Doctoral Dissertation

Nonintegrability and Related Dynamics of Ordinary Differential Equations

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Chapter 1

Introduction

The problem of *integrability* for ordinary differential equations is one of classical and important topics in dynamical systems, as seen in Poincaré's famous work [51, 52] on the restricted three-body problem in 1890's. The notion of integrability for Hamiltonian systems has been well established and is called *Liouville integrableity*: If an m-degree-of-freedom Hamiltonian system with $m \in \mathbb{N}$ is integrable in this sense, then it has m (Poisson commuting) first integrals (see [9, 43] for its precise definition). The Liouville-Arnold theorem (see, e.g., Section 49 of [9]) states that such a Hamiltonian system can be solved by quadrature. In his paper [15] published in 1998, Bogoyavlenskij extended the concept of integrability to general ordinary differential equations including non-Hamiltonian systems, using commutative vector fields as well as first integrals.

From the late 20th century to early 21st century, the theory of integrability for ordinary differential equations has progressed remarkably. In 1980's, Ziglin [79] developed a method called the *Ziglin analysis* to obtain a sufficient condition for meromorphical nonintegrability of complex Hamiltonian systems in which both the dependent and independent variables are extended to complex ones. In his method, monodoromy matrices of the variational equations, i.e., linearized equations, along particular nonconstant solutions such as periodic and homoclinic orbits are computed and their noncommutativity detects the nonintegrability of Hamiltonian systems. Subsequently, Morales-Ruiz and Ramis [43, 47] improved his method drastically using differential Galois theory [23, 53], and developed a stronger method which is now called the *Morales-Ramis theory.* In their method, the differential Galois groups of the variational equations along particular nonconstant solutions are used and the noncommutativity of their identity components detects the meromorphic or rational nonintegrability of Hamiltonian systems. Moreover, Ayoul and Zung [12] used a simple trick called the *cotangent lift* (see also Section 3.3 below) and showed that the Morales-Ramis theory is also applicable to nonintegrability of non-Hamiltonian systems in the sense of Bogoyavlenskij. These methods have been successfully applied to numerous examples including general three-body problem [18, 19, 58, 59]. See [37, 45, 48] and references therein for these results.

On the other hand, since the mid 20th century, including Ueda and his coworkers (e.g., [29, 60]) on the forced Duffing oscillator, there has been many researches reporting that nonlinear oscillators exhibit chaotic behavior in numerical simulations and/or experiments. See [26, 62] and references therein. However, the nonintegrability and existence of chaos in these nonlinear oscillators have not been mathematically proved except in special cases. It has also been reported that complicated dynamics such as chaos occur in many noningrable systems (see, e.g., [38–40, 46, 70, 76, 77]), but there are many unknown parts on the relationship between the nonintegrability of differential equations and their dynamics.

In this thesis, we consider a class of ordinary differential equations including non-Hamiltonian systems and nonlinear oscillators with parametric or external forcing, and develop theories for their nonintegrability and related dynamics. In particular, we give several necessary conditions for persistence of periodic and homoclinic orbits, first integrals and commutative vector fields in perturbed systems, and for real-analytic integrability of nearly integrable systems in the sense of Bogoyavlenskij such that the first integrals and commutative vector fields also depend analytically on the perturbation parameter.

The rest of this thesis consists of the following chapters.

In Chapter 2, we discuss nonintegrability of parametrically forced nonlinear oscillators which are represented by second-order homogeneous differential equations with trigonometric coefficients and contain the Duffing and van der Pol oscillators as special cases. Specifically, we give sufficient conditions for their rational nonintegrability in the meaning of Bogoyavlenskij, using Kovacic's algorithm [34] as well as an extension of the Morales-Ramis theory due to Ayoul and Zung. In application of the extended Morales-Ramis theory, for the associated variational equations, the identity components of their differential Galois groups are shown to be not commutative even if the differential Galois groups are triangularizable, i.e., they can be solved by quadrature. The obtained results are very general and reveal their rational nonintegarbility for the wide class of parametrically forced nonlinear oscillators. We also give two examples for the van der Pol and Duffing oscillators to demonstrate our results.

In Chapter 3, we study persistence of periodic and homoclinic orbits, first integrals and commutative vector fields in dynamical systems depending on a small parameter and give several necessary conditions for their persistence. Here we treat homoclinic orbits not only to equilibria but also to periodic orbits. We also discuss some relationships of these results with the standard subharmonic and homoclinic Melnikov methods [26, 42, 62, 65] for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems, and with another version of the homoclinic Melnikov method [61] for autonomous perturbations of multi-degree-of-freedom Hamiltonian systems. In particular, we show that a first integral which converges to the Hamiltonian or another first integral as the perturbation tends to zero does not exist near the unperturbed periodic or homoclinic orbits in the perturbed systems if the subharmonic or homoclinic Melnikov functions are not identically zero on connected open sets. We illustrate our theory for four examples: The periodically forced Duffing oscillator, two identical pendula coupled with a harmonic oscillator, a periodically forced rigid body and a three-mode truncation of a buckled beam.

In Chapter 4, we study the existence of first integrals and integrability for perturbations of integrable systems in the sense of Bogoyavlenskij including non-Hamiltonian ones. We especially assume that there exists a family of periodic orbits on a regular level set of the first integrals which has a connected and compact component and give sufficient conditions for nonexistence of the same number of first integrals in the perturbed systems as the unperturbed ones and for their nonintegrability near the level set such that the first integrals and commutative vector fields depend analytically on the small parameter. We compare our results with the classical results of Poincaré [51, 52] and Kozlov [35, 36] for systems written in action and angle coordinates and discuss their relationships with the subharmonic and homoclinic Melnikov methods for periodic perturbations of single-degree-of-freeedom Hamiltonian systems. We illustrate our theory for three examples containing the periodically forced Duffing oscillator.

Finally, we give concluding remarks and some comments on future work in Chapter 5.

Chapter 2

Nonintegrability of Parametrically Forced Nonlinear Oscillators

2.1 Introduction

We consider second-order homogeneous differential equations with trigonometric coefficients of the form

$$\ddot{x} + f(x, \dot{x}, \cos\omega t, \sin\omega t)x + g(x, \dot{x}, \cos\omega t, \sin\omega t)\dot{x} = 0,$$

or equivalently as a first-order system,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -f(x_1, x_2, \cos\omega t, \sin\omega t)x_1 - g(x_1, x_2, \cos\omega t, \sin\omega t)x_2, \tag{2.1.1}$$

where the dot represents differentiation with respect to t; $\omega > 0$ is a constant; and $f, g : \mathbb{R}^4 \to \mathbb{R}$ are rational and satisfy

$$f(0, 0, y_1, y_2) = f_0 + f_1 y_1, \quad g(0, 0, y_1, y_2) = g_0 + g_1 y_1 + g_2 y_2 \tag{2.1.2}$$

for any $y_1, y_2 \in \mathbb{R}$ with constants $f_0, f_1, g_0, g_1, g_2 \in \mathbb{R}$. Even if $f(0, 0, y_1, y_2) = f_0 + f_1 y_1 + f_2 y_2$ with a constant $f_2 \neq 0$, then we can redefine the functions f and g so that condition (2.1.2) holds, by shifting the independent variable t. Equation (2.1.1) represents several parametrically forced nonlinear oscillators. For example, it reduces to the Duffing oscillator when $f(x_1, x_2, y_1, y_2) = f_0 + \bar{f}_1 x_1^2$ and $g(x_1, x_2, y_1, y_2) = g_0$, and the van der Pol oscillator when $f(x_1, x_2, y_1, y_2) = f_0$ and $g(x_1, x_2, y_1, y_2) = g_0(1 - x_1^2)$, where $\bar{f}_1$ as well as f_0, g_0 is a constant. These nonlinear oscillators often exhibit chaotic dynamics when they are parametrically excited. See, e.g., [7, 28, 55]. A global perturbation technique called Melnikov's method [26, 62] allows us to detect the existence of horseshoe dynamics yielding chaos in some cases [7] but no mathematical explanation for the occurrence of chaos has still been given in the other cases.

In this chapter we discuss the nonintegrability of (2.1.1), which provides another reason why it may exhibit complicated dynamics such as chaotic motions. For general Hamiltonian systems, Morales-Ruiz and Ramis [47] developed a strong method to present a sufficient condition for their meromorphic or rational nonintegrability. Their theory, which is now called the Morales-Ramis theory, states that complex Hamiltonian systems are meromorphically or rationally nonintegrable if the identity components of the differential Galois groups [23, 53] for their variational equations (VEs) or normal variational equations (NVEs) around particular nonconstant solutions such as periodic orbits are not commutative. See also [43]. In application of the Morales-Ramis theory, a useful algorithm which was proposed by Kovacic [34] for linear second order differential equations and is now called the Kovacic algorithm has been frequently used to determine the differential Galois groups of the associated VEs and NVEs. Moreover, Ayoul and Zung [12] showed that the Morales-Ramis theory is also applicable for detection of meromorphic or rational nonintegrability of non-Hamiltonian systems in the meaning of Bogoyavlenskij [15] (see Definition 2.2.1).

Specifically, using the extended Morales-Ramis theory and the Kovacic algorithm, we prove that an autonomous system corresponding to (2.1.1) (see Eq. (2.2.1)) is rationally nonintegrable in the meaning of Bogoyavlenskij if $(f_0, f_1) \neq (0,0)$, $(f_1, g_1, g_2) \neq (0,0,0)$ and $\omega > 0$. This means that Eq. (2.1.1) has at most one rational first integral of $x_1, x_2, \cos\omega t$ and $\sin\omega t$. Moreover, it is shown to have no such rational first integral if $f(x_1, x_2, y_1, y_2)$ is independent of x_2, $g(x_1, x_2, y_1, y_2) \equiv 0$, $f_1 \neq 0$ and $\omega > 0$ under an additional condition (see Theorem 2.2.3 below). In application of the extended Morales-Ramis theory, for the associated VEs, the identity components of their differential Galois groups are shown to be not commutative even if the differential Galois groups are triangularizable, i.e., they can be solved by quadratures. We should also mention that Acosta-Humánez [3] studied their nonintegrability for a class of single-degree-of-freedom nonautonomous Hamiltonian systems which contains a special case of (2.1.1), using the Morales-Ramis theory, and that Stachowiak [57] discussed the existence of hypergeometric first integrals for autonomous Duffing and van der Pol oscillators.

The rest of this chapter is as follows: In Section 2.2 we state our main results. We provide prerequisites on the differential Galois theory and the extension of the Morales-Ramis theory due to Ayoul and Zung [12] in Section 2.3, and prove the main results in Sections 2.4 and 2.5. Finally, we give two examples for the van der Pol and Duffing oscillators with parametric forcing to demonstrate our results in Section 2.6. For the reader's convenience the Kovacic algorithm is also outlined in a minimum necessary form for our application in Appendix 2.A.

2.2 Main Results

In this section we give our main results. We first complexify (2.1.1) and rewrite it as

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\hat{f}(x_1, x_2, y)x_1 - \hat{g}(x_1, x_2, y)x_2, \quad \dot{y} = i\omega y, \tag{2.2.1}$$

where

$$\begin{aligned}\hat{f}(x_1, x_2, y) =& f\left(x_1, x_2, \tfrac{1}{2}(y + y^{-1}), -\tfrac{1}{2}i(y - y^{-1})\right),\\ \hat{g}(x_1, x_2, y) =& g\left(x_1, x_2, \tfrac{1}{2}(y + y^{-1}), -\tfrac{1}{2}i(y - y^{-1})\right).\end{aligned}$$

Equation (2.2.1) is not Hamiltonian in general. We adopt the following definition of integrability due to Bogoyavlenskij [15].

Definition 2.2.1 (Bogoyavlenskij). *Consider systems of of the form*

$$\dot{x} = v(x), \quad x \in D \subset \mathbb{C}^n, \tag{2.2.2}$$

where $n > 0$ is an integer, D is a region in $\mathbb{C}^n$ and $v : D \to \mathbb{C}^n$ is holomorphic. Let q be an integer such that $1 \le q \le n$. Equation (2.2.2) *is called* $(q, n-q)$-integrable *or simply* integrable *if there exist q vector fields $v_1(x)(:= v(x)), v_2(x), \dots, v_q(x)$ and $n-q$ scalar-valued functions $F_1(x), \dots, F_{n-q}(x)$ such that the following two conditions hold:*

(i) *$v_1, \dots, v_q$ are linearly independent almost everywhere and commute with each other, i.e., $[v_j, v_k] := (\mathrm{D}v_k)v_j - (\mathrm{D}v_j)v_k = 0$ for $j, k = 1, \dots, q$;*

(ii) *$\mathrm{D}F_1, \dots, \mathrm{D}F_{n-q}$ are linearly independent almost everywhere and $F_1, \dots, F_{n-q}$ are first integrals of $v_1, \dots, v_q$, i.e., $(\mathrm{D}F_k)v_j = 0$ for $j = 1, \dots, q$ and $k = 1, \dots, n-q$.*

If $v_1, v_2, \dots, v_q$ and $F_1, \dots, F_{n-q}$ are meromorphic and rational, respectively, then Eq. (2.2.2) *is said to be* meromorphically *and* rationally integrable.

Definition 2.2.1 is regarded as a generalization of the Liouville integrability for Hamiltonian systems since if a Hamiltonian system with n degrees of freedom is Liouville integrable, then there exist n functionally independent first integrals and n linearly independent vector fields corresponding to the first integrals (almost everywhere). The statement similar to that of the Liouville-Arnold theorem [9] also holds for integrable systems in the meaning of Bogoyavlenskij. See [15] for more details.

We now state our first main result as follows.

Theorem 2.2.2. *Suppose that $(f_0, f_1) \neq (0, 0)$, $(f_1, g_1, g_2) \neq (0, 0, 0)$ and $\omega > 0$. Then Eq.* (2.2.1) *is rationally nonintegrable.*

We prove Theorem 2.2.2 in Section 2.4. The statement of the theorem is very general and it is applicable to many parametrically forced nonlinear oscillators including classical ones stated in Section 2.1. It also says that even linear oscillators subjected to parametric forcing is rationally nonintegrable if the hypothesis of the theorem holds. Moreover, it means that the two-dimensional nonautonomous system (2.1.1) has at most one rational first integral of $x_1, x_2, \cos\omega t$ and $\sin\omega t$ under the general condition even if it is linear. Here we say that $F(x_1, x_2, \cos\omega t, \sin\omega t)$ is a first integral of (2.1.1) if $F(x_1(t), x_2(t), \cos\omega t, \sin\omega t)$ is independent of t when $(x_1(t), x_2(t))$ is a solution to (2.1.1). However, Eq. (2.1.1) has no such rational first integral in a restricted case as follows.

Theorem 2.2.3. *Suppose that $f(x_1, x_2, y_1, y_2)$ is independent of x_2, $g(x_1, x_2, y_1, y_2) \equiv 0$, $f_1 \neq 0$ and $\omega > 0$. Let*

$$V(x, y) = \int \hat{f}(x, y) x \, \mathrm{d}x, \tag{2.2.3}$$

where we write $\hat{f}(x_1, y)$ instead of $\hat{f}(x_1, x_2, y)$. If $V(x, y)$ is a rational function of x and y, then Eq. (2.1.1) *has no rational first integral of $x_1, x_2, \cos\omega t$ and $\sin\omega t$.*

We prove Theorem 2.2.3 in Section 2.5. The hypothesis of the theorem also means that Eq. (2.1.1) is an nonautonomous Hamiltonian system.

2.3 Prerequisites

In this section, we provide prerequisites on differential Galois theory for linear differential equations (see [23,53] for more details) and the generalization of the Morales-Ramis theory [43, 47] due to Ayoul and Zung [12].

2.3.1 Differential Galois Theory

Let $\mathbb{K}$ be a differential field with a derivation ∂ and let n be a positive integer. Consider the n-dimensional system of linear first-order differential equations

$$\partial\xi = A\xi, \quad \xi \in \mathbb{C}^n, \tag{2.3.1}$$

where A is an $n \times n$ matrix with entries in $\mathbb{K}$. For higher-order differential equations, one can transform them into the form (2.3.1). Here the differential field $\mathbb{K}$ is also referred to as the *coefficient field* for (2.3.1). The subfield $C_{\mathbb{K}} = \operatorname{Ker}\partial \subset \mathbb{K}$ is called the *constant field* of $\mathbb{K}$. A *differential field extension* $\mathbb{L} \supset \mathbb{K}$ is a field extension such that $\mathbb{L}$ is a differential field and the derivations on $\mathbb{L}$ and $\mathbb{K}$ coincide on $\mathbb{K}$. It is also called a *Picard-Vessiot extension* for (2.3.1) if it is generated by $\mathbb{K}$ and the entries of a fundamental matrix Ξ of (2.3.1), and its constant field coincides with $C_{\mathbb{K}}$.

Fix a Picard-Vessiot extension $\mathbb{L} \supset \mathbb{K}$ and a fundamental matrix Ξ. A $\mathbb{K}$-*differential automorphism* σ of $\mathbb{L}$ is a field automorphism of $\mathbb{L}$ such that

$$\forall a \in \mathbb{K},\ \sigma(a) = a \quad \text{and} \quad \forall a \in \mathbb{L},\ \partial(\sigma(a)) = \sigma(\partial a),$$

and all $\mathbb{K}$-differential automorphisms of $\mathbb{L}$ form a group. We call the group the *differential Galois group* of (2.3.1) and denote it by $\mathrm{DGal}(\mathbb{L}/\mathbb{K})$. The differential Galois group $\mathrm{DGal}(\mathbb{L}/\mathbb{K})$ is also regarded as a linear algebraic subgroup of $\mathrm{GL}(n, C_{\mathbb{K}})$ and determined up to conjugation by the choice of Ξ. A unique maximal connected linear algebraic subgroup of $\mathrm{DGal}(\mathbb{L}/\mathbb{K})$ containing the identity element is called its *identity component*, which is denoted by $\mathrm{DGal}(\mathbb{L}/\mathbb{K})^0$.

The differential Galois theory can determine whether one can solve differential equations by quadratures, as algebraic Galois theory can for solving algebraic equations by radicals. Roughly speaking, a differential equation can be solved by quadratures if and only if the identity component of its differential Galois group is solvable. We recommend the reader who is interested in the details of the theory to consult, e.g., the books [23,53].

2.3.2 Generalization of the Morales-Ramis Theory

We turn to the general system (2.2.2). Let $x = \phi(t)$, $t \in \mathbb{C}$, be its nonstationary particular solution. The VE of (2.2.2) along $x = \phi(t)$ is given by

$$\dot{\xi} = \mathrm{D}v(\phi(t))\xi, \quad \xi \in \mathbb{C}^n. \tag{2.3.2}$$

Let Γ be a curve given by $x = \phi(t)$. Assume that its closure $\bar{\Gamma}$ in $\mathbb{P}^n$ contains points at infinity and the vector field $v(x)$ can be meromorphically extended to a region containing $\bar{\Gamma}$ in $\mathbb{P}^n$. We take the meromorphic function field on $\bar{\Gamma}$ as the coefficient field $\mathbb{K}$ of (2.3.2). Using arguments given by Morales-Ruiz and Ramis [43, 47] and Ayoul and Zung [12], we have the following result.

