psi_s^{(1)} + \cdots + \alpha_n\psi_s^{(n)}$ for each $s \in [0, 1]$, then*

$$\mathfrak{M}_t[\varphi_s] = \alpha_1\mathfrak{M}_t[\psi_s^{(1)}] + \cdots + \alpha_n\mathfrak{M}_t[\psi_s^{(n)}]$$

holds for $t \in [0, 1]$, and also $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ is a transpose symmetric path of weighted $\mathfrak{M}[\varphi_s]$-means.

A transpose symmetric path of weighted $\mathfrak{M}[\varphi_s]$-means given in Theorem 3.1 includes some weighted operator means, for example, the weighted Heron mean, the weighted logarithmic mean and the weighted Heinz mean.

(i) Weighted logarithmic mean: Firstly we discuss $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ for the representing function φ of the logarithmic mean. Let $\varphi(x) = LM(x)$. Then the representing function of $\mathfrak{M}_t[LM]$ is

$$m_t[LM](x) = \frac{1}{\log x}\left\{\frac{1-t}{t}(x^t - 1) + \frac{t}{1-t}x^t(x^{1-t} - 1)\right\} = LM_t(x),$$

in particular

$$m_{\frac{1}{2}}[LM](x) = \frac{x-1}{\log x} = LM(x).$$

Therefore $m_t[LM]$ makes a transpose symmetric path of weighted $\mathfrak{LM}$-means. This weighted $\mathfrak{LM}$-mean coincides with the weighted logarithmic mean $\mathfrak{LM}_t$ introduced by Pal, Singh, Moslehian and Aujla [12], that is, $\mathfrak{M}_t[LM] = \mathfrak{LM}_t$.

Moreover, we consider $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ for the representing function of the weighted logarithmic mean, that is, $\varphi_s(x) = LM_s(x)$. The representing function of $\mathfrak{M}_t[LM_s]$ is

$$\begin{aligned} m_t[LM_s](x) = &\frac{1-t}{t\log x}\left\{\frac{s}{1-s}(x^{t(1-s)}-1)+\frac{1-s}{s}(x^t - x^{t(1-s)})\right\} \\ &+ \frac{tx^t}{(1-t)\log x}\left\{\frac{1-s}{s}(x^{(1-t)s}-1)+\frac{s}{1-s}(x^{1-t}-x^{(1-t)s})\right\}, \end{aligned}$$

in particular

$$\begin{aligned} m_{\frac{1}{2}}[LM_s](x) = &\frac{1}{\log x}\left\{\frac{s}{1-s}(x^{\frac{1-s}{2}}-1)+\frac{1-s}{s}(x^{\frac{1}{2}}-x^{\frac{1-s}{2}})\right\} \\ &+ \frac{x^{\frac{1}{2}}}{\log x}\left\{\frac{1-s}{s}(x^{\frac{s}{2}}-1)+\frac{s}{1-s}(x^{\frac{1}{2}}-x^{\frac{s}{2}})\right\}. \end{aligned}$$

(ii) Weighted Heron mean: We discuss $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ for the representing function φ_s of the weighted arithmetic mean, that is, $\varphi_s(x) = A_s(x)$. The representing function of $\mathfrak{M}_t[A_s]$ is

$$m_t[A_s](x) = (1-s)x^t + s\{(1-t)+tx\} = K_{t,[s]}(x),$$

in particular

$$m_{\frac{1}{2}}[A_s](x) = (1-s)x^{\frac{1}{2}} + s\frac{x+1}{2} = K_{[s]}(x).$$

Therefore $m_t[A_s]$ makes a transpose symmetric path of weighted $\mathfrak{K}_{[s]}$-means. This weighted $\mathfrak{K}_{[s]}$-mean coincides with the weighted Heron mean $\mathfrak{K}_{t,[s]}$, that is, $\mathfrak{M}_t[A_s] = \mathfrak{K}_{t,[s]}$.