Theorem 2.3.1. *Let G be the differential Galois group of* (2.3.2). *Suppose that the VE* (2.3.2) *has irregular singularities at infinity. If Eq.* (2.2.2) *is rationally integrable near* Γ*, then the identity component* G^0 *of* G *is commutative.*

Remark 2.3.2.

(i) In the above statement, assume that the VE (2.3.2) *has no irregular singularity at infinity. Then we can replace the word "rationally" with "meromorphically" in the conclusion. See Section* 4.2 *of* [43] *or Section* 5.2 *of* [47] *for the details.*

(ii) Ayoul and Zung [12] *treated the VE* (2.3.2) *only on* Γ*, but we can easily extend their result to our situation using arguments given in Section* 4.2 *of* [43] *or Section* 5.2 *of* [47] *although the obtained result is weaker: only rational integrability is mentioned instead of meromorphic one, as stated above. See also Section* 2 *of* [6].

As described in [4,75], if Eq. (2.3.2) has an invariant manifold containing the particular solution $\phi(t)$, then it becomes easier to check whether G^0 is not commutative. For simplicity, assume that the m-dimensional plane $\{x = (x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 = \cdots = x_{n-m} = 0\}$ is invariant for some positive integer $m < n$. Then the VE (2.3.2) can be written as

$$\dot{\xi} = \begin{pmatrix} A_0(t) & 0 \\ A_1(t) & A_2(t) \end{pmatrix} \xi, \tag{2.3.3}$$

where $A_0(t)$, $A_1(t)$ and $A_2(t)$ are, respectively, $(n-m)\times(n-m)$, $m\times(n-m)$ and $m\times m$ matrices. Let $\hat{G}$ be a differential Galois group of the NVE

$$\dot{\bar{\xi}} = A_0(t)\bar{\xi},$$

where $\bar{\xi} = (\xi_1, \dots, \xi_{n-m})$. We have the following proposition.

Proposition 2.3.3. *If G^0 is commutative, then so is $\hat{G}^0$.*

A proof of this proposition is essentially found in that of Theorem 3.4 of [4] although only Hamiltonian systems were treated there. Thus, if $\hat{G}^0$ is not commutative, then so is G^0 and by Theorem 2.3.1 and Remark 2.3.2(i) Eq. (2.2.2) is rationally or meromorphically nonintegrable near Γ under the condition that the VE (2.3.2) has an irregular singularity at infinity or not.

2.4 Proof of Theorem 2.2.2

We now prove Theorem 2.2.2 using the theory of Section 2.3.2. Henceforth we assume that $(f_0, f_1) \neq (0,0)$, $(f_1, g_1, g_2) \neq (0,0,0)$ and $\omega > 0$, and take $\omega = 1$ without loss of generality. Actually, if $\omega \neq 1$, then we only have to replace t, f_j and g_j with ωt, f_j/ω^2 and g_j/ω for $j = 0, 1$ or $0, 1, 2$.

First, we easily see that $(x_1, x_2, y) = (0, 0, \mathrm{e}^{it})$ is a particular solution to (2.2.1), and obtain the VE of (2.2.1) around it as

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = -(f_0 + f_1 \cos t)\xi_1 - (g_0 + g_1 \cos t + g_2 \sin t)\xi_2, \quad \dot{\xi}_3 = i\xi_3 \tag{2.4.1}$$

under condition (2.1.2). The y-axis is invariant for (2.2.1) and the VE (2.4.1) has the form (2.3.3) with $m=1$. So the associated NVE is given by

$$\dot{\xi}_1=\xi_2,\quad \dot{\xi}_2=-(f_0+f_1\cos t)\xi_1-(g_0+g_1\cos t+g_2\sin t)\xi_2. \tag{2.4.2}$$

Equation (2.4.2) has no singular point in $\mathbb{C}$ and an irregular singular point at $t=\infty$. Hence, it follows from Theorem 2.3.1 and Proposition 2.3.3 that if the identity component $\hat{G}^0$ of the differential Galois group $\hat{G}$ for (2.4.2) is not commutative, then Eq. (2.2.1) is rationally nonintegrable near the y-axis in $\mathbb{C}^3$. In the following we use the Kovacic algorithm [34] to show that $\hat{G}^0$ is not commutative. A necessary part of the algorithm is briefly described in Appendix 2.A. See [23, 34] for more details on the algorithm.

We rewrite (2.4.2) as the second-order differential equation

$$\ddot{\xi}_1+(g_0+g_1\cos t+g_2\sin t)\dot{\xi}_1+(f_0+f_1\cos t)\xi_1=0. \tag{2.4.3}$$

By changing the independent variable as $z=\mathrm{e}^{it}$, Eq. (2.4.3) becomes

$$\xi_1''-\left(i\hat{g}_1+\frac{ig_0-1}{z}+\frac{i\hat{g}_{-1}}{z^2}\right)\xi_1'-\left(\frac{f_1}{2z}+\frac{f_0}{z^2}+\frac{f_1}{2z^3}\right)\xi_1=0, \tag{2.4.4}$$

where the prime represents differentiation with respect to z and

$$\hat{g}_1=\tfrac{1}{2}(g_1-ig_2),\quad \hat{g}_{-1}=\tfrac{1}{2}(g_1+ig_2).$$

The identity component of the differential Galois group of (2.4.4) with $\mathbb{K}=\mathbb{C}(z)$ coincides with that of (2.4.3) with $\mathbb{K}=\mathbb{C}(e^{it})$ (see, e.g., Proposition 1 of [5]). Finally, by changing the dependent variable as

$$\xi_1=\exp\left(\tfrac{1}{2}\int\left(i\hat{g}_1+\frac{ig_0-1}{z}+\frac{i\hat{g}_{-1}}{z^2}\right)dz\right)\zeta,$$

Eq. (2.4.4) reduces to the form (2.A.1) with

$$\begin{aligned}r(z)=&\tfrac{1}{4}\left(i\hat{g}_1+\frac{ig_0-1}{z}+\frac{i\hat{g}_{-1}}{z^2}\right)^2-\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left(i\hat{g}_1+\frac{ig_0-1}{z}+\frac{i\hat{g}_{-1}}{z^2}\right)+\left(\frac{f_1}{2z}+\frac{f_0}{z^2}+\frac{f_1}{2z^3}\right)\\ =&r_4z^{-4}+r_3z^{-3}+r_2z^{-2}+r_1z^{-1}+r_0,\end{aligned} \tag{2.4.5}$$

where

$$\begin{aligned}&r_4=-\tfrac{1}{4}\hat{g}_{-1}^2,\quad r_3=\tfrac{1}{2}(f_1-(g_0-i)\hat{g}_{-1}),\\ &r_2=-\tfrac{1}{4}(g_0^2+1)-\tfrac{1}{2}\hat{g}_1\hat{g}_{-1}+f_0,\quad r_1=\tfrac{1}{2}(f_1-(g_0+i)\hat{g}_1),\quad r_0=-\tfrac{1}{4}\hat{g}_1^2.\end{aligned}$$

We now apply the Kovacic algorithm. The function $r(z)$ of (2.4.5) has a pole only at $z=0$, and its order e_0 is 3 or 4, depending on whether $\hat{g}_{-1}=0$ or not, since $(f_1,\hat{g}_{-1})\neq(0,0)$. Note that $\hat{g}_{-1}=0$ if and only if $\hat{g}_1=0$, since $g_1,g_2\in\mathbb{R}$. The order e_∞ of $r(z)$ at ∞ is 1 if $\hat{g}_1=0$, and it is 0 otherwise. Using Proposition 2.A.2, we obtain the following observations on cases (a)-(c) of Proposition 2.A.1:

(i) If $f_1\neq 0$ and $\hat{g}_1,\hat{g}_{-1}=0$, i.e., $g_1,g_2=0$, then cases (a) and (c) do not occur but case (b) may occur;

(ii) If $(f_0, f_1) \neq (0,0)$ and $\hat{g}_1, \hat{g}_{-1} \neq 0$, i.e., $(g_1, g_2) \neq (0,0)$, then cases (b) and (c) do not occur but case (a) may occur.

Note that the identity components of the differential Galois groups for (2.A.1) with (2.4.5) and for (2.4.3) are not commutative if cases (a)-(c) do not occur, and that they may be noncommutative even if case (a) occurs.

For case (i) we use the algorithm of Appendic 2.A to show that case (d) of Proposition 2.A.1 occurs, since $e_0 = 3$ and $e_\infty = 1$ so that

$$d_e = \tfrac{1}{2}(1-3) = -1,$$

which means no occurrence of case (b). Thus, Eq. (2.2.1) is rationally nonintegrable when $f_1 \neq 0$ and $g_1, g_2 = 0$.

We turn to case (ii) and apply the algorithm of Appendix 2.A. We have the Laurent series expansions of $\sqrt{r(z)}$ at $z = 0$ and ∞ as

$$\sqrt{r(z)} = -\frac{i\hat{g}_{-1}}{2z^2} + \cdots$$

and

$$\sqrt{r(z)} = -\frac{i\hat{g}_1}{2} + \cdots,$$

respectively. We calculate (2.A.2) and (2.A.3) as

$$\begin{aligned}
\kappa_0^+ &= \tfrac{1}{2}\left(-\frac{2r_3}{i\hat{g}_{-1}} + 2\right) = \frac{f_1 g_2}{g_1^2 + g_2^2} + \tfrac{1}{2} + i\left(\frac{f_1 g_1}{g_1^2 + g_2^2} - \tfrac{1}{2}g_0\right),\\
\kappa_0^- &= \tfrac{1}{2}\left(\frac{2r_3}{i\hat{g}_{-1}} + 2\right) = -\frac{f_1 g_2}{g_1^2 + g_2^2} + \tfrac{3}{2} - i\left(\frac{f_1 g_1}{g_1^2 + g_2^2} - \tfrac{1}{2}g_0\right),\\
\kappa_\infty^\pm &= \mp\frac{r_1}{i\hat{g}_1} = \mp\frac{f_1 g_2}{g_1^2 + g_2^2} \pm \tfrac{1}{2} \pm i\left(\frac{f_1 g_1}{g_1^2 + g_2^2} - \tfrac{1}{2}g_0\right)
\end{aligned}$$

and

$$d_s = \begin{cases}
-\dfrac{2f_1 g_2}{g_1^2 + g_2^2} & \text{for } (s(0), s(\infty)) = (+,+);\\
-1 - i\left(\dfrac{2f_1 g_1}{g_1^2 + g_2^2} - g_0\right) & \text{for } (s(0), s(\infty)) = (+,-);\\
-1 + i\left(\dfrac{2f_1 g_1}{g_1^2 + g_2^2} - g_0\right) & \text{for } (s(0), s(\infty)) = (-,+);\\
\dfrac{2f_1 g_2}{g_1^2 + g_2^2} - 2 & \text{for } (s(0), s(\infty)) = (-,-),
\end{cases}$$

respectively. Thus, d_s can be a nonnegative integer if and only if

$$\frac{2f_1 g_2}{g_1^2 + g_2^2} \in \mathbb{Z} \setminus \{1\}. \tag{2.4.6}$$

Hence, according to the recipe of the Kovacic algorithm stated in Appendix 2.A, if condition (2.4.6) does not hold, then case (d) of Proposition 2.A.1 occurs and consequently the identity component $\hat{G}^0$ of the differential Galois group $\hat{G}$ for (2.4.2) is not commutative.

We also have the following.

Lemma 2.4.1. *If* $(f_0, f_1), (g_1, g_2) \neq (0,0)$ *and case* (a) *of Proposition* 2.A.1 *occurs for* (2.A.1) *with* (2.4.5)*, then* $\hat{G}^0$ *is not commutative.*

See Appendix 2.B for a proof of this lemma. Thus, we prove that $\hat{G}^0$ is not commutative and consequently Eq. (2.2.1) is rationally nonintegrable when $(f_0, f_1) \neq (0,0)$ and $(g_1, g_2) \neq (0,0)$, whether case (a) or case (d) of Proposition 2.A.1 occurs. This completes the proof of Theorem 2.2.2. □

2.5 Proof of Theorem 2.2.3

We next prove Theorem 2.2.3. Henceforth we assume that the hypotheses of the theorem hold: $f(x_1, x_2, y_1, y_2)$ is independent of x_2, $g(x_1, x_2, y_1, y_2) \equiv 0$, $f_1 \neq 0$, $\omega > 0$ and $V(x, y)$ is a rational function of x and y (see Eq. (2.2.3) for the definition of $V(x, y)$). We also take $\omega = 1$ without loss of generality as in Section 4. We rewrite (2.1.1) and (2.2.1) as

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -f(x_1, \cos t, \sin t)x_1 \tag{2.5.1}$$

and

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\hat{f}(x_1, y)x_1, \quad \dot{y} = iy, \tag{2.5.2}$$

respectively, where

$$\hat{f}(x_1, y) = f\left(x_1, \tfrac{1}{2}(y + y^{-1}), -\tfrac{1}{2}i(y - y^{-1})\right).$$

Define the Hamiltonian function

$$H(x_1, x_2, y_1, y_2) = \tfrac{1}{2}x_2^2 + V(x_1, y_1) + iy_1y_2.$$

The associated Hamiltonian system is given by

$$\begin{aligned} &\dot{x}_1 = x_2, \quad \dot{x}_2 = -\hat{f}(x_1, y_1)x_1, \\ &\dot{y}_1 = iy_1, \quad \dot{y}_2 = -\frac{\partial V}{\partial y_1}(x_1, y_1) - iy_2, \end{aligned} \tag{2.5.3}$$

whose (x_1, x_2, y_1)-components have the same form as (2.5.2). We easily see that $(x_1, x_2, y_1, y_2) = (0, 0, \mathrm{e}^{i\omega t}, \mathrm{e}^{-i\omega t})$ is a particular solution to (2.5.3), along which the VE and NVE are give by

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = -(f_0 + f_1 \cos t)\xi_1, \quad \dot{\xi}_3 = i\xi_3, \quad \dot{\xi}_4 = -i\xi_4. \tag{2.5.4}$$

and

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = -(f_0 + f_1 \cos t)\xi_1, \tag{2.5.5}$$

respectively. Note that

$$\frac{\partial^2 V}{\partial x_1 \partial y_1}(0, y_1) = \frac{\partial^2 V}{\partial y_1^2}(0, y_1) = 0 \quad \text{for } y_1 \neq 0.$$

Equation (2.5.5) is a special case of (2.4.2) with $g_0, g_1, g_2 = 0$. Hence, it follows from the proof of Theorem 2.2.2 (for case (i)) in Section 2.4 that the identity component of its differential Galois group is not commutative. This implies via the standard Morales-Ramis theory [43, 47] that the Hamiltonian system (2.5.3) is rationally nonintegrable in the meaning of Liouville [9, 43], i.e., it has no additional rational first integral.

Suppose that Eq. (2.5.1) has a rational first integral $F(x_1, x_2, \cos\omega t, \sin\omega t)$. Then $F(x_1, x_2, \frac{1}{2}(y_1 + y_1^{-1}), -\frac{1}{2}i(y_1 - y_1^{-1}))$ is also a first integral for (2.5.3) although it is functionally independent of H. This contradicts the rational nonintegrability of (2.5.3). Thus, we obtain the desired result. □

2.6 Examples

Finally, we give two examples for the van der Pol and Duffing oscillators to demonstrate our results.

2.6.1 Van der Pol Oscillator

We first consider the van der Pol oscillator subjected to parametric forcing:

$$\ddot{x} + (g_0(1 - x^2) + g_1 \cos\omega t + g_2 \sin\omega t)\dot{x} + (1 + f_1 \cos\omega t)x = 0, \tag{2.6.1}$$

where f_1, g_0, g_1, g_2 and $\omega > 0$ are constants as above. Equation (2.6.1) is a special case of (2.1.1) with

$$f(x_1, x_2, y_1, y_2) = 1 + f_1 y_1, \quad g(x_1, x_2, y_1, y_2) = g_0(1 - x_1^2) + g_1 y_1 + g_2 y_2.$$

Using Theorem 2.2.2 with $f_0 = 1 \neq 0$, we see that if $(f_1, g_1, g_2) \neq (0, 0, 0)$, then the associated three-dimensional system is rationally nonintegrable and Eq. (2.6.1) has at most one rational first integral of $x, \dot{x}, \cos\omega t$ and $\sin\omega t$. This statement is true even if $g_0 = 0$, i.e., Eq. (2.6.1) is linear.

2.6.2 Duffing Oscillator

We next consider the parametrically forced Duffing oscillator:

$$\ddot{x} + (g_0 + g_1 \cos\omega t + g_2 \sin\omega t)\dot{x} + (f_0 + f_1 \cos\omega t + \bar{f}_1 x^2)x = 0, \tag{2.6.2}$$

where $\bar{f}_1, f_0, f_1, g_0, g_1, g_2$ and $\omega > 0$ are constants. Equation (2.6.2) is a special case of (2.1.1) with

$$f(x_1, x_2, y_1, y_2) = f_0 + f_1 y_1 + \bar{f}_1 x_1^2, \quad g(x_1, x_2, y_1, y_2) = g_0 + g_1 y_1 + g_2 y_2.$$

Using Theorem 2.2.2 and 2.2.3, we obtain the following results:

(i) Case of $f_0 \neq 0$: If $(f_1, g_1, g_2) \neq (0, 0, 0)$, then the associated three-dimensional system is rationally nonintegrable and Eq. (2.6.2) has at most one rational first integral of $x, \dot{x}, \cos\omega t$ andf $\sin\omega t$.

(ii) Case of $f_0 = 0$: If $f_1 \neq 0$, then the associated three-dimensional system is rationally nonintegrable and Eq. (2.6.2) has at most one rational first integral of $x, \dot{x}, \cos\omega t$ andf $\sin\omega t$.

(iii) Case of $g_0, g_1, g_2 = 0$: If $f_1 \neq 0$, then the associated three-dimensional system is rationally nonintegrable and Eq. (2.6.2) has no rational first integral of $x, \dot{x}, \cos\omega t$ and $\sin\omega t$. Note that the function $V(x, y)$ defined in (2.2.3) is rational on x and y.

We also remark that the above three statements hold even if $\bar{f}_1 = 0$, i.e., Eq. (2.6.2) is linear.

2.A Kovacic Algorithm

Consider second-order differential equations of the form

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2} = r(z)\zeta, \quad \zeta \in \mathbb{C}, \tag{2.A.1}$$

where $r(z) \in \mathbb{C}(z) \setminus \mathbb{C}$. Let G be the differential Galois group of (2.A.1) over $\mathbb{C}(z)$. We easily see that G is an algebraic subgroup of $\mathrm{SL}(2, \mathbb{C})$ (e.g., [33]). We also have the following four cases for G (see [23, 34] for a proof).

Proposition 2.A.1 (Kaplansky [33], Kovacic [34])**.** *One of the following four cases occurs*:

(a) *G is triangularizable*;

(b) *G is conjugate to a subgroup of*

$$D^\dagger = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C}^* \right\}$$

but it is not triangularizable;

(c) *G is finite but neither triangulariable nor conjugate to $D^\dagger$*;

(d) *$G = \mathrm{SL}(2, \mathbb{C})$*.

If one of cases (a)-(c) of Proposition 2.A.1 occurs, then the connected identity component G^0 of G is solvable so that Eq. (2.A.1) is solvable. On the other hand, if cases (a)-(c) do not occur, then $G^0 = G = \mathrm{SL}(2, \mathbb{C})$.

Let $r(z) = r_1(z)/r_2(z)$, where $r_1(z), r_2(z) \in \mathbb{C}[z]$ are relatively prime. Poles of $r(z)$ are zeros of $r_2(z)$ and their orders are the multiplicities of the zeros of $r_2(z)$. By the *order of $r(z)$ at ∞*, we also mean the order of ∞ as a zero of $r(z)$, i.e., $\deg r_2(z) - \deg r_1(z)$. We have the following necessary conditions for cases (a)-(c) in the above proposition to occur (see [23, 34] for a proof).

Proposition 2.A.2 (Kovacic [34])**.** *The following conditions are necessary for the corresponding cases in Proposition* 2.A.1 *to occur.*

(a) *No pole of $r(z)$ is of odd order greater than* 1, *and the order of $r(z)$ at ∞ is not an odd less than* 2.

(b) *There exists a pole of $r(z)$ such that its order is* 2 *or an odd greater than* 2.

(c) *No pole of* $r(z)$ *is of order greater than* 2 *and the order of* $r(z)$ *at* ∞ *is greater than* 1.

Note that cases (a), (b) and (c) of Proposition 2.A.1 may not occur even though conditions (a), (b) and (c) hold, respectively. Especially, Proposition 2.A.2 only states fewer conditions for case (c) than the original version in [34] since they are enough for our application in Section 2.4. Kovacic [34] also gave an algorithm for finding a "closed-form" solution and detecting whether cases (a)-(c) in Proposition 2.A.1 really occur or not under conditions (a), (b) and (c), respectively. See also [23]. In the following we describe necessary parts of his algorithm for cases (a) and (b) in the setting required in Section 2.4: $r(z)$ has a pole only at $z = 0$, its order is $e_0 > 2$, and the order of $r(z)$ at ∞ is $e_\infty < 2$. See [23, 34] for the full version of his algorithm and its proof.

2.A.1 Case (a)

Additionally, assume that condition (a) in Proposition 2.A.2 holds. For simplicity we also assume that $e_0 = 4$ and $e_\infty = 0$, which hold in our application of Section 2.4.

Let b_0 and b_∞ be the coefficients of z^{-3} and z^{-1} in the Laurent series expansions of $r(z)$ at $z = 0$ and ∞, respectively. The Laurent series expansions of $\sqrt{r(z)}$ at $z = 0$ and ∞ are also written as

$$\sqrt{r(z)} = \frac{a_0}{z^2} + \cdots$$

and

$$\sqrt{r(z)} = a_\infty + \cdots ,$$

respectively. Let

$$\kappa_0^\pm = \tfrac{1}{2}\left(\pm\frac{b_0}{a_0} + 2\right), \quad \kappa_\infty^\pm = \pm\frac{b_\infty}{2a_\infty}, \tag{2.A.2}$$

where the upper and lower signs are taken for the superscripts '+' and '−', respectively. For each sequence $s = \{s(c)\}_{c\in\{0,\infty\}}$ with $s(c) = +$ or $-$, define

$$d_s = \kappa_\infty^{s(\infty)} - \kappa_0^{s(0)}. \tag{2.A.3}$$

If d_s is not a nonnegative integer for any sequence s, then case (a) of Proposition 2.A.1 does not occur.

Assume that d_s is a nonnegative integer for some sequence s. If there exists a monic polynomial $P(z)$ of degree d_s satisfying the linear differential equation

$$P'' + 2\theta(z)P' + (\theta'(z) + \theta^2(z) - r(z))P = 0, \tag{2.A.4}$$

then

$$\zeta = P(z)\exp\left(\int \theta(z)\mathrm{d}z\right)$$

is a solution to (2.A.1) and hence case (a) of Proposition 2.A.1 occurs, where

$$\theta(z) = \frac{s(0)a_0}{z^2} + \frac{\kappa_0^{s(0)}}{z} + s(\infty)a_\infty. \tag{2.A.5}$$

If such a polynomial is not found for any sequence s, then it does not occur.

2.A.2 Case (b)

Additionally, assume that condition (b) in Proposition 2.A.2 holds. Let

$$d_e = \frac{1}{2}(e_\infty - e_0).$$

If d_e is not a nonnegative integer, then case (b) of Proposition 2.A.1 does not occur. We also have a further recipe similar to that of case (a) to determine whether it occurs or not when d_e is a nonnegative integer although it is not given here since it is not necessary for our purpose. The reader who is interested in it should consult [23, 34].

2.B Proof of Lemma 2.4.1

Suppose that the hypothesis of the lemma holds. Let $\tilde{G}$ be the differential Galois group of (2.4.4). Then $\tilde{G}$ is triangularizable. We only have to show that its identity component $\tilde{G}^0$ is not commutative in this situation.

We first note that condition (2.4.6) holds and either d_+ or d_- is a nonnegative integer, where

$$d_+ = -\frac{2f_1g_2}{g_1^2+g_2^2}, \quad d_- = \frac{2f_1g_2}{g_1^2+g_2^2} - 2.$$

Moreover, Eq. (2.A.4) has a d_sth degree monic polynomial solution $P(z)$ and Eq. (2.A.1) with (2.4.5) has a solution of the form

$$\zeta = P(z)\exp\left(\int\left(\mp\frac{i\hat{g}_1}{2} + \frac{\kappa_0^\pm}{z} \mp \frac{i\hat{g}_{-1}}{2z^2}\right)\mathrm{d}z\right),$$

where the upper or lower sign is taken, depending on whether d_+ or d_- is a nonnegative integer. Let

$$\tilde{d}_+ = -\frac{2f_1g_1}{g_1^2+g_2^2}, \quad \tilde{d}_- = \frac{2f_1g_1}{g_1^2+g_2^2} - 2.$$

If $f_1 = 0$, then $d_s = d_+ = 0$, $\tilde{d}_+ = 0$ and the function $\theta(z)$ defined in (2.A.5) becomes

$$\theta(z) = -\frac{i\hat{g}_{-1}}{2z^2} - \frac{ig_0 - 1}{2z} - \frac{i\hat{g}_1}{2},$$

so that $P(z) = 1$ is not a solution to (2.A.4) since $f_0 \neq 0$ and

$$\theta(z)^2 + \theta'(z) - r(z) = -\frac{f_0}{z^2}.$$

Hence, we can assume that $f_1 \neq 0$.

We rewrite

$$\kappa_0^+ = -\tfrac{1}{2}(d_+ - 1 + i(\tilde{d}_+ + g_0)), \quad \kappa_0^- = -\tfrac{1}{2}(d_- - 1 + i(\tilde{d}_- + 2 - g_0)).$$

We have a solution $\xi_1 = \tilde{\xi}_1(z)$ to (2.4.4) given by

$$\tilde{\xi}_1(z) = P(z)\exp\left(-\int\frac{d_+ + i\tilde{d}_+}{2z}\mathrm{d}z\right) = P(z)z^{-(d_+ + i\tilde{d}_+)/2}$$

if $d_+ \in \mathbb{Z}_{\geq 0} = \{d \in \mathbb{Z} \mid d \geq 0\}$, and

$$\begin{aligned}\tilde{\xi}_1(z) =& P(z)\exp\left(\int\left(i\hat{g}_1 + \frac{-(d_- + i\tilde{d}_-) + 2i(g_0-1)}{2z} + \frac{i\hat{g}_{-1}}{z^2}\right)\mathrm{d}z\right)\\ =& P(z)z^{-(d_-+i\tilde{d}_-)/2+i(g_0-1)}e^{i(\hat{g}_1 z - \hat{g}_{-1}/z)}\end{aligned}$$

if $d_- \in \mathbb{Z}_{\geq 0}$. Substituting $\xi_1 = \tilde{\xi}_1(z)u(z)$ into (2.4.4), we have

$$u'' + \left(\frac{2\tilde{\xi}_1'(z)}{\tilde{\xi}_1(z)} - i\hat{g}_1 - \frac{ig_0 - 1}{z} - \frac{i\hat{g}_{-1}}{z^2}\right)u' = 0.$$

We solve the above equation to obtain

$$u(z) = \int \tilde{\xi}_1(z)^{-2} z^{ig_0-1} e^{i(\hat{g}_1 z - \hat{g}_{-1}/z)}\mathrm{d}z, \tag{2.B.1}$$

which yields another solution to (2.4.4) given by

$$\tilde{\xi}_2(z) = \tilde{\xi}_1(z)\int \tilde{\xi}_1(z)^{-2} z^{ig_0-1} e^{i(\hat{g}_1 z - \hat{g}_{-1}/z)}\mathrm{d}z.$$

In the following, we show that $\tilde{G}^0$ is not commutative for $d_+ \in \mathbb{Z}_{\geq 0}$ and $d_- \in \mathbb{Z}_{\geq 0}$ separately, to complete the proof.

2.B.1 Case of $d_+ \in \mathbb{Z}_{\geq 0}$

For $\sigma \in \tilde{G}$ we compute

$$\begin{aligned}\frac{\sigma(\tilde{\xi}_1(z))'}{\sigma(\tilde{\xi}_1(z))} =& \sigma\left(\frac{\tilde{\xi}_1'(z)}{\tilde{\xi}_1(z)}\right) = \sigma\left(\frac{P'(z)}{P(z)} - \frac{d_+ + i\tilde{d}_+}{2z}\right)\\ =& \frac{P'(z)}{P(z)} - \frac{d_+ + i\tilde{d}_+}{2z} = \frac{\tilde{\xi}_1'(z)}{\tilde{\xi}_1(z)},\end{aligned}$$

which yields

$$\sigma(\tilde{\xi}_1(z)) = C_1\tilde{\xi}_1(z), \quad C_1 \in \mathbb{C}^*. \tag{2.B.2}$$

Let

$$\varphi(z) = z^{ig_0-1}e^{i(\hat{g}_1 z - \hat{g}_{-1}/z)}.$$

Since

$$\sigma(\varphi(z)) = C_2\varphi(z) \tag{2.B.3}$$

for $C_2 \in \mathbb{C}^*$ similarly, we have

$$\sigma'(u(z)) = \sigma(u'(z)) = C_1^{-2}C_2\tilde{\xi}_1(z)^{-2}\varphi(z) = C_1^{-2}C_2u'(z), \tag{2.B.4}$$

so that for some $C_3 \in \mathbb{C}$

$$\sigma(u(z)) = C_1^{-2}C_2u(z) + C_3 \tag{2.B.5}$$

and

$$\sigma(\tilde{\xi}_2(z)) = C_1^{-1}C_2\tilde{\xi}_2(z) + C_1C_3\tilde{\xi}_1(z). \tag{2.B.6}$$

Assume that $C_3 = 0$ for any $\sigma \in \tilde{G}$. Let $w(z) = u'(z)/u(z)$. Then by (2.B.4) and (2.B.5) we have $\sigma(w(z)) = w(z)$ for any $\sigma \in \tilde{G}$ so that $w(z) \in \mathbb{C}(z)$. Moreover,

$$u(z) = C_4 \exp\left(\int w(z)\mathrm{d}z\right)$$

for some $C_4 \in \mathbb{C}^*$ and consequently

$$u'(z) = C_4 w(z) \exp\left(\int w(z)\mathrm{d}z\right). \tag{2.B.7}$$

On the other hand, it follows from (2.B.1) that

$$u'(z) = P(z)^{-2} z^{d_+-1+i(g_0+\tilde{d}_+)} e^{i(\hat{g}_1 z - \hat{g}_{-1}/z)}. \tag{2.B.8}$$

Comparing (2.B.7) and (2.B.8), we obtain

$$\begin{aligned} w(z) =& i\left(\hat{g}_1 + \frac{\hat{g}_{-1}}{z^2}\right) + \frac{\alpha_0}{z} + \sum_{j=1}^N \frac{\alpha_j}{z - z_j} \\ =& C_4^{-1} P(z)^{-2} z^{-\alpha_0+d_+-1+i(\tilde{d}_++g_0)} \prod_{j=1}^N (z-z_j)^{-\alpha_j}, \end{aligned} \tag{2.B.9}$$

where $N \in \mathbb{Z}_{\geq 0}$ and $\alpha_0, \alpha_j, z_j \in \mathbb{C}$, $j = 1, \ldots, N$, such that $z_j \neq z_k$ for $j \neq k$ and $\alpha_j, z_j \neq 0$ for $j = 1, \ldots, N$ while α_0 and N are allowed to be zero. Hence

$$P(z)^2 = \frac{z^{-\alpha_0+d_++1+i(\tilde{d}_++g_0)} \prod_{j=1}^N (z-z_j)^{1-\alpha_j}}{C_4 \hat{P}(z)}, \tag{2.B.10}$$

where

$$\hat{P}(z) = (i\hat{g}_1 z^2 + \alpha_0 z + i\hat{g}_{-1}) \prod_{j=1}^N (z - z_j) + \sum_{k=1}^N \alpha_k \prod_{j=1,\, j\neq k}^N (z-z_j) z^2.$$

Since $\hat{g}_1, \hat{g}_{-1} \neq 0$ by $(g_1, g_2) \neq (0,0)$, we have $\hat{P}(0), \hat{P}(z_j) \neq 0$, $j = 1, \ldots, N$. So the right hand side of (2.B.10) is not a polynomial even if all the powers, $-\alpha_0 + d_+ + 1 + i(\tilde{d}_+ + g_0)$ and $1 - \alpha_j$, $j = 1, \ldots, N$, in the numerator are nonnegative integers. This contradicts the assumption that $P(z)$ is a polynomial. Hence, $C_3 \neq 0$ for some $\sigma \in \tilde{G}$, so that $\tilde{G}$ is not diagonalizable.

We represent $\tilde{G}$ as a subgroup of $\mathrm{GL}(2, \mathbb{C})$ with the fundamental solutions $\tilde{\xi}_j(z)$, $j = 1, 2$. Suppose that $\tilde{d}_+ \neq 0$, i.e., $g_1 \neq 0$. Then $\tilde{\xi}_1(z)$ is transcendental and consequently there exists an element $\sigma \in \tilde{G}$ for any $C_1 \in \mathbb{C}^*$ such that the relation (2.B.2) holds. Since $u'(z)$ is transcendental, we can take $C_1^{-2}C_2$ in (2.B.4) for some $\sigma \in \tilde{G}$ such that it is not a root of the unity. Noting that $\tilde{G}$ is not diagonalizable, we see that

$$\tilde{G} = \left\{ \begin{pmatrix} c_1 & \mu \\ 0 & c_2 \end{pmatrix} \,\middle|\, c_1, c_2 \in \mathbb{C}^*, \mu \in \mathbb{C} \right\},$$

so that $\tilde{G}^0$ is not commutative.

Next, suppose that $\tilde{d}_+ = 0$, i.e., $g_1 = 0$. Then $d_+ \neq 0$ by $f_1 g_2 \neq 0$, and $C_1 = 1$ or $C_1 = -1$ in (2.B.2) since $\tilde{\xi}_1(z)$ or $\tilde{\xi}_1(z)^2$ becomes a rational function. However, since $\hat{g}_1, \hat{g}_{-1} \neq 0$, $\varphi(z)$ is transcendental and consequently there exists an element $\sigma \in \tilde{G}$ for any $C_2 \in \mathbb{C}^*$ such that the relation (2.B.3) holds. Since $\tilde{G}$ is not diagonalizable, it follows from (2.B.6) that

$$\tilde{G}^0 = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & c \end{pmatrix} \middle| \, c \in \mathbb{C}^*, \mu \in \mathbb{C} \right\},$$

which is not commutative.

2.B.2 Case of $d_- \in \mathbb{Z}_{\geq 0}$

As in Section 2.B.1, we easily see that there exists an element $\sigma \in \tilde{G}$ for any $C_1 \in \mathbb{C}^*$ (resp. $C_2 \in \mathbb{C}^*$) such that the relation (2.B.2) (resp. (2.B.3)) holds, since $\tilde{\xi}_1(z)$ (resp. $\varphi(z)$) is transcendental by $\hat{g}_1, \hat{g}_{-1} \neq 0$. In addition, the relation (2.B.6) holds for some $C_3 \in \mathbb{C}$. We see that if $C_3 = 0$ for any $\sigma \in \tilde{G}$, then instead of (2.B.9)

$$\begin{aligned} w(z) =& -i\left(\hat{g}_1 + \frac{\hat{g}_{-1}}{z^2}\right) + \frac{\alpha_0}{z} + \sum_{j=1}^{N} \frac{\alpha_j}{z - z_j} \\ =& C_4^{-1} P(z)^{-2} z^{-\alpha_0 + d_- - 1 + i(\tilde{d}_- - g_0 + 2)} \prod_{j=1}^{N} (z - z_j)^{-\alpha_j}, \end{aligned}$$

holds for some $C_4 \in \mathbb{C}^*$ and a contradiction also occurs. So $\tilde{G}$ is not diagonalizable. Moreover, since $u'(z) = \varphi(z)/\tilde{\xi}_1(z)^2$ is transcendental, we proceed in the same way as when $\tilde{d}_+ \neq 0$ in Section 2.B.1 to show that $\tilde{G}^0$ is not commutative. □

Chapter 3

Persistence of Periodic and Homoclinic Orbits, First Integrals and Commutative Vector Fields

3.1 Introduction

Let $\mathscr{M}$ be an n-dimensional paracompact oriented C^4 real manifold for $n \geq 2$. Here we require its paracompactness and orientedness for defining integrals on $\mathscr{M}$. Consider dynamical systems of the form

$$\dot{x} = X_\varepsilon(x), \quad x \in \mathscr{M}, \tag{3.1.1}$$

where ε is a small parameter such that $0 < \varepsilon \ll 1$, and the vector field X_ε is C^3 with respect to x and ε. Let $X_\varepsilon(x) = X^0(x) + \varepsilon X^1(x) + O(\varepsilon^2)$ for $\varepsilon > 0$ sufficiently small. The system (3.1.1) becomes

$$\dot{x} = X^0(x) \tag{3.1.2}$$

when $\varepsilon = 0$, and it is regarded as a perturbation of (3.1.2). Assume that the unperturbed system (3.1.2) has a periodic or homoclinic orbit and a first integral or commutative vector field. Here we are mainly interested in their persistence in (3.1.1) for $\varepsilon > 0$ sufficiently small.

Bogoyavlenskij [15] extended a concept of Liouville integraility [9,43], which is defined for Hamiltonian systems, and proposed a definition of integrability for general systems. For (3.1.1), its integrability means that there exist $k\,(\geq 1)$ commutative vector fields containing X_ε and $n - k\,(\geq 0)$ first integrals for them such that the vector fields and first integrals are, respectively, linearly and functionally independent over a dense open set in $\mathscr{M}$. For integrable systems in this meaning, we have a statement similar to the Liouville-Arnold theorem for Hamiltonian systems (e.g., Section 49 in Chapter 10 of [9]): The flow on a level set of the first integrals is diffeomorphically conjugate to a linear flow on the k-dimensional torus $\mathbb{T}^k$ if the level set is a k-dimensional compact manifold (see Proposition 2 of [15]). Thus, the existence of first integrals and commutative vector fields is closely related to integrability of (3.1.1).