(iii) Weighted Heinz mean: Here, we replace the parameter q of $HZ_{[q]}(a,b)$ as $\overline{HZ}_{[s]}(a,b) = HZ_{[\frac{1-s}{2}]}(a,b)$. Then $\overline{HZ}_{[0]}(a,b) = G(a,b)$ and $\overline{HZ}_{[1]}(a,b) = A(a,b)$ hold, and also $\overline{HZ}_{[s]}(a,b)$ is increasing for $s \geq 0$. We discuss $\{\mathfrak{M}_t[\varphi_s]\}_{t\in[0,1]}$ for the representing function φ_s of the weighted geometric mean, that is, $\varphi_s(x) = G_s(x)$. The representing function of $\mathfrak{M}_t[G_s]$ is

$$m_t[G_s](x) = (1-t)x^{(1-s)t} + tx^{t+s(1-t)} = x^{(1-s)t}\{(1-t)+tx^s\},$$

in particular

$$m_{\frac{1}{2}}[G_s](x) = \frac{x^{\frac{1-s}{2}} + x^{\frac{1+s}{2}}}{2} = \overline{HZ}_{[s]}(x).$$

Therefore $m_t[G_s]$ makes a transpose symmetric path of weighted $\overline{\mathfrak{HZ}}_{[s]}$-means, so that we can newly define the weighted Heinz mean $\overline{\mathfrak{HZ}}_{t,[s]}$ whose representing function is $\overline{HZ}_{t,[s]}(x)$ as

$$\overline{\mathfrak{HZ}}_{t,[s]} = \mathfrak{M}_t[G_s], \quad \overline{HZ}_{t,[s]}(x) = m_t[G_s](x) = (1-t)x^{t-st} + tx^{t+s(1-t)}.$$

By Theorem 3.2, we have

$$\mathfrak{M}_t[G_s] \leq \mathfrak{M}_t[LM_s] \leq \mathfrak{M}_t[A_s] \tag{3.1}$$

for $t, s \in [0,1]$. Since $x^t \leq m_t[G_s](x)$ holds for all $x > 0$, (3.1) ensures a refinement of (1.1) as follows:

Theorem 3.4. *For $t, s \in [0,1]$, the inequalities*

$$\mathfrak{H}_t \leq \mathfrak{G}_t \leq \overline{\mathfrak{H}\mathfrak{Z}}_{t,[s]} \leq \mathfrak{M}_t[LM_s] \leq \mathfrak{K}_{t,[s]} \leq \mathfrak{A}_t$$

hold. In particular, we have

$$\mathfrak{H}_t \leq \mathfrak{G}_t \leq \overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{1}{2}]} \leq \mathfrak{L}\mathfrak{M}_t \leq \mathfrak{K}_{t,[\frac{1}{2}]} \leq \mathfrak{A}_t. \tag{3.2}$$

Moreover, we can obtain the following estimations of $\mathfrak{L}\mathfrak{M}_t$ via the power difference mean, which are more precise than (3.2). We omit its proof.

Theorem 3.5. *For $t \in [0,1]$ and natural numbers n such that $n \geq 2$, the inequalities*

$$\begin{aligned}\overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{1}{2}]} &\leq \mathfrak{M}_t[J_{[\frac{-1}{3}]}] \leq \cdots \leq \mathfrak{M}_t[J_{[\frac{-1}{n}]}] \leq \mathfrak{M}_t[J_{[\frac{-1}{n+1}]}] \leq \cdots \\ &\leq \mathfrak{L}\mathfrak{M}_t \leq \cdots \leq \mathfrak{M}_t[J_{[\frac{1}{n+1}]}] \leq \mathfrak{M}_t[J_{[\frac{1}{n}]}] \leq \cdots \leq \mathfrak{M}_t[J_{[\frac{1}{2}]}] \leq \mathfrak{K}_{t,[\frac{1}{2}]}\end{aligned}$$

hold, where $\mathfrak{M}_t[J_{[\frac{1}{n}]}]$ and $\mathfrak{M}_t[J_{[\frac{-1}{n}]}]$ are the weighted operator means such that

$$\begin{aligned}\mathfrak{M}_t[J_{[\frac{1}{n}]}] &= \frac{1}{n+1}\left(\mathfrak{A}_t + \overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{n-1}{n}]} + \cdots + \overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{1}{n}]} + \mathfrak{G}_t\right), \\ \mathfrak{M}_t[J_{[\frac{-1}{n}]}] &= \frac{1}{n-1}\left(\overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{n-1}{n}]} + \overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{n-2}{n}]} + \cdots + \overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{2}{n}]} + \overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{1}{n}]}\right).\end{aligned}$$

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