Even if the unperturbed system (3.1.2) is integrable, the perturbed system (3.1.1) is generally believed to be nonintegrable for $\varepsilon > 0$ small. For example, when the system

(3.1.1) is analytic and Hamiltonian for $\varepsilon \geq 0$, a famous result of Poincaré [52] says that its analytic Liouville integrability does not persist for $\varepsilon > 0$ under some generic assumptions. This means that not only first integrals independent of the Hamiltonian but also (Hamiltonian) vector fields commutative with X_0 do not persist in general. See also [35] for a more general result on nonexistence of first integrals, which was extended to non-Hamiltonian systems in [36]. Moreover, Morales-Ruiz [44] studied time-periodic Hamiltonian perturbations of single-degree-of-freedom Hamiltonian systems with homoclinic orbits, and showed a relationship between their nonintegrability and a version due to Ziglin [78] of the Melnikov method [42] by taking the time t and small parameter ε as state variables. Here the Melnikov method enables us to detect transversal self-intersection of complex separatrices of periodic orbits unlike the standard version [26, 42, 62]. More concretely, under some restrictive conditions, he essentially proved that they are meromorphically nonintegrable in the Bogoyavlenskij sense if the Melnikov functions are not identically zero, when a generalization due to Ayoul and Zung [12] of the Morales-Ramis theory [43, 47], which provides a sufficient condition for nonintegrability of dynamical systems, is applied. See Section 3.4.1 below for more details. On the other hand, to the authors' knowledge, the persistence of first integrals and commutative vector fields, especially when the unperturbed system (3.1.2) is nonintegrable, in non-Hamiltonian systems has attracted little attention.

In this chapter, we give several necessary conditions for persistence of periodic or homoclinic orbits, first integrals or commutative vector fields in (3.1.1). In particular, we treat homoclinic orbits not only to equilibria but also to periodic orbits. Moreover, we see that persistence of periodic or homoclinic orbits and first integrals or commutative vector fields near them have the same necessary conditions (cf. Theorems 3.2.1-3.2.4, 3.3.5, 3.3.8, 3.3.10 and 3.3.12). This indicates close relationships between the dynamics and geometry of the perturbed systems. We also discuss some relationships of these results with the standard subharmonic and homoclinic Melnikov methods [26, 42, 62, 65] for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems as in [44], and with another version of the homoclinic Melnikov method due to Wiggins [61] for autonomous Hamiltonian perturbations of multi-degree-of-freedom integrable Hamiltonian systems. The subharmonic Melnikov method provides a sufficient condition for persistence of periodic orbits in the perturbed system: If the subharmonic Melnikov functions have a simple zero, then such orbits persist. For the latter homoclinic Melnikov method, we restrict ourselves to the case in which the unperturbed systems have invariant manifolds consisting of periodic orbits to which there exist homoclinic orbits since only such a situation can be treated in our result, although the technique was developed for more general systems. These versions of the Melnikov methods are described shortly in Section 3.4 below. In particular, we show that a first integral which converges to the Hamiltonian or another first integral as $\varepsilon \to 0$ does not exist near the unperturbed periodic or homoclinic orbits in the perturbed system for $\varepsilon > 0$ sufficiently small if the subharmonic or homoclinic Melnikov functions are not identically zero on connected open sets. We illustrate our theory for four examples: The periodically forced Duffing oscillator [26, 62], two identical pendula coupled with a harmonic oscillator, a periodically forced rigid body [72] and a three-mode truncation of a buckled beam [67]. The persistence of first integrals is discussed in the first and second examples, the persistence of a first integral and periodic orbits in the third one and the persistence of commutaive vector fields in the fourth one.

The outline of this chapter is as follows: In Sections 3.2 and 3.3, we present our main results for first integrals and commutative vector fields, respectively, as well as for both of periodic and homoclinic orbits. For the reader's convenience, in Appendix 3.A, we collect basic notions and facts on connections of vector bundles and linear differential equations as auxiliary materials for Section 3.3. In Section 3.4, we describe some relationships of the main results with the subharmonic and homoclinic Melnikov methods when the unperturbed system (3.1.2) is a single-degree-of-freedom Hamiltonian system. Finally, we give the four examples to illustrate our theory in Section 3.5.

3.2 First Integrals

In this section, we discuss persistence of periodic and homoclinic orbits and first integrals for (3.1.1). In the discussion here, less smoothness of $\mathscr{M}$ and X_ε is needed: $\mathscr{M}$ and X_ε are C^3 and C^2, respectively.

3.2.1 Periodic orbits

We begin with a case in which the unperturbed system (3.1.2) has a periodic orbit in (3.1.2). We make the following assumptions on (3.1.2):

(A1) There exists a T-periodic orbit $\gamma(t)$ for some constant $T>0$ in (3.1.2);

(A2) There exists a non-constant C^3 first integral $F(x)$ of (3.1.2), i.e.,

$$dF(X^0)=0,$$

near $\Gamma=\{\gamma(t)\mid t\in[0,T)\}$.

Define

$$\mathscr{I}_{F,\gamma}:=\int_0^T dF(X^1)(\gamma(t))dt. \tag{3.2.1}$$

We state our main results for persistence of periodic orbits and first integrals.

Theorem 3.2.1. *Assume that (A1) and (A2) hold. If the perturbed system* (3.1.1) *has a T_ε-periodic orbit γ_ε depending C^2-smoothly on ε such that $T_0=T$ and $\gamma_0=\gamma$, then the integral $\mathscr{I}_{F,\gamma}$ must be zero.*

Proof. Assume that (A1) and (A2) hold and the system (3.1.1) has a periodic orbit $\gamma_\varepsilon=\gamma+O(\varepsilon)$ for $\varepsilon>0$. Since γ_ε is a T_ε-periodic orbit in (3.1.1), we compute

$$\int_0^{T_\varepsilon} dF(X_\varepsilon)(\gamma_\varepsilon(t))dt=F(\gamma_\varepsilon(T_\varepsilon))-F(\gamma_\varepsilon(0))=0.$$

On the other hand, since F is a first integral of X^0, we have $dF(X^0)=0$, so that

$$dF(X_\varepsilon)(\gamma_\varepsilon(t))=\varepsilon dF(X^1)(\gamma(t))+O(\varepsilon^2).$$

Since $T_\varepsilon=T+O(\varepsilon)$, we see by the above two equations that

$$\int_0^{T_\varepsilon} dF(X_\varepsilon)(\gamma_\varepsilon(t))dt=\varepsilon\int_0^T dF(X^1)(\gamma(t))dt+O(\varepsilon^2)=\varepsilon\mathscr{I}_{F,\gamma}+O(\varepsilon^2)=0.$$

Thus, we obtain $\mathscr{I}_{F,\gamma}=0$. $\square$

Theorem 3.2.2. *Assume that (A1) and (A2) hold. If the perturbed system* (3.1.1) *has a* C^3 *first integral* $F_\varepsilon(x)$ *depending* C^2*-smoothly on* ε *near* Γ *such that* $F_0(x) = F(x)$*, then the integral* $\mathscr{I}_{F,\gamma}$ *must be zero.*

Proof. Assume that (A1) and (A2) hold and the system has a first integral $F_\varepsilon = F + \varepsilon F^1 + O(\varepsilon^2)$ near Γ. Since γ is a T-periodic orbit in (3.1.2), we have

$$\int_0^T dF_\varepsilon(X_0)(\gamma(t))dt = F_\varepsilon(\gamma(T)) - F_\varepsilon(\gamma(0)) = 0. \tag{3.2.2}$$

On the other hand, since $dF_\varepsilon(X_\varepsilon) = 0$ and

$$dF_\varepsilon(X_\varepsilon) = dF_\varepsilon(X^0) + \varepsilon dF_\varepsilon(X^1) + O(\varepsilon^2),$$

we have

$$dF_\varepsilon(X^0) = -\varepsilon dF_\varepsilon(X^1) + O(\varepsilon^2) \tag{3.2.3}$$

near Γ. From (3.2.2) and (3.2.3) we obtain

$$\begin{aligned}\int_0^T dF_\varepsilon(X^0)(\gamma(t))dt &= -\,\varepsilon \int_0^T dF_\varepsilon(X^1)(\gamma(t))dt + O(\varepsilon^2)\\ &= -\,\varepsilon \mathscr{I}_{F,\gamma} + O(\varepsilon^2) = 0,\end{aligned}$$

which yields the desired result. $\square$

Theorems 3.2.1 and 3.2.2 mean that if $\mathscr{I}_{F,\gamma} \neq 0$, then neither the periodic orbit γ nor first integral F persists in (3.1.1) for $\varepsilon > 0$.

3.2.2 Homoclinic orbits

We next consider a case in which the unperturbed system (3.1.2) has a homoclinic orbit to an equilibrium or to a periodic orbit in (3.1.2). Instead of (A1) and (A2), we assume the following on (3.1.2):

(A1') There exists a homoclinic orbit $\gamma^{\mathrm{h}}(t)$ to a T-periodic orbit $\gamma^{\mathrm{p}}(t)$ in (3.1.2);

(A2') There exists a non-constant C^3 first integral $F(x)$ of (3.1.2) near $\Gamma^{\mathrm{h}} = \{\gamma^{\mathrm{h}}(t) \mid t \in \mathbb{R}\} \cup \Gamma^{\mathrm{p}}$, where $\Gamma^{\mathrm{p}} = \{\gamma^{\mathrm{p}}(t) \mid t \in [0, T)\}$.

In assumption (A1') γ^{p} may be an equilibrium. As seen below we have statements similar to Theorems 3.2.1 and 3.2.2 in this case but another idea is needed for their proofs since the situation is not simple when $\gamma^{\mathrm{h}}(t)$ is a homoclinic orbit to a periodic orbit.

We first define an integral which plays a similar role as $\mathscr{I}_{F,\gamma}$ in Section 3.2.1 (see Eq. (3.2.1)). Let γ^{p} be not an equilibrium. Choose a point $x_0 = \gamma^{\mathrm{p}}(0)$ and take an $(n-1)$-dimensional hypersurface Σ as the Poincaré section such that γ^{p} intersects Σ transversely at x_0. Restricting Σ to a sufficiently small neighborhood of x_0 if necessary, we can assume that $\gamma^{\mathrm{h}}(t)$ intersects Σ transversely infinitely many times, say at $T_j \in \mathbb{R}$

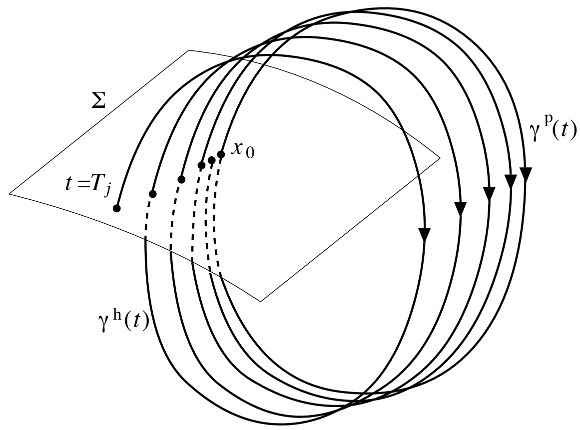

Figure 3.1: Poincaré section Σ.

with $T_{j-1} < T_j$, $j \in \mathbb{Z}$, such that $\lim_{j\to-\infty} T_j = -\infty$ and $\lim_{j\to+\infty} T_j = +\infty$, since it converges to $\gamma^{\rm p}(t)$. In particular,

$$\lim_{j\to\pm\infty} \gamma^{\rm h}(T_j) = x_0.$$

See Fig. 3.1. So we formally define

$$\tilde{\mathscr{I}}_{F,\gamma^{\rm h}} := \lim_{k\to+\infty} \int_{T_{-k}}^{T_k} dF(X^1)(\gamma^{\rm h}(t))dt. \tag{3.2.4}$$

If $\gamma_\varepsilon^{\rm p}$ is an equilibrium, then Eq. (3.2.4) is reduced to

$$\tilde{\mathscr{I}}_{F,\gamma^{\rm h}} := \int_{-\infty}^{\infty} dF(X^1)(\gamma^{\rm h}(t))dt \tag{3.2.5}$$

by taking any sequence $\{T_j\}_{-\infty}^{\infty}$ such that $\lim_{j\to\pm\infty} T_j = \pm\infty$.

We now state our main results for persistence of homoclinic orbits and first integrals.

Theorem 3.2.3. *Assume that (A1') and (A2') hold and that there exists a periodic orbit $\gamma_\varepsilon^{\rm p}$ depending C^2-smoothly on ε in* (3.1.1) *such that $\gamma_0^{\rm p} = \gamma^{\rm p}$. If the perturbed system* (3.1.1) *has a homoclinic orbit $\gamma_\varepsilon^{\rm h}$ to $\gamma_\varepsilon^{\rm p}$ depending C^2-smoothly on ε such that $\gamma_0^{\rm h} = \gamma^{\rm h}$, then the limit in the right hand side of* (3.2.4) *exists and $\tilde{\mathscr{I}}_{F,\gamma^{\rm h}} = 0$.*

Proof. Assume that the hypotheses of this theorem hold, $\gamma^{\rm p}$ is not an equilibrium but periodic orbit, and the system (3.1.1) has a homoclinic orbit $\gamma_\varepsilon^{\rm h} = \gamma^{\rm h} + O(\varepsilon)$ to a periodic orbit $\gamma_\varepsilon^{\rm p} = \gamma^{\rm p} + O(\varepsilon)$. For $\varepsilon > 0$ sufficiently small, the periodic orbit $\gamma_\varepsilon^{\rm p}$ intersects the Poincaré section Σ transversely, say at $t = 0$. Similarly, $\gamma_\varepsilon^{\rm h}$ intersects Σ transversely infinitely many times, say at $T_j^\varepsilon \in \mathbb{R}$ with $T_{j+1}^\varepsilon < T_j^\varepsilon$, $j \in \mathbb{Z}$, such that $\lim_{j\to\pm\infty} T_j^\varepsilon = \pm\infty$. Moreover,

$$\lim_{j\to\pm\infty} \gamma^{\rm h}(T_j^\varepsilon) = x_\varepsilon := \gamma_\varepsilon^{\rm p}(0).$$

We easily see that

$$\lim_{k\to+\infty} \int_{T_{-k}^\varepsilon}^{T_k^\varepsilon} dF(X_\varepsilon)(\gamma_\varepsilon^{\rm h}(t))dt = \lim_{k\to+\infty} \big(F(\gamma_\varepsilon^{\rm h}(T_k^\varepsilon)) - F(\gamma_\varepsilon^{\rm h}(T_{-k}^\varepsilon))\big) = 0. \tag{3.2.6}$$

Introduce a metric in a neighborhood of x_0 using the standard Euclidean one in the coordinates. For $\delta > 0$ sufficiently small, let $k > 0$ be an integer such that $\gamma^{\mathrm{h}}(T_{\pm j})$ lie in a δ-neighborhood of x_0 for $j > k$. We can choose $\varepsilon > 0$ sufficiently small such that on $[T^\varepsilon_{-k}, T^\varepsilon_k]$

$$\gamma^{\mathrm{h}}_\varepsilon(t) = \gamma^{\mathrm{h}}(t) + O(\varepsilon),$$

which yields $T^\varepsilon_j = T_j + O(\varepsilon)$ for $|j| \le k$ and

$$dF(X_\varepsilon)(\gamma^{\mathrm{h}}_\varepsilon(t)) = \varepsilon dF(X^1)(\gamma^{\mathrm{h}}(t)) + O(\varepsilon^2)$$

since $dF(X^0) = 0$. Hence,

$$\int_{T^\varepsilon_{-k}}^{T^\varepsilon_k} dF(X_\varepsilon)(\gamma^{\mathrm{h}}_\varepsilon(t))dt = \varepsilon \int_{T^\varepsilon_{-k}}^{T^\varepsilon_k} dF(X^1)(\gamma^{\mathrm{h}}(t))dt + O(\varepsilon^2). \tag{3.2.7}$$

Taking $\delta \to 0$, we have $T^\varepsilon_{\pm k} \to \pm\infty$, so that by (3.2.6) and (3.2.7) the limit in the right hand side of (3.2.4) exists and it must be zero. □

Theorem 3.2.4. *Assume that (A1') and (A2') hold. If the perturbed system* (3.1.1) *has a C^3 first integral F_ε depending C^2-smoothly on ε near Γ^{h} such that $F_0 = F$, then the limit in the right hand side of* (3.2.4) *exists and* $\tilde{\mathscr{I}}_{F,\gamma^{\mathrm{h}}} = 0$.

Proof. Assume that the hypotheses of the theorem hold, γ^{p} is not an equilibrium but periodic orbit, and the system (3.1.1) has a first integral $F_\varepsilon = F + \varepsilon F^1 + O(\varepsilon^2)$ near Γ^{h}. We compute

$$\lim_{k\to+\infty} \int_{T_{-k}}^{T_k} dF_\varepsilon(X^0)(\gamma^{\mathrm{h}}(t))dt = \lim_{k\to+\infty} \big(F_\varepsilon(\gamma^{\mathrm{h}}(T_k)) - F_\varepsilon(\gamma^{\mathrm{h}}(T_{-k}))\big) = 0. \tag{3.2.8}$$

On the other hand, by (3.2.3)

$$\int_{T_{-k}}^{T_k} dF(X^0)(\gamma^{\mathrm{h}}(t))dt = -\varepsilon \int_{T_{-k}}^{T_k} dF(X^1)(\gamma^{\mathrm{h}}(t))dt + O(\varepsilon^2). \tag{3.2.9}$$

As in the proof of Theorem 3.2.2, it follows from (3.2.8) and (3.2.9) that the limit in the right hand side of (3.2.4) exists and it must be zero. □

Remark 3.2.5.

(i) In the proofs of Theorems 3.2.3 and 3.2.4, when γ^{p} is an equilibrium, we only have to choose any strictly monotonically increasing and diverging sequences $\{T^\varepsilon_j\}^\infty_{-\infty}$, $\{T_j\}^\infty_{-\infty}$ such that $T^\varepsilon_j = T_j + O(\varepsilon)$, $j \in \mathbb{Z}$, and to apply the same arguments.

(ii) In Theorem 3.2.3, if the periodic orbit (or equilibrium) γ^{p} is hyperbolic, then the condition on existence of $\gamma^{\mathrm{p}}_\varepsilon$ is not needed since such a periodic orbit (or equlibrium) necessarily exists.

Theorems 3.2.3 and 3.2.4 mean that if $\tilde{\mathscr{I}}_{F,\gamma^{\mathrm{h}}} \neq 0$, then neither the homoclinic orbit γ^{h} nor first integral F persists in (3.1.1) for $\varepsilon > 0$.

3.3 Commutative Vector Fields

In this section, we discuss persistence of periodic and homoclinic orbits and commutative vector fields for (3.1.1).

3.3.1 Variational and adjoint variational equations

Before stating the main results, we give some preliminary results on variational and adjoint variational equations. A similar treatment in a complex setting are found in [11, 22, 47]. For the reader's convenience, some auxiliary materials are provided in Appendix 3.A.

Let $\mathscr{M}$ be an n-dimensional paracompact oriented C^3 real manifold as in Section 3.2. Let X be a C^2 vector field on $\mathscr{M}$ and let Γ_ϕ be an integral curve given by a non-stationary solution $x = \phi(t)$ to the associated differential equation

$$\dot{x} = X(x). \tag{3.3.1}$$

The immersion $i : \Gamma_\phi \to \mathscr{M}$ induces a subbundle $T_{\Gamma_\phi} := i^* T\mathscr{M}$ of the vector bundle $T\mathscr{M}$, where i^* represents the pullback of i. Let $s : \Gamma_\phi \to T_{\Gamma_\phi}$ be a C^1 section of T_{Γ_ϕ}. We define the *variational equation* (VE) of X along Γ_ϕ as

$$\nabla s := dt \otimes \mathcal{L}_X Y|_{\Gamma_\phi} = 0, \tag{3.3.2}$$

where "$\otimes$" represents the tensor product, Y is any C^1 vector field extension of s to $\mathscr{M}$, $\mathcal{L}_X$ represents the Lie derivative along X, and "$dt\otimes$" is frequently omitted in references. Here ∇ is a connection of T_{Γ_ϕ}, and s is a horizontal section of ∇ if it satisfies the VE (3.3.2) (see Appendix 3.A.1). Locally, Eq. (3.3.2) is expressed as

$$\frac{d\hat{U}}{dt} = \frac{\partial X}{\partial x}(\phi(t))\hat{U} \tag{3.3.3}$$

in the frame $\left(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}\right)$ associated with the coordinates $(x_1, ..., x_n)$, where

$$s = \sum_{j=1}^{n} \Xi_j \frac{\partial}{\partial x_j}.$$

See Appendix 3.A.2 for the derivation of (3.3.3).

Let $T^*_{\Gamma_\phi}$ be the dual bundle of T_{Γ_ϕ}, and let $\alpha : \Gamma_\phi \to T^*_{\Gamma_\phi}$ be a C^1 section of $T^*_{\Gamma_\phi}$.

Lemma 3.3.1. *The dual connection ∇^* of ∇ (see Appendix 3.A.1) is given by*

$$\nabla^* \alpha = dt \otimes \mathcal{L}_X \omega|_{\Gamma_\phi}, \tag{3.3.4}$$

where $\omega : \mathscr{M} \to T^*\mathscr{M}$ is any C^1 differential 1-form extension of α.

Proof. Let s be a section of $T_{\Gamma_\phi}$and let Y be its vector field extension as above. The Lie derivative $\mathcal{L}_X$ satisfies

$$\mathcal{L}_X \langle Y, \omega \rangle = \langle \mathcal{L}_X Y, \omega \rangle + \langle Y, \mathcal{L}_X \omega \rangle,$$

which yields

$$dt \otimes \mathcal{L}_X \langle Y, \omega \rangle|_{\Gamma_\phi} = \langle \nabla s, \alpha \rangle + \langle s, dt \otimes \mathcal{L}_X \omega|_{\Gamma_\phi} \rangle$$

when restricted to Γ_ϕ, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing by the duality. On the other hand, since Γ_ϕ is an integral curve of X, we have

$$d\langle s, \alpha \rangle = d\langle Y, \omega \rangle|_{\Gamma_\phi} = dt \otimes X(\langle Y, \omega \rangle)|_{\Gamma_\phi} = dt \otimes \mathcal{L}_X \langle Y, \omega \rangle|_{\Gamma_\phi}.$$

By definition, we obtain (3.3.4). □

We call

$$\nabla^* \alpha = 0 \tag{3.3.5}$$

the *adjoint variational equation* (AVE) of X along Γ_ϕ. Thus, α is a horizontal section of ∇^* if it satisfies the AVE (3.3.5). Locally, Eq. (3.3.5) is expressed as

$$\frac{d\eta}{dt} = -\left(\frac{\partial X}{\partial x}(\phi(t)) \right)^{\mathrm{T}} \eta \tag{3.3.6}$$

in the frame $(dx_1, \dots, dx_n)$, where the superscript "T" represents the transpose operator and

$$\alpha = \sum_{j=1}^{n} \eta_j dx_j.$$

See Appendix 3.A.2 for the derivation of (3.3.6).

Lemma 3.3.2. (i) *If X has a first integral F, then the following hold:*

(ia) The section $\alpha = dF|_{\Gamma_\phi}$ of $T^*_{\Gamma_\phi}$ satisfies the AVE (3.3.5) *of X along Γ_ϕ;*

(ib) $\langle s, dF|_{\Gamma_\phi} \rangle$ is a first integral of the VE (3.3.2) *of X along Γ_ϕ, i.e.,*

$$d\langle s, dF|_{\Gamma_\phi} \rangle = 0$$

if the section s of T_{Γ_ϕ} satisfies (3.3.2).

(ii) *If X has a commutative vector field Z, i.e.,*

$$[X, Z] = 0,$$

where $[\cdot, \cdot]$ denotes the Lie bracket, then the following hold:

(iia) The section $s = Z|_{\Gamma_\phi}$ of T_{Γ_ϕ} satisfies the VE (3.3.2);

(iib) $\langle Z|_{\Gamma_\phi}, \alpha \rangle$ is a first integral of the AVE (3.3.5), *i.e.,*

$$d\langle Z|_{\Gamma_\phi}, \alpha \rangle = 0$$

if the section α of $T^*_{\Gamma_\phi}$ satisfies (3.3.5).

Proof. Let s and α satisfy the VE (3.3.2) and AVE (3.3.5), respectively. Since

$$d\langle s, \alpha\rangle = \langle \nabla s, \alpha\rangle + \langle s, \nabla^* \alpha\rangle = \langle 0, \alpha\rangle + \langle s, 0\rangle = 0,$$

we see that $\langle s, \alpha\rangle$ is a constant. Hence, parts (ib) and (iib) immediately follow from (iia) and (ia), respectively.

Now we show (ia) and (iia). If X has a first integral F, then by Cartan's formula (see, e.g., Theorem 4.2.3 of [41]) we have

$$\mathcal{L}_X dF = d(i_X(dF)) + i_X(d^2F) = 0,$$

where i_X denotes the interior product of X. This yields (ia) when restricted to Γ_ϕ. If X has a commutative vector field Z, then we obtain (iia) since $\mathcal{L}_X Z|_{\Gamma_\phi} = [X, Z]|_{\Gamma_\phi} = 0$. □

Similar results to Lemma 3.3.2 for symplectic connections can be proven by using the musical isomorphism of symplectic forms (see Lemma 4.1 of [47] and Chapter 4 of [52]).

3.3.2 Periodic orbits

We turn to the issue of persistence of periodic orbits and commutative vector fields in (3.1.1). Instead of (A2), we assume the following on (3.1.2):

(A3) The unperturbed system (3.1.2) has a C^3 commutative vector field Z, i.e.,

$$[X^0, Z] = 0,$$

near Γ, such that it is linearly independent of X^0.

Lemma 3.3.3. *Under assumption (A1), the connection ∇^* of T^*_Γ has a nontrivial horizontal section $\omega : \Gamma \to T^*_\Gamma$, i.e., ω satisfies the AVE (3.3.5) of X^0 along Γ.*

Proof. Let ψ^t denote the flow of X^0 and let $x_0 = \gamma(0) \in \Gamma$. For any $p \in \Gamma$, there exists a unique time $t_p \in [0, T)$ such that $\gamma(t_p) = \psi^{t_p}(x_0) = p$. Define $\theta : \Gamma \to [0, T)$ by $\theta(p) := t_p$. Since $\theta \circ \gamma = \mathrm{id}$, we have

$$\frac{d}{dt}\theta(\gamma(t)) = 1, \tag{3.3.7}$$

where id represents the identity map. On the other hand, by the tubular neighborhood theorem (e.g., Theorem 5.2 in Chapter 4 of [30]), there is a neighborhood $\mathscr{N}(\Gamma)$ of Γ which is diffeomorphic to the normal bundle N_Γ of Γ in $\mathscr{M}$. Let $f : \mathscr{N}(\Gamma) \to N_\Gamma$ be the diffeomorphism, and let $\pi : N_\Gamma \to \Gamma$ be the natural projection. Define a map $\Theta : \mathscr{N}(\Gamma) \to \mathbb{R}$ by $\Theta := f^*\pi^*\theta$. Since $f|_\Gamma = \mathrm{id}$ and $\pi|_\Gamma = \mathrm{id}$, we have

$$\Theta|_\Gamma = \theta. \tag{3.3.8}$$

Using (3.3.7) and (3.3.8), we show that for $x \in \Gamma$

$$(i_X d\Theta)_x = (\mathcal{L}_X \Theta)_x = \lim_{t\to 0} \frac{\Theta(\psi^t(x)) - \Theta(x)}{t} = \lim_{t\to 0} \frac{\theta(\psi^t(x)) - \theta(x)}{t} = 1,$$

which yields

$$\mathcal{L}_X(d\Theta)|_\Gamma = (i_X d^2\Theta + d i_X d\Theta)|_\Gamma = d((i_X d\Theta)|_\Gamma) = 0$$

by Cartan's formula. Hence, by Lemma 3.3.1 we see that $\omega = d\Theta|_\Gamma$ is a nontrivial horizontal section of ∇^* since θ is not a constant. □

Remark 3.3.4.

(i) *Let $\mathscr{M} = \mathbb{R}^n$. We see that $\dot\gamma(t)$ is a periodic solution to the VE* (3.3.3) *and consequently its Floquet exponents (see, e.g., Section 2.4 of [21]) include one. Hence, the AVE* (3.3.6) *possesses one as its Floquet exponent and consequently it has a periodic solution, which provides a horizontal section of ∇^* as guaranteed by Lemma 3.3.3.*

(ii) *Assume that (A1) and (A2) hold. From Lemma 3.3.2 (ia) we see that $dF|_\Gamma$ is a horizontal section of ∇^*.*

Let ω be a horizontal section of ∇^* as stated in Lemma 3.3.3, and define the integral

$$\mathscr{I}_{\omega,Z,\gamma} := \int_0^T \omega([X^1, Z])(\gamma(t))dt. \tag{3.3.9}$$

We now state our result on persistence of commutative vector fields.

Theorem 3.3.5. *Assume that (A1) and (A3) hold. If the perturbed system* (3.1.1) *has a C^3 commutative vector field Z_ε depending C^2-smoothly on ε near Γ such that $Z_0 = Z$, then the integral $\mathscr{I}_{\omega,Z,\gamma}$ is zero.*

For the proof of Theorems 3.3.5 we use the cotangent lift trick [12], and rewrite (3.1.1) as a Hamiltonian system. In this situation the persistence of commutative vector fields of (3.1.1) is reduced to that of first integrals of the lifted Hamiltonian system. We first explain the trick in a general setting, following [12]. See Chapter 5 of [41] for necessary information on Hamiltonian mechanics.

Let $T^*\mathscr{M}$ be the cotangent bundle of $\mathscr{M}$ and let $\pi : T^*\mathscr{M} \to \mathscr{M}$ be the natural projection. Define a differential 1-form $\lambda : T^*\mathscr{M} \to T^*(T^*\mathscr{M})$, which is often called a *(Poincaré-)Liouville form*, as

$$\lambda_z = p(d\pi_z(\cdot)),$$

where $z = (x, p) \in T^*\mathscr{M}$. Letting $\Omega_0 = d\lambda$, we have a symplectic manifold $(T^*\mathscr{M}, \Omega_0)$. In the local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$, λ and Ω_0 are written as

$$\lambda = \sum_{k=1}^n p_k dx_k \quad \text{and} \quad \Omega_0 = \sum_{k=1}^n dp_k \wedge dx_k,$$

respectively.

Let X be a smooth vector field on $\mathscr{M}$, and define a function $h_X : T^*\mathscr{M} \to \mathbb{R}$ as

$$h_X(x, p) = \langle p, X(x)\rangle, \tag{3.3.10}$$

where $(x, p) \in T^*\mathscr{M}$. Then the Hamiltonian vector field $\hat X$ with the Hamiltonian h_X on the symplectic manifold $(T^*\mathscr{M}, \Omega_0)$ is called the *cotangent lift* of X. Note that the smoothness of $\hat X$ is less by one than that of X. In the local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$, the vector field $\hat X$ is expressed as

$$\frac{dx}{dt} = X(x)\left(= \frac{\partial h_X}{\partial p}\right), \quad \frac{dp}{dt} = -\frac{\partial X(x)}{\partial x}^{\mathrm{T}} p\left(= -\frac{\partial h_X}{\partial x}\right), \tag{3.3.11}$$

the second equation of which has the same form as the AVE (3.3.6) when $x = \phi(t)$.

Lemma 3.3.6. *For any vector fields X and Z on $\mathscr{M}$ we have*

$$\{h_X, h_Z\} = h_{[X,Z]}$$

(see Eq. (3.3.10)*), where $\{\cdot,\cdot\}$ denotes the Poisson bracket for the symplectic form Ω_0.*

Proof. In the local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$, we write

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}, \quad Z = \sum_{j=1}^{n} Z_j \frac{\partial}{\partial x_j}, \quad p = \sum_{l=1}^{n} p_l \, dx^l.$$

We compute

$$\begin{aligned}
\{h_X, h_Z\} =& \{\langle p, X(x)\rangle, \langle p, Z(x)\rangle\} = \left\{ \sum_{i=1}^{n} p_i X_i(x), \sum_{j=1}^{n} p_j Z_j(x) \right\} \\
=& \sum_{k=1}^{n} \left(X_k \sum_{i=1}^{n} p_i \frac{\partial Z_i}{\partial x_k} - Z_k \sum_{j=1}^{n} p_j \frac{\partial X_j}{\partial x_k} \right) \\
=& \sum_{i=1}^{n} p_i \sum_{k=1}^{n} \left(X_k \frac{\partial Z_i}{\partial x_k} - Z_k \frac{\partial X_i}{\partial x_k} \right) = \langle p, [X, Z]\rangle = h_{[X,Z]},
\end{aligned}$$

which yields the desired result. □

We also need the following fact, which was used in the proof of Proposition 2 of [12].

Lemma 3.3.7. *If Z is a commutative vector field of X, then h_Z is a first integral for the cotangent lift $\hat{X}$ of X.*

Proof. It follows from Lemma 3.3.6 that $dh_Z(\hat{X}) = \{h_X, h_Z\} = h_{[X,Z]}$. Hence, $dh_Z(\hat{X}) = 0$ if $[X, Z] = 0$. □

We are now in a position to prove Theorem 3.3.5.

Proof of Theorem 3.3.5. Assume that the hypotheses of the theorem hold. Let $\hat{X}_\varepsilon$ be the cotangent lift of X_ε. By Lemma 3.3.3 there exists a section ω of T^*_Γ satisfying the AVE (3.3.5) of X^0 along Γ and $\hat{\gamma}(t) = (\gamma(t), \omega_{\gamma(t)})$ is a T-periondic solution for $\hat{X}_0$. Moreover, by Lemma 3.3.7 $\hat{X}_0$ has a first integral h_Z.

Suppose that the system (3.1.1) has a commutative vector field $Z_\varepsilon = Z + O(\varepsilon)$ near Γ. Then by Lemma 3.3.7 $h_{Z_\varepsilon} = h_Z + O(\varepsilon)$ is a first integral of $\hat{X}_\varepsilon$ near $\hat{\Gamma} = \{\hat{\gamma}(t) \mid t \in [0, T)\}$. Using Lemma 3.3.6, we compute

$$\begin{aligned}
\mathscr{I}_{h_Z,\hat{\gamma}} =& \int_0^T dh_Z(\hat{X}^1)(\hat{\gamma}(t))dt = \int_0^T \{h_{X^1}, h_Z\}(\hat{\gamma}(t))dt \\
=& \int_0^T h_{[X^1,Z]}(\hat{\gamma}(t))dt = \int_0^T \langle \omega, [X^1, Z]\rangle_{\gamma(t)} dt \\
=& \int_0^T \omega([X^1, Z])_{\gamma(t)} dt = \mathscr{J}_{\omega,Z,\gamma}
\end{aligned} \tag{3.3.12}$$

for $\hat{X}_\varepsilon$. We apply Theorem 3.2.2 to complete the proof. □

Theorem 3.3.5 means that if $\mathscr{I}_{\omega,Z,\gamma} \neq 0$, then the commutative vector field Z does not persist in (3.1.1) for $\varepsilon > 0$.

As in the proof of Theorem 3.3.5, we see that if $\gamma_\varepsilon(t)$ is a T_ε-periodic orbit in (3.1.1), then by Lemma 3.3.3 there exists a section $\omega_\varepsilon = \omega + O(\varepsilon)$ of $T^*_{\Gamma_\varepsilon}$ satisfying the AVE (3.3.5) of X_ε along Γ_ε and $\hat{\gamma}(t) = (\gamma_\varepsilon(t), \omega_{\varepsilon,\gamma_\varepsilon(t)})$ is a T_ε-periodic orbit for the cotangent lift $\hat{X}_\varepsilon$ of X_ε, where $\Gamma_\varepsilon = \{\gamma_\varepsilon(t) \mid t \in [0, T_\varepsilon)\}$. Here the section ω of T^*_Γ satisfies the AVE (3.3.5) of X^0 along Γ. Applying Theorem 3.2.1 to $\hat{X}_\varepsilon$ and using (3.3.12), we obtain the following result on persistence of periodic orbits.

Theorem 3.3.8. *Assume that (A1) and (A3) hold. If the perturbed system* (3.1.1) *has a T_ε-periodic orbit γ_ε depending C^2-smoothly on ε such that $T_0 = T$ and $\gamma_0 = \gamma$, then the integral $\mathscr{I}_{\omega,Z,\gamma}$ is zero for some section ω of T^*_Γ satisfying the AVE* (3.3.5) *of X^0 along Γ.*

Theorem 3.3.8 means that if $\mathscr{I}_{\omega,Z,\gamma} \neq 0$ for any horizontal section ω of ∇^*, then the periodic orbit γ does not persist in (3.1.1) for $\varepsilon > 0$.

3.3.3 Homoclinic orbits

We next discuss the persistence of homoclinic orbits and commutative vector fields in (3.1.1). Instead of (A3) we assume the following on (3.1.2):

(A3') The unperturbed system (3.1.2) has a C^3 commutative vector field Z near Γ^{h}, such that it is linearly independent of X^0.

In the proof of Lemma 3.3.3, we did not essentially use the fact that $\gamma(t)$ is periodic. So we prove the following lemma similarly.

Lemma 3.3.9. *Under assumption (A1'), the connection ∇^* of $T^*_{\Gamma^{\mathrm{h}\prime}}$ has a nontrivial horizontal section $\omega^{\mathrm{h}} : \Gamma^{\mathrm{h}\prime} \to T^*_{\Gamma^{\mathrm{h}\prime}}$, i.e., ω^{h} satisfies the AVE* (3.3.5) *of X^0 along $\Gamma^{\mathrm{h}\prime}$ where $\Gamma^{\mathrm{h}\prime} := \{\gamma^{\mathrm{h}}(t) \mid t \in \mathbb{R}\}$.*

Let ω^{h} be such a horizontal section of ∇^* as stated in Lemma 3.3.9, and define

$$\tilde{\mathscr{I}}_{\omega^{\mathrm{h}},Z,\gamma^{\mathrm{h}}} := \lim_{k\to+\infty} \int_{T_{-k}}^{T_k} \omega^{\mathrm{h}}([X^1, Z])_{\gamma^{\mathrm{h}}(t)} dt, \tag{3.3.13}$$

where the sequence $\{T_j\}_{j=-\infty}^{\infty}$ is taken as in (3.2.4). If γ^{p} is an equilibrium, then Eq. (3.3.13) is reduced to

$$\tilde{\mathscr{I}}_{\omega^{\mathrm{h}},Z,\gamma^{\mathrm{h}}} = \int_{-\infty}^{\infty} \omega^{\mathrm{h}}([X^1, Z])_{\gamma^{\mathrm{h}}(t)} dt \tag{3.3.14}$$

like (3.2.5).

Theorem 3.3.10. *Assume that (A1') and (A3') hold. If the perturbed system* (3.1.1) *has a C^3 commutative vector field Z_ε depending C^2-smoothly on ε near Γ^{h} such that $Z_0 = Z$, then the limit in the right hand side of* (3.3.13) *exists and $\tilde{\mathscr{I}}_{\omega^{\mathrm{h}},Z,\gamma} = 0$.*

Proof. If γ^{p} is a periodic orbit, then by Lemma 3.3.3 there exists a horizontal section ω^{p} of $T^*_{\Gamma^{\mathrm{p}}}$ satisfying the AVE (3.3.5) of X^0 along Γ^{p} and $(\gamma^{\mathrm{p}}(t), \omega^{\mathrm{p}}_{\gamma^{\mathrm{p}}(t)})$ is a periodic orbit for the cotangent lift $\hat{X}^0$ of X^0. Similarly, by assumptions (A1') and Lemma 3.3.9, we have a homoclinic orbit $(\gamma^{\mathrm{h}}(t), \omega^{\mathrm{h}}_{\gamma^{\mathrm{h}}(t)})$ to the periodic orbit $(\gamma^{\mathrm{p}}(t), \omega^{\mathrm{p}}_{\gamma^{\mathrm{p}}(t)})$ for $\hat{X}^0$, where ω^{h} is a horizontal section of ∇^* for $T^*_{\Gamma^{\mathrm{h}}}$. By applying Theorem 3.2.4 to the cotangent lift $\hat{X}_\varepsilon$ of X_ε, the rest of the proof is done similarly as in Theorem 3.3.5. $\square$

Theorem 3.3.10 means that if $\mathscr{J}_{\omega^{\mathrm{h}},Z,\gamma^{\mathrm{h}}} \neq 0$, then the commutative vector field Z does not persist in (3.1.1) for $\varepsilon > 0$.

Remark 3.3.11. *Using Theorems 3.2.2, 3.2.4, 3.3.5 and 3.3.10, we can determine whether given first integrals and commutative vector fields do not persist in* (3.1.1) *but there still exist a sufficient number of first integrals and commutative vector fields depending smoothly on the parameter ε. For example, the unperturbed system* (3.1.2) *may have different first integrals and commutative vector fields which persist. So we have to overcome this difficulty to extend the results of Poincaré [52] and Kozlov [35, 36] and obtain a sufficient condition for such nonintegrability of the perturbed systems.*

As in the proof of Theorem 3.3.10, if $\gamma^{\mathrm{p}}_\varepsilon = \gamma^{\mathrm{p}} + O(\varepsilon)$ is a T_ε-periodic orbit with $T_\varepsilon = T + O(\varepsilon)$ and $\gamma^{\mathrm{h}}_\varepsilon = \gamma^{\mathrm{h}} + O(\varepsilon)$ is a homoclinic orbit to $\gamma^{\mathrm{p}}_\varepsilon$ in (3.1.1), then by Lemmas 3.3.3 and 3.3.9 there exist horizontal sections $\omega^{\mathrm{p}}_\varepsilon = \omega^{\mathrm{p}} + O(\varepsilon)$ and $\omega^{\mathrm{h}}_\varepsilon = \omega^{\mathrm{h}} + O(\varepsilon)$ of ∇^* for $\Gamma^{\mathrm{p}}_\varepsilon = \{\gamma^{\mathrm{h}}_\varepsilon(t) \mid t \in [0, T_\varepsilon)\}$ and $\Gamma^{\mathrm{h}\prime}_\varepsilon = \{\gamma^{\mathrm{h}}_\varepsilon(t) \mid t \in \mathbb{R}\}$, respectively, so that for the cotangent lift $\hat{X}_\varepsilon$ of X_ε $(\gamma^{\mathrm{p}}_\varepsilon(t), \omega^{\mathrm{p}}_{\varepsilon,\gamma^{\mathrm{p}}(t)})$ is a periodic orbit to which $(\gamma^{\mathrm{h}}_\varepsilon(t), \omega^{\mathrm{h}}_{\varepsilon,\gamma^{\mathrm{h}}(t)})$ is a homoclinic orbit. Here the section ω^{h} of ∇^* satisfies the AVE (3.3.5) along Γ^{h}. Applying Theorem 3.2.1 to $\hat{X}_\varepsilon$, we obtain the following.

Theorem 3.3.12. *Assume that (A1') and (A3') hold and that there exists a periodic orbit $\gamma^{\mathrm{p}}_\varepsilon$ depending C^2-smoothly on ε such that $\gamma^{\mathrm{p}}_0 = \gamma^{\mathrm{p}}$. If the perturbed system* (3.1.1) *has a homoclinic orbit $\gamma^{\mathrm{h}}_\varepsilon$ depending C^2-smoothly on ε in* (3.1.1) *such that $\gamma^{\mathrm{h}}_0 = \gamma^{\mathrm{h}}$, then the limit in the right hand side of* (3.3.13) *exists and $\tilde{\mathscr{J}}_{\omega^{\mathrm{h}},Z,\gamma} = 0$ for some section ω of $T^*_{\Gamma^{\mathrm{h}}}$ satisfying the AVE* (3.3.5) *along Γ^{h}.*

Theorem 3.3.12 means that if $\tilde{\mathscr{J}}_{\omega^{\mathrm{h}},Z,\gamma^{\mathrm{h}}} \neq 0$ for any horizontal section ω^{h} of ∇^* for Γ^{h}, then the homoclinic orbit γ^{h} does not persists in (3.1.1) for $\varepsilon > 0$.

3.4 Some relationships with the Melnikov Methods

In this section, we discuss some relationships of the main results in Sections 3.2 and 3.3 with the standard, subharmonic and homoclinic Melnikov methods [26, 42, 62, 65], which provide sufficient conditions for persistence of periodic and homoclinic orbits, respectively, in time-periodic perturbations of single-degree-of-freedom Hamiltonian systems, and with another version of the homoclinic Melnikov method due to Wiggins [61] for autonomous perturbations of multi-degree-of-freedom Hamiltonian systems.

3.4.1 Standard Melnikov methods

We first review the standard Melnikov methods for subharmonic and homoclinic orbits. See [26, 62, 65] for more details.

We consider systems of the form

$$\dot{x} = J_2 DH(x) + \varepsilon g(x,t), \quad x \in \mathbb{R}^2, \tag{3.4.1}$$

where ε is a small parameter as in the previous sections, $H : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ are, respectively, C^3 and C^2 in x, $g(x,t)$ is T-periodic in t with $T > 0$ a constant, and J_2 is the 2×2 symplectic matrix,

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

When $\varepsilon = 0$, Eq. (4.5.1) becomes a single-degree-of-freedom Hamiltonian system with the Hamiltonian $H(x)$,

$$\dot{x} = J_2 DH(x). \tag{3.4.2}$$

Let $\theta = t \mod T$ so that $\theta \in \mathbb{S}_T^1$, where $\mathbb{S}_T^1 = \mathbb{R}/T\mathbb{Z}$. We rewrite (4.5.1) as an autonomous system,

$$\dot{x} = J_2 DH(x) + \varepsilon g(x,\theta), \quad \dot{\theta} = 1. \tag{3.4.3}$$

We begin with the subharmonic Melnikov method [26, 62, 65], and make the following assumption:

(M) The unperturbed system (4.5.2) possesses a one-parameter family of periodic orbits $q^\alpha(t)$ with period T^α, $\alpha \in (\alpha_1, \alpha_2)$, for some $\alpha_1 < \alpha_2$.

Fix the value of $\alpha \in (\alpha_1, \alpha_2)$ such that

$$lT^\alpha = mT \tag{3.4.4}$$

for some relatively prime integers $l, m > 0$. When $\varepsilon = 0$, Eq. (4.5.3) has a one-parameter family of mT-periodic orbits $(x,\theta) = (q^\alpha(t-\tau), t)$, $\tau \in [0,T)$. Note that $(x,\theta) = (q^\alpha(t-\tau-jT), t)$ represents the same periodic orbit in the phase space $\mathbb{R}^2 \times \mathbb{S}_T^1$ for $j = 0, 1, \ldots, m-1$. Define the *subharmonic Melnikov function* as

$$M^{m/l}(\tau) := \int_0^{mT} DH(q^\alpha(t)) \cdot g(q^\alpha(t), t+\tau)dt, \tag{3.4.5}$$

where the dot '$\cdot$' represents the standard inner product in $\mathbb{R}^2$. We have the following (see [26, 62, 65] for the proof).

Theorem 3.4.1. *If the subharmonic Melnikov function $M^{l/m}(\tau)$ has a simple zero at $\tau = \tau_0 \in \mathbb{S}_T^1$, then for $\varepsilon > 0$ sufficiently small Eq.* (4.5.3) *has a periodic orbit of period mT near the unperturbed periodic orbit $(x,\theta) = (q^\alpha(t-\tau_0), t)$ satisfying* (3.4.4).

Theorem 3.4.1 means that the periodic orbit $(x,\theta) = (q^\alpha(t-\tau_0), t)$ persists in (4.5.3) for $\varepsilon > 0$ sufficiently small if $M^{m/l}(\tau)$ has a simple zero at $\tau = \tau_0$. The stability of the perturbed periodic orbit can be also determined easily [65]. Moreover, several bifurcations of the periodic orbits were discussed in [65, 68, 69].

We next review the homoclinic Melnikov method [26, 42, 62] and assume the following instead of (M):

(M') The unperturbed system (4.5.2) possesses a hyperbolic saddle point p connected to itself by a homoclinic orbit $q^{\mathrm{h}}(t)$.

When $\varepsilon = 0$, Eq. (4.5.3) has a hyperbolic T-periodic orbit $(x, \theta) = (p, t)$ with a one-parameter family of homoclinic orbits $(x, \theta) = (q^{\mathrm{h}}(t - \tau), t)$, $\tau \in \mathbb{S}^1_T$. Note that $(x, \theta) = (q^{\mathrm{h}}(t - \tau - jT), t)$ represents the same homoclinic orbit in the phase space $\mathbb{R}^2 \times \mathbb{S}^1_T$ for $j = 0, 1, \ldots, m - 1$. We easily show that there exists a hyperbolic periodic orbit near $(x, \theta) = (p, t)$ (see [26, 62] for the proof). Define the *homoclinic Melnikov function* as

$$M(\tau) := \int_{-\infty}^{\infty} DH(q^{\mathrm{h}}(t)) \cdot g(q^{\mathrm{h}}(t), t + \tau) dt \tag{3.4.6}$$

We have the following (see [26, 42, 62] for the proof).

Theorem 3.4.2. *If the homoclinic Melnikov function $M(\tau)$ has a simple zero at $\tau = \tau_0 \in \mathbb{S}^1_T$, then for $\varepsilon > 0$ sufficiently small Eq. (4.5.3) has a transverse homoclinic orbit to the hyperbolic periodic orbit near $(x, \theta) = (q^{\mathrm{h}}(t - \tau_0), t)$.*

Theorem 3.4.2 means that the homoclinic orbit $(x, \theta) = (q^{\mathrm{h}}(t - \tau_0), t)$ persists in (4.5.3) for $\varepsilon > 0$ sufficiently small if $M(\tau)$ has a simple zero at $\tau = \tau_0$. By the Smale-Birkhoff theorem [26,62], the existence of transverse homoclinic orbits to hyperbolic periodic orbits implies that chaotic motions occur in (4.5.3), i.e., in (4.5.1).

We now describe some relationships of our results on persistence of first integrals with the standard Melnikov methods for (4.5.3), which has the Hamiltonian $H(x)$ is a first integral when $\varepsilon = 0$. We first state the relationship for the subharmonic Melnikov method.

Theorem 3.4.3. *Suppose that assumption (M) and the resonance condition $lT^\alpha = mT$ hold for $l, m > 0$ relatively prime integers. If Eq. (4.5.3) has a C^3 first integral $F_\varepsilon(x, t) = H(x) + O(\varepsilon)$ depending C^2-smoothly on ε in a neighborhood of*

$$\Gamma^\alpha_{\tau_0} = \{(q^\alpha(t - \tau_0), t) \mid t \in [0, mT)\}$$

with $\tau_0 \in \mathbb{S}^1_T$, then there exists a connected open set $\Pi \subset \mathbb{S}^1_T$ such that $\tau_0 \in \Pi$ and the subharmonic Melnikov function $M^{m/l}(\tau)$ is zero on Π.

Proof. Assume that the hypotheses of the theorem hold and F_ε is a first integral of (4.5.3). Then $\hat{\gamma}^{m/l}_\tau(t) = (q^\alpha(t - \tau), t)$ is an mT-periodic orbit in (4.5.3) with $\varepsilon = 0$ for any $\tau \in [0, T)$. Letting $F = H$, we write the integral (3.2.1) as

$$\begin{aligned} \mathscr{I}_{H, \hat{\gamma}^{m/l}_\tau} &= \int_0^{mT} DH(q^\alpha(t - \tau)) \cdot g(q^\alpha(t - \tau), t) dt \\ &= \int_0^{mT} DH(q^\alpha(t)) \cdot g(q^\alpha(t), t + \tau) dt, \end{aligned}$$

which coincides with $M^{m/l}(\tau)$. We choose a connected open set $\Pi \subset \mathbb{S}^1_T$ such that the neighborhood of $\Gamma^\alpha_{\tau_0}$ contains $\bigcup_{\tau \in \Pi} \Gamma^\alpha_\tau$. Applying Theorem 3.2.2 to the unperturbed periodic orbit $\hat{\gamma}^{m/l}_\tau$ for $\tau \in \Pi$, we obtain the desired result. $\square$

Theorem 3.4.3 means that if there exists a connected open set $\Pi \subset \mathbb{S}^1_T$ such that $M^{m/l}(\tau) \not\equiv 0$ on Π, then the first integral H does not persist near $\bigcup_{\tau\in\Pi} \Gamma^\alpha_\tau$ in (4.5.3) for $\varepsilon > 0$.

Remark 3.4.4. *Under the hypotheses of Theorem 3.4.3 the following hold:*

(i) It follows from Theorem 3.2.1 that if the periodic orbit $(x, \theta) = (q^\alpha(t-\tau), t)$ persists in (4.5.3)*, then $M^{m/l}(\tau) = 0$;*

(ii) If Eq. (4.5.3) *has such a first integral near $\bigcup_{\tau\in\mathbb{S}^1_T} \Gamma^\alpha_\tau$, then $M^{m/l}(\tau)$ is identically zero on $\mathbb{S}^1_T$;*

(iii) If H, g are analytic and Eq. (4.5.3) *has such a first integral near $\Gamma^\alpha_{\tau_0}$ with some $\tau_0 \in \mathbb{S}^1_T$, then $M^{m/l}(\tau)$ is identically zero on $\mathbb{S}^1_T$.*

The statement of part (i) consists with Theorem 3.4.1. Part (iii) follows from the identity theorem (e.g., Theorem 3.2.6 of [1]) since $M^{m/l}(\tau)$ is also analytic.

Similarly, we have the following result for the homoclinic Melnikov method.

Theorem 3.4.5. *Suppose that assumption (M') holds. If Eq.* (4.5.3) *has a C^3 first integral $F_\varepsilon(x, t) = H(x) + O(\varepsilon)$ depending C^2-smoothly on ε in a neighborhood of*

$$\Gamma^{\mathrm{h}}_{\tau_0} = \{(q^{\mathrm{h}}(t-\tau_0), t) \mid t \in \mathbb{R}\}$$

with $\tau_0 \in \mathbb{S}^1_T$, then there exists a connected open set $\Pi \subset \mathbb{S}^1_T$ such that $\tau_0 \in \Pi$ and the homoclinic Melnikov function $M(\tau)$ is zero on Π.

Proof. Assume that (M') holds. Then in (4.5.3) with $\varepsilon = 0$, (p, t) represents a periodic orbit, to which $\hat{\gamma}^{\mathrm{h}}_\tau(t) = (q^{\mathrm{h}}(t-\tau), t)$ is a homoclinic orbit, for any $\tau \in [0, T)$. We take the Poincaré section $\Sigma = \{(x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1_T \mid \theta = 0\}$ and set $T_j = jT$, $j \in \mathbb{Z}$. Letting $F = H$, we write the integral in (3.2.4) as

$$\int_{-jT}^{jT} DH(q^{\mathrm{h}}(t-\tau)) \cdot g(q^{\mathrm{h}}(t-\tau), t)dt = \int_{-jT}^{jT} DH(q^{\mathrm{h}}(t)) \cdot g(q^{\mathrm{h}}(t), t+\tau)dt,$$

which converges to $M(\tau)$ as $j \to \infty$. We choose a connected open set $\Pi \subset \mathbb{S}^1_T$ such that the neighborhood of $\Gamma^{\mathrm{h}}_{\tau_0}$ contains $\bigcup_{\tau\in\Pi} \Gamma^{\mathrm{h}}_\tau$. Applying Theorem 3.2.4 to the unperturbed homoclinic orbit $\hat{\gamma}^{\mathrm{h}}_\tau$ for $\tau \in \Pi$, we obtain the desired result. □

Theorem 3.4.5 means that if there exists a connected open set $\Pi \in \mathbb{S}^1_T$ such that $M(\tau) \not\equiv 0$ on Π, then the first integral H does not persist near $\bigcup_{\tau\in\Pi} \Gamma^{\mathrm{h}}_\tau$ in (4.5.3) for $\varepsilon > 0$.

Remark 3.4.6. *Under the hypotheses of Theorem 3.4.5 the following hold, as in Remark 3.4.4:*

(i) It follows from Theorem 3.2.3 that if the homoclinic orbit $(x, \theta) = (q^{\mathrm{h}}(t-\tau), t)$ persists in (4.5.3)*, then $M(\tau) = 0$;*

(ii) If Eq. (4.5.3) *has such a first integral near $\bigcup_{\tau\in\mathbb{S}^1_T} \Gamma^{\mathrm{h}}_\tau$, then $M(\tau)$ is identically zero on $\mathbb{S}^1_T$;*

(iii) If H, g are analytic and Eq. (4.5.3) *has such a first integral near $\Gamma^{\mathrm{h}}_{\tau_0}$ with some $\tau_0 \in \mathbb{S}_T$, then $M(\tau)$ is identically zero on $\mathbb{S}^1_T$.*

The statement of part (i) consists with Theorem 3.4.2.

3.4.2 Another version of the homoclinic Melnikov method

We next consider $(m+1)$-degree-of-freedom Hamiltonian systems of the form

$$\begin{aligned}
&\dot{x} = J_{2m} D_x H^0(x,I) + \varepsilon J_{2m} D_x H^1(x,I,\theta),\\
&\dot{I} = -\varepsilon D_\theta H^1(x,I,\theta), \qquad\qquad (x,I,\theta) \in \mathbb{R}^{2m} \times V \times \mathbb{S}^1_{2\pi},\\
&\dot{\theta} = D_I H^0(x,I) + \varepsilon D_I H^1(x,I,\theta),
\end{aligned} \tag{3.4.7}$$

for which $H_\varepsilon(x,I,\theta) = H^0(x,I) + \varepsilon H^1(x,I,\theta)$ is the Hamiltonian, where $m \geq 1$ is an integer, $V \subset \mathbb{R}$ is an open interval, $H^0(x,I), H^1(x,I,\theta)$ are C^3 in (x,I,θ), and J_{2m} is the $2m \times 2m$ symplectic matrix given by

$$J_{2m} = \begin{pmatrix} 0 & \mathrm{id}_m \\ -\mathrm{id}_m & 0 \end{pmatrix},$$

where id_m is the $m \times m$ identity matrix. When $\varepsilon = 0$, Eq. (3.4.7) becomes

$$\dot{x} = J_{2m} D_x H^0(x,I), \quad \dot{I} = 0, \quad \dot{\theta} = D_I H^0(x,I). \tag{3.4.8}$$

Note that I and θ are scalar variables. We assume the following on the unperturbed system (3.4.8):

(W1) For each $I \in V$, the first equation is has m C^3 first integrals $F_j(x,I)$, $j = 1,\dots,m$, with $F_1(x,I) = H^0(x,I)$ such that $D_x F_j(x,I)$, $j = 1,\dots,m$, are linearly independent except at equilibria and they are in involution, i.e., $\{F_i(x,I), F_j(x,I)\} := D_x F_i(x,I) \cdot J_{2m} D_x F_j(x,I) = 0$, $i,j = 1,\dots,m$.

(W2) For each $I \in V$ the first equation has a hyperbolic equilibrium x^I and an $(m-1)$-parameter family of homoclinic orbits $q^I(t;\alpha)$, $\alpha \in \bar{V} \subset \mathbb{R}^{m-1}$, to x^I, where x^I and $q^I(t;\alpha)$ depend C^2-smoothly on I and α, and $\bar{V}$ is an connected open in $\mathbb{R}^{m-1}$.

(W3) $D_I H^0(q^I(t;\alpha), I) > 0$ for $(I,\alpha) \in V \times \bar{V}$.

Obviously, I is a first integral of (3.4.8) as well as $F_j(x,I)$, $j = 1,\dots,m$, so that the Hamiltonian system (3.4.7) is Liouville integrable [9,43]. Thus, Eq. (3.4.7) is a special case in a class of systems called “System III” in Chapter 4 of [61], in which very wide classes of systems containing more general Hamiltonian systems, especially having multiple action and angular variables such as the scalar variables I and θ in (3.4.7), were discussed.

In (3.4.8) $\mathscr{N}_0 = \{(x^I, I, \theta) \mid I \in V, \theta \in \mathbb{S}^1_{2\pi}\}$ is a two-dimensional normally hyperbolic invariant manifold with boundary whose stable and unstable manifolds coincide along the homoclinic manifold

$$\bar{\Gamma}^{\mathrm{h}} = \{(q^I(t;\alpha), I, \theta) \mid I \in V, \alpha \in \bar{V}, \theta \in \mathbb{S}^1_{2\pi}\}.$$

Here “normal hyperbolicity” means that the expansive and contraction rates of the flow generated by (3.4.8) normal to $\mathscr{N}_0$ dominate those tangent to $\mathscr{N}_0$. Note that $(x,I,\theta) = (x^{I_0}, I_0, D_I H^0(x^{I_0}, I_0)t + \theta_0)$ represents a periodic orbit on $\mathscr{N}_0$ for $(I_0,\theta_0) \in V \times \mathbb{S}^1_{2\pi}$. Using the invariant manifold theory [63], we show that when $\varepsilon \neq 0$ Eq. (3.4.7) also has a two-dimensional normally hyperbolic invariant manifold $\mathscr{N}_\varepsilon$ near $\mathscr{N}_0$ and its stable and unstable manifolds are close to those of $\mathscr{N}_0$. Moreover, the invariant manifold $\mathscr{N}_\varepsilon$ consists of periodic orbits $\gamma^{\mathrm{p}}_{I,\varepsilon}$, which are given as intersections between $\mathscr{N}_\varepsilon$ and the level sets $H_\varepsilon(x, I, \theta) = \text{const.}$ since $\mathrm{D}_I H^0(x^I, I) > 0$ by (W3), near $\gamma^{\mathrm{p}}_I = \{(x^I, I, \theta) | \theta \in \mathbb{S}^1_{2\pi}\}$ for $I \in V$. Note that $\mathscr{N}_\varepsilon$ can be invariant by taking two periodic orbits as its boundary as in Proposition 2.1 of [66].

Let $\theta = \theta^I(t; \alpha)$ denote the solution to

$$\dot{\theta} = D_I H^0(q^I(t; \alpha), I)$$

with $\theta(0) = 0$, i.e.,

$$\theta^I(t; \alpha) = \int_0^t D_I H^0(q^I(t; \alpha), I) dt.$$

Then $\gamma^{\mathrm{h}}_{I,\alpha,\theta_0}(t) = (q^I(t; \alpha), I, \theta^I(t; \alpha) + \theta_0)$ is a homoclinic orbit to the periodic orbit γ^{p}_I in (3.4.8) for any $\theta_0 \in \mathbb{S}^1_{2\pi}$. Let $\{T^{I,\alpha}_j\}_{j=-\infty}^{\infty}$ be a sequence for $(I, \alpha) \in V \times \bar{V}$ such that

$$\theta^I(T^{I,\alpha}_j; \alpha) = 0, \quad j \in \mathbb{Z}, \quad \text{and} \quad \lim_{j \to \pm\infty} T^{I,\alpha}_j = \pm\infty. \tag{3.4.9}$$

By assumption (W3) there exists such an sequence $\{T^{I,\alpha}_j\}_{j=-\infty}^{\infty}$. Define the *Melnikov functions* for (3.4.7) as

$$\bar{M}^I_1(\theta_0, \alpha) = \lim_{j \to \infty} \int_{T^{I,\alpha}_{-j}}^{T^{I,\alpha}_j} D_\theta H^1(q^I(t; \alpha), I, \theta^I(t; \alpha) + \theta_0) dt \tag{3.4.10}$$

and

$$\begin{aligned}
&\bar{M}^I_k(\theta_0, \alpha) \\
&= \lim_{j \to \infty} \int_{T^{I,\alpha}_{-j}}^{T^{I,\alpha}_j} \big(D_x F_k(q^I(t; \alpha), I) \cdot J_{2m} D_x H^1(q^I(t; \alpha), I, \theta^I(t; \alpha) + \theta_0) \\
&\qquad - D_I F_k(q^I(t; \alpha), I) D_\theta H^1(q^I(t; \alpha), I, \theta^I(t; \alpha) + \theta_0) \big) dt
\end{aligned} \tag{3.4.11}$$

for $k = 2, \ldots, m$. Note that the definitions of the Melnikov functions $\bar{M}^I_k$, $k \geq 2$, are different from the original ones of [61]. We call $\bar{M}^I = (\bar{M}^I_1, \ldots, \bar{M}^I_m)$ the *Melnikov vector*. From Theorem 4.1.19 of [61] we obtain the following result for (3.4.7).

Theorem 3.4.7. *Suppose that assumptions (W1)-(W3) hold. If for some* $I \in V$

(i) $\bar{M}^I(\theta, \alpha) = 0$*;*

(ii) $\det D\bar{M}^I(\theta, \alpha) \neq 0$

at $(\theta, \alpha) = (\theta_0, \alpha_0)$*, then the* $(m + 1)$*-dimensional stable and unstable manifolds* $W^{\mathrm{s}}(\gamma^{\mathrm{p}}_{I,\varepsilon})$ *and* $W^{\mathrm{u}}(\gamma^{\mathrm{p}}_{I,\varepsilon})$ *intersect transversely near* $(x, I, \theta) = (q^I(0; \alpha_0), I, \theta_0)$ *on the level set of* $H_\varepsilon(\gamma^{\mathrm{p}}_{I,\varepsilon})$*.*

Proof. Assume that the hypotheses of Theorem 3.4.7 hold. Let $\tilde{M}^I_1(\theta, \alpha) = \bar{M}^I_1(\theta, \alpha)$ and

$$\tilde{M}^I_k(\theta, \alpha) = \bar{M}^I_k(\theta, \alpha) + D_I F_k(x_I, I) \bar{M}^I_1(\theta, \alpha), \quad k = 2, \ldots, m,$$

and let $\tilde{M}^I = (\tilde{M}_1^I, \dots, \tilde{M}_m^I)$, which is the original Melnikov vector defined in [61] for (3.4.7). Note that in [61], although a time sequence does not appear in its formulas (4.1.84) and (4.1.85) or (4.1.101) and (4.1.102), such conditional convergences of the integrals as (3.4.10) and (3.4.11) are implicitly assumed (see his arguments on system III in part iii) of Section 4.1d of [61]). We see that if $\bar{M}^I(\theta, \alpha)$ satisfies conditions (i) and (ii) at $(\theta, \alpha) = (\theta_0, \alpha_0)$, then $\tilde{M}^I(\theta_0, \alpha_0) = 0$ and $\det D\tilde{M}^I(\theta_0, \alpha_0) \neq 0$, since $\det D\tilde{M}^I(\theta_0, \alpha_0) = \det D\bar{M}^I(\theta_0, \alpha_0)$, which follows from

$$D\tilde{M}^I(\theta_0, \alpha_0) = \begin{pmatrix} D_\theta \bar{M}_1^I(\theta_0, \alpha_0), D_\alpha \bar{M}_1^I(\theta_0, \alpha_0) \\ D_\theta \bar{M}_2^I(\theta_0, \alpha_0), D_\alpha \bar{M}_2^I(\theta_0, \alpha_0) \\ \vdots \\ D_\theta \bar{M}_m^I(\theta_0, \alpha_0), D_\alpha \bar{M}_m^I(\theta_0, \alpha_0) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ D_I F_2(x_I, I)(D_\theta \bar{M}_1^I(\theta_0, \alpha_0), D_\alpha \bar{M}_1^I(\theta_0, \alpha_0)) \\ \vdots \\ D_I F_m(x_I, I)(D_\theta \bar{M}_1^I(\theta_0, \alpha_0), D_\alpha \bar{M}_1^I(\theta_0, \alpha_0)) \end{pmatrix}.$$

We obtain the desired result from Theorem 4.1.19 of [61]. □

Remark 3.4.8. *The Melnikov vector $\bar{M}^I(\theta, \alpha)$ does not depend on the choice of time sequence $\{T_j^{I,\alpha}\}_{j=-\infty}^{\infty}$. Actually, letting $\{\hat{T}_j^{I,\alpha}\}_{j=-\infty}^{\infty}$ be a different time sequence satisfying*

$$\theta^I(\hat{T}_j^{I,\alpha}; \alpha) = \hat{\theta}_0, \quad j \in \mathbb{Z}, \quad \textit{and} \quad \lim_{j \to \pm\infty} \hat{T}_j^{I,\alpha} = \pm\infty$$

instead of (3.4.9), *we have*

$$\begin{aligned} \hat{M}_1^I(\theta, \alpha) :=& \lim_{j \to \infty} \int_{\hat{T}_{-j}^{I,\alpha}}^{\hat{T}_j^{I,\alpha}} D_\theta H^1(q^I(t; \alpha), I, \theta^I(t; \alpha) + \theta) dt \\ =& \bar{M}_1^I(\theta, \alpha) + \lim_{j \to \infty} \left(\int_{\hat{T}_{-j}^{I,\alpha}}^{T_{-j}^{I,\alpha}} + \int_{T_j^{I,\alpha}}^{\hat{T}_j^{I,\alpha}} \right) D_\theta H^1(q^I(t; \alpha), I, \theta^I(t; \alpha) + \theta) dt \\ =& \bar{M}_1^I(\theta, \alpha) \end{aligned}$$

since

$$\begin{aligned} &\lim_{t \to \pm\infty} D_\theta H^1(q^I(t; \alpha), I, \theta) = D_\theta H^1(x^I, I, \theta), \\ &\lim_{t \to \pm\infty} D_I H^0(q^I(t; \alpha), I) = D_I H^0(x^I, I) \end{aligned}$$

and

$$\begin{aligned} &\left(\int_{\hat{T}_{-j}^{I,\alpha}}^{T_{-j}^{I,\alpha}} + \int_{T_j^{I,\alpha}}^{\hat{T}_j^{I,\alpha}} \right) D_\theta H^1(x^I, I, \theta^I(t; \alpha) + \theta) D_I H^0(q^I(t; \alpha), I) dt \\ &= \left(\int_{\hat{T}_{-j}^{I,\alpha}}^{T_{-j}^{I,\alpha}} + \int_{T_j^{I,\alpha}}^{\hat{T}_j^{I,\alpha}} \right) \frac{d}{dt} H^1(x^I, I, \theta^I(t; \alpha) + \theta) \, dt = 0. \end{aligned}$$

Similarly, we show

$$\begin{aligned}\hat{M}_k^I(\theta,\alpha) :=& \lim_{j\to\infty}\int_{\hat{T}_{-j}^{I,\alpha}}^{\hat{T}_j^{I,\alpha}} \big(D_x F_k(q^I(t;\alpha),I)\cdot J_{2m} D_x H^1(q^I(t;\alpha),I,\theta^I(t;\alpha)+\theta)\\ &\qquad - D_I F_k(q^I(t;\alpha),I) D_\theta H^1(q^I(t;\alpha),I,\theta^I(t;\alpha)+\theta)\big)dt\\ =&\bar{M}_k^I(\theta,\alpha),\quad k=2,\dots,m.\end{aligned}$$

Theorem 3.4.7 means that the homoclinic orbit $\gamma^{\mathrm{h}}_{I,\alpha,\theta_0}(t)$ persists in (3.4.7) for $\varepsilon>0$ sufficiently small if the Melnikov vector $\bar{M}^I(\theta,\alpha)$ satisfies its hypotheses. By the Smale-Birkhoff theorem [26, 62], such transverse intersection between the stable and unstable manifolds of periodic orbits implies that chaotic motions occur in (3.4.7).

We now describe a relationship of our results on persistence of first integrals with the homoclinic Melnikov methods for (3.4.7), in which the Hamiltonian $H_\varepsilon(x,I,\theta)$ is always a persisting first integral. We have the following result.

Theorem 3.4.9. *Suppose that assumptions (W1)-(W3) hold. If the Hamiltonian system* (3.4.7) *has a C^3 first integral $F_{k,\varepsilon}(x,I,\theta)=F_k(x,I)+O(\varepsilon)$ (resp. $F_{m+1,\varepsilon}(x,I,\theta)=I+O(\varepsilon)$) depending C^2 smoothly on ε in a neighborhood of*

$$\bar{\Gamma}^{\mathrm{h}}_{I_0,\theta_0,\alpha_0}=\{(q^{I_0}(t;\alpha_0),I_0,\theta^{I_0}(t;\alpha_0))\mid t\in\mathbb{R}\}$$

for some $k=2,\dots,m$, then there exists a connected open set $\bar{\Pi}\subset V\times\mathbb{S}^1_{2\pi}\times\bar{V}$ such that $(I_0,\theta_0,\alpha_0)\in\bar{\Pi}$ and the Melnikov function $\bar{M}_k^I(\theta,\alpha)=0$ (resp. $\bar{M}_1^I(\theta,\alpha)=0$) on $\bar{\Pi}$.

Proof. Assume that (W1)-(W3) hold. We choose the Poincaré section $\Sigma=\{(x,I,\theta)\in\mathbb{R}^{2m}\times V\times\mathbb{S}^1_{2\pi}\mid\theta=\theta_0\}$ and take $T_j=T_j^{I,\alpha}$, $j\in\mathbb{Z}$ (cf. Eq. (3.4.9)). Letting $F=F_k$ for $k=2,\dots,m$ (resp. $F=I$), we write the integral in (3.2.4) as

$$\begin{aligned}&\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}} \big(D_x F_k(q^I(t;\alpha),I)\cdot J_{2m} D_x H^1(q^I(t;\alpha),I,\theta^I(t;\alpha)+\theta_0)\\ &\qquad - D_I F_k(q^I(t;\alpha),I) D_\theta H^1(q^I(t;\alpha),I,\theta^I(t;\alpha)+\theta_0)\big)dt\\ &\left(\text{resp.}\quad\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}} D_\theta H^1(q^I(t;\alpha),I,\theta^I(t;\alpha)+\theta_0)dt\right)\end{aligned}$$

for the homoclinic orbit $\gamma^{\mathrm{h}}_{I,\theta,\alpha_0}(t)$. We choose a connected open set $\bar{\Pi}\subset V\times\mathbb{S}^1_{2\pi}\times\bar{V}$ such that the neighborhood of $\bar{\Gamma}^{\mathrm{h}}_{I_0,\theta_0,\alpha_0}$ contains $\bigcup_{(I,\theta,\alpha)\in\bar{\Pi}}\bar{\Gamma}^{\mathrm{h}}_{I,\theta,\alpha}$. Applying Theorem 3.2.4 to the unperturbed homoclinic orbit $\gamma^{\mathrm{h}}_{I,\theta,\alpha}(t)$ for $(I,\theta,\alpha)\in\bar{\Pi}$, we obtain the desired result. □

Remark 3.4.10. *Under the hypotheses of Theorem 4.5.5 the following hold as in Remarks 3.4.4 and 3.4.6:*

(i) It follows from Theorem 3.2.3 that if the homoclinic orbit $\gamma^{\mathrm{h}}_{I_0,\theta_0,\alpha_0}(t)$ persists in (3.4.7)*, then $\bar{M}^{I_0}(\theta_0,\alpha_0)=0$;*

(ii) If Eq. (3.4.7) *has such a first integral near $\bar{\Gamma}^{\mathrm{h}}$, then the corresponding Melnikov function is identically zero on $V\times\mathbb{S}^1_{2\pi}\times\bar{V}$;*

(iii) If H^0,H^1 are analytic and Eq. (3.4.7) *has such a first integral except for H_ε near $\bar{\Gamma}^{\mathrm{h}}_{I_0,\theta_0,\alpha_0}$ with some $(I_0,\theta_0,\alpha_0)\in\bar{\Pi}$, then $\bar{M}^I(\theta)=0$ is identically zero on $\tilde{\Gamma}^{\mathrm{h}}$.*

The statement of part (i) consists with Theorem 3.4.7.

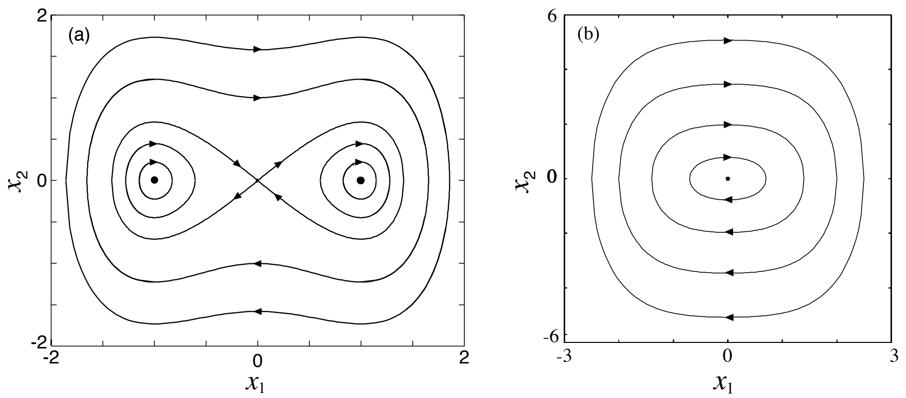

Figure 3.2: Phase portraits of (4.6.3) with $\varepsilon = 0$: (a) $a = 1$; (b) $a = -1$.

3.5 Examples

We now illustrate the above theory for four examples: The periodically forced Duffing oscillator [26, 62, 64, 65], two identical pendula coupled with a harmonic oscillator, a periodically forced rigid body [72] and a three-mode truncation of a buckled beam [67].

3.5.1 Periodically forced Duffing oscillator

We first consider the periodically forced Duffing oscillator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = ax_1 - x_1^3 + \varepsilon(\beta \cos \omega t - \delta x_2), \tag{3.5.1}$$

where $x_1, x_2 \in \mathbb{R}$, $a = 1$ or -1, and β, δ, ω are positive constants. When $\varepsilon = 0$, Eq. (4.6.3) becomes a single-degree-of-freedom Hamiltonian system with the Hamiltonian

$$H = -\frac{1}{2}ax_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2, \tag{3.5.2}$$

and it is a special case of (4.5.3). See Fig. 3.2 for the phase portraits of (4.6.3) with $\varepsilon = 0$.

We begin with the case of $a = 1$. When $\varepsilon = 0$, in the phase plane there exist a pair of homoclinic orbits

$$q_\pm^{\mathrm{h}}(t) = (\pm\sqrt{2}\,\mathrm{sech}\, t, \mp\sqrt{2}\,\mathrm{sech}\, t\, \tanh t),$$

a pair of one-parameter families of periodic orbits

$$\begin{aligned} q_\pm^k(t) = &\Bigg(\pm\frac{\sqrt{2}}{\sqrt{2-k^2}}\,\mathrm{dn}\left(\frac{t}{\sqrt{2-k^2}}\right),\\ &\mp\frac{\sqrt{2}k^2}{2-k^2}\,\mathrm{sn}\left(\frac{t}{\sqrt{2-k^2}}\right)\mathrm{cn}\left(\frac{t}{\sqrt{2-k^2}}\right)\Bigg), \quad k \in (0,1), \end{aligned}$$

inside each of them, and a one-parameter periodic orbits

$$\begin{aligned} \tilde{q}^k(t) = &\Bigg(\frac{\sqrt{2}k}{\sqrt{2k^2-1}}\,\mathrm{cn}\left(\frac{t}{\sqrt{2k^2-1}}\right),\\ &-\frac{\sqrt{2}k}{2k^2-1}\,\mathrm{sn}\left(\frac{t}{\sqrt{2k^2-1}}\right)\mathrm{dn}\left(\frac{t}{\sqrt{2k^2-1}}\right)\Bigg), \quad k \in \big(1/\sqrt{2}, 1\big), \end{aligned}$$

outside of them, as shown in Fig. 3.2(a), where sn, cn and dn represent the Jacobi elliptic functions with the elliptic modulus k. See [20] for general information on elliptic functions. The periods of $q_\pm^k(t)$ and $\tilde{q}^k(t)$ are given by $T^k = 2K(k)\sqrt{2-k^2}$ and $\tilde{T}^k = 4K(k)\sqrt{2k^2-1}$, respectively, where $K(k)$ is the complete elliptic integral of the first kind. See also [26,62].

Assume that the resonance conditions

$$lT^k = \frac{2\pi m}{\omega}, \quad \text{i.e.,} \quad \omega = \frac{2\pi m}{2lK(k)\sqrt{2-k^2}}, \tag{3.5.3}$$

and

$$l\tilde{T}^k = \frac{2\pi m}{\omega}, \quad \text{i.e.,} \quad \omega = \frac{2\pi m}{4lK(k)\sqrt{2k^2-1}}, \tag{3.5.4}$$

hold for $q_\pm^k(t)$ and $\tilde{q}^k(t)$, respectively, with $l, m > 0$ relatively prime integers. We compute the subharmonic Melnikov function (4.5.5) for $q_\pm^k(t)$ and $\tilde{q}^k(t)$ as

$$M_\pm^{m/l}(\tau) = -\delta J_1(k,l) \pm \beta J_2(k,m,l)\sin\tau$$

and

$$\tilde{M}^{m/l}(\tau) = -\delta \tilde{J}_1(k,l) + \beta \tilde{J}_2(k,m,l)\sin\tau,$$

respectively, where

$$\begin{aligned}
J_1(k,l) &= \frac{4l[(2-k^2)E(k) - 2k'^2K(k)]}{3(2-k^2)^{3/2}},\\
J_2(k,m,l) &= \begin{cases} \sqrt{2}\pi\omega\,\mathrm{sech}\left(\dfrac{m\pi K(k')}{K(k)}\right) & (\text{for } l=1);\\ 0 & (\text{for } l \neq 1),\end{cases}\\
\tilde{J}_1(k,l) &= \frac{8l[(2k^2-1)E(k) + k'^2K(k)]}{3(2k^2-1)^{3/2}},\\
\tilde{J}_2(k,m,l) &= \begin{cases} 2\sqrt{2}\pi\omega\,\mathrm{sech}\left(\dfrac{m\pi K(k')}{2K(k)}\right) & (\text{for } l=1 \text{ and } m \text{ odd});\\ 0 & (\text{for } l \neq 1 \text{ or } m \text{ even}).\end{cases}
\end{aligned}$$

Here $E(k)$ is the complete elliptic integral of the second kind and $k' = \sqrt{1-k^2}$ is the complimentary elliptic modulus. We see that the subharmonic Melnikov functions $M_\pm^{m/l}(\tau)$ and $\tilde{M}^{m/l}(\tau)$ are not identically zero on any connected open set in $\mathbb{S}_T^1$. We also compute the homoclinic Melnikov function (4.5.7) for $q_\pm^{\mathrm{h}}(t)$ as

$$M_\pm(\tau) = -\frac{4}{3}\delta \pm \sqrt{2}\pi\omega\beta\,\mathrm{csch}\left(\frac{\pi\omega}{2}\right)\sin\tau,$$

which is not identically zero on any connected open set in $\mathbb{S}_T^1$. See also [26, 62] for the computations of the Melnikov functions.

Let

$$\begin{aligned}
R &= \{k \in (0,1) \mid k \text{ satisfies (4.6.5) for } m, l \in \mathbb{N}\},\\
\tilde{R} &= \left\{k \in \left(1/\sqrt{2}, 1\right) \mid k \text{ satisfies (4.6.6) for } m, l \in \mathbb{N}\right\},
\end{aligned}$$

and let

$$\begin{aligned}
S^k_\pm &= \{(x,\theta)\in\mathbb{R}^2\times\mathbb{S}^1_T \mid x=q^k_\pm(t),\},\\
\tilde{S}^k &= \{(x,\theta)\in\mathbb{R}^2\times\mathbb{S}^1_T \mid x=\tilde{q}^k(t)\},\\
S^{\mathrm{h}}_\pm &= \{(x,\theta)\in\mathbb{R}^2\times\mathbb{S}^1_T \mid x=q^{\mathrm{h}}_\pm(t)\}.
\end{aligned}$$

Applying Theorems 3.4.3 and 3.4.5, we obtain the following.

Proposition 3.5.1. *The first integral* (3.5.2) *does not persist near $S^k_\pm$ for $k\in R$, $\tilde{S}^k$ for $k\in\tilde{R}$, and $S^{\mathrm{h}}_\pm$ in* (4.6.3) *with $a=1$ for $\varepsilon>0$.*

Remark 3.5.2. *When $\beta>0$ but $\delta=0$, so that Eq.* (4.6.3) *is Hamiltonian, the statement of Proposition 3.5.1 still holds near $S^k_\pm$ for $k\in R_1$, $\tilde{S}^k$ for $k\in\tilde{R}_{\mathrm{o}}$, and $S^{\mathrm{h}}_\pm$, where*

$$\begin{aligned}
R_1 &= \{k\in(0,1)\mid k \textit{ satisfies } (4.6.5) \textit{ with } l=1\},\\
\tilde{R}_{\mathrm{o}} &= \bigl\{k\in\bigl(1/\sqrt{2},1\bigr)\mid k \textit{ satisfies } (4.6.6) \textit{ with } l=1 \textit{ and } m \textit{ odd}\bigr\},
\end{aligned}$$

We turn to the case of $a=-1$. When $\varepsilon=0$, in the phase plane there exists a one-parameter family of periodic orbits

$$\begin{aligned}
\gamma^k(t)=&\biggl(\frac{\sqrt{2}k}{\sqrt{1-2k^2}}\,\mathrm{cn}\left(\frac{t}{\sqrt{1-2k^2}}\right),\\
&-\frac{\sqrt{2}k}{1-2k^2}\,\mathrm{sn}\left(\frac{t}{\sqrt{1-2k^2}}\right)\mathrm{dn}\left(\frac{t}{\sqrt{1-2k^2}}\right)\biggr),\quad k\in\bigl(0,1/\sqrt{2}\bigr),
\end{aligned}$$

as shown in Fig. 3.2(b), and their period is given by $\hat{T}^k=4K(k)\sqrt{1-2k^2}$. See also [64,65]. Assume that the resonance conditions

$$l\hat{T}^k=\frac{2\pi m}{\omega},\quad\text{i.e.,}\quad \omega=\frac{\pi m}{2lK(k)\sqrt{1-2k^2}} \tag{3.5.5}$$

holds for $l,m>0$ relatively prime integers. We compute the subharmonic Melnikov function (4.5.5) for $\gamma^k(t)$ as

$$\hat{M}^{m/l}(\tau)=-\delta\hat{J}_1(k,l)\pm\beta\hat{J}_2(k,m,l)\sin\tau,$$

where

$$\begin{aligned}
\hat{J}_1(k,l)&=\frac{8l[(2k^2-1)E(k)+k'^2K(k)]}{3(1-2k^2)^{3/2}},\\
\hat{J}_2(k,m,l)&=\begin{cases}\dfrac{\sqrt{2}\pi^2 m}{K(k)\sqrt{1-2k^2}}\,\mathrm{sech}\left(\dfrac{\pi mK(k')}{2K(k)}\right) & (\text{for } l=1 \text{ and } m \text{ odd});\\
0 & (\text{for } l\neq1 \text{ or } m \text{ even}).\end{cases}
\end{aligned}$$

See also [64, 65] for the computations of the Melnikov function. Thus, the Melnikov function $\bar{M}^{m/l}(\tau)$ is not identically zero on any connected open set in $\mathbb{S}^1_T$.

Let

$$\hat{R}=\bigl\{k\in\bigl(0,1/\sqrt{2}\bigr)\mid k\text{ satisfies (4.6.8) for } m,l\in\mathbb{N}\bigr\}$$

and let

$$\hat{S}^k=\{(x,\theta)\in\mathbb{R}^2\times\mathbb{S}^1_T\mid x=\gamma^k(t)\}.$$

Applying Theorem 3.4.3, we obtain the following.

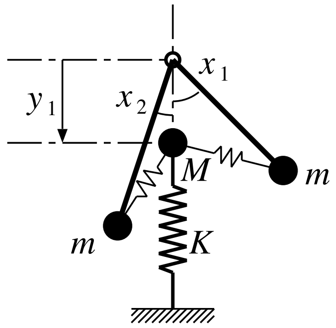

Figure 3.3: Two identical pendula coupled with a harmonic oscillator.

Proposition 3.5.3. *The first integral* (3.5.2) *does not persist near* $\hat{S}^k$ *for* $k \in \hat{R}$ *in* (4.6.3) *with* $a = -1$ *for* $\varepsilon > 0$.

Remark 3.5.4. *When* $\beta > 0$ *but* $\delta = 0$, *i.e., Eq.* (4.6.3) *is Hamiltonian, the statement of Proposition 4.6.6 still holds near* $\hat{S}^k$ *for* $k \in \hat{R}_{\mathrm{o}}$, *where*

$$\hat{R}_{\mathrm{o}} = \left\{k \in \left(0, 1/\sqrt{2}\right) \mid k \text{ satisfies } (4.6.8) \text{ with } l = 1 \text{ and } m \text{ odd}\right\}.$$

3.5.2 Two pendula coupled with a harmonic oscillator

We next consider the three-degree-of-freedom Hamiltonian system

$$\begin{aligned} \dot{x}_1 &= x_3, \quad \dot{x}_3 = -\sin x_1 - \varepsilon y_1 \sin x_1, \\ \dot{x}_2 &= x_4, \quad \dot{x}_4 = -\sin x_2 - \varepsilon y_1 \sin x_2, \\ \dot{y}_1 &= y_2, \quad \dot{y}_2 = -\omega_0^2 y_1 + \varepsilon(\cos x_1 + \cos x_2) \end{aligned} \tag{3.5.6}$$

with the Hamiltonian

$$H = -\cos x_1 - \cos x_2 + \frac{1}{2}(x_3^2 + x_4^2 + \omega_0^2 y_1^2 + y_2^2) - \varepsilon y_1(\cos x_1 + \cos x_2),$$

where $x_1, x_2 \in \mathbb{S}^1_{2\pi}$, $x_3, x_4, y_1, y_2 \in \mathbb{R}$ and ω_0 is a positive constant. The system (3.5.6) represents non-dimensionalized equations of motion for two identical pendula coupled with a harmonic oscillator shown in Fig. 3.3. Here the gravitational force acts downwards, and the spring K generates a restoring force Ky_1, where y_1 is the displacement of the mass $M = m$ from the pivot of the pendula. Linear restoring forces with a spring constant of $O(\varepsilon)$ and zero natural length also occur between the two masses m and the mass M. In particular, $\omega_0^2 = K\ell/Mg + O(\varepsilon)$, where g is the gravitational acceleration and ℓ is the length from the pivot to the mass m.

Introduce the action-angle coordinates $(I, \theta) \in \mathbb{R}_+ \times \mathbb{S}^1_{2\pi}$ such that

$$y_1 = \sqrt{\frac{2I}{\omega_0}} \sin\theta, \quad y_2 = \sqrt{2\omega_0 I} \cos\theta$$

and rewrite (3.5.6) as

$$\begin{aligned}
&\dot{x}_1 = x_3, \quad \dot{x}_3 = -\sin x_1 - \varepsilon\sqrt{\frac{2I}{\omega_0}}\sin\theta\sin x_1,\\
&\dot{x}_2 = x_4, \quad \dot{x}_4 = -\sin x_2 - \varepsilon\sqrt{\frac{2I}{\omega_0}}\sin\theta\sin x_2,\\
&\dot{I} = \varepsilon\sqrt{\frac{2I}{\omega_0}}\cos\theta(\cos x_1 + \cos x_2),\\
&\dot{\theta} = \omega_0 - \varepsilon\frac{\sin\theta}{\sqrt{2\omega_0 I}}(\cos x_1 + \cos x_2),
\end{aligned} \tag{3.5.7}$$

which has the form (3.4.7) with

$$\begin{aligned}
H^0(x, I) &= -\cos x_1 - \cos x_2 + \frac{1}{2}(x_3^2 + x_4^2) + \omega_0 I,\\
H^1(x, I, \theta) &= -\sqrt{\frac{2I}{\omega_0}}\sin\theta(\cos x_1 + \cos x_2).
\end{aligned}$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. When $\varepsilon = 0$, the x-component of (3.5.7) has a first integral

$$F_2(x, I) = -\cos x_1 + \frac{1}{2}x_3^2$$

and a hyperbolic equilibrium $x^I = (\pi, \pi, 0, 0)$ to which there exist four one-parameter families of homoclinic orbits

$$\begin{aligned}
&q^I_{\pm,+}(t;\alpha) = (\pm 2\arcsin(\tanh t), 2\arcsin(\tanh(t+\alpha)), \pm 2\operatorname{sech} t, 2\operatorname{sech}(t+\alpha)),\\
&q^I_{\pm,-}(t;\alpha) = (\pm 2\arcsin(\tanh t), -2\arcsin(\tanh(t+\alpha)) \pm 2\operatorname{sech} t, -2\operatorname{sech}(t+\alpha)),
\end{aligned}$$

where $\alpha \in \mathbb{R}$. Thus, assumptions (W1)-(W3) hold with $m = 2$.

We compute (3.4.11) for the homocloinic orbits $(x, I, \theta) = (q^I_{\pm,\pm}(t;\alpha), I, \omega_0 t + \theta_0)$ as

$$\begin{aligned}
\bar{M}^I_2(\theta_0, \alpha) &= -\sqrt{\frac{2I}{\omega_0}}\int_{-\infty}^{\infty} 2\sin(\omega_0 t + \theta_0)\operatorname{sech} t\ \sin(2\arcsin(\tanh t))dt\\
&= -4\sqrt{\frac{2I}{\omega_0}}\cos\theta_0\int_{-\infty}^{\infty}\operatorname{sech}^2 t\ \tanh t\ \sin\omega_0 t\, dt\\
&= -\pi\sqrt{8\omega_0^3 I}\ \operatorname{csch}\left(\frac{\pi\omega_0}{2}\right)\cos\theta_0.
\end{aligned}$$

On the other hand, letting $\{T_j^{I,\alpha}\}_{j=-\infty}^{\infty}$ be a time sequence satisfying (3.4.9), we write the integral in (3.4.10) as

$$
\begin{aligned}
&-\sqrt{\frac{2I}{\omega_0}}\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}} \cos(\omega_0 t+\theta_0)(\cos(2\arcsin(\tanh t))\\
&\qquad +\cos(2\arcsin(\tanh(t+\alpha))))dt\\
&=-\sqrt{\frac{2I}{\omega_0}}\biggl(\cos\theta_0\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}(1-2\tanh^2 t)\cos\omega_0 t\,dt\\
&\qquad -\sin\theta_0\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}(1-2\tanh^2 t)\sin\omega_0 t\,dt\\
&\qquad +\cos(\theta_0-\alpha\omega)\int_{T_{-j}^{I,\alpha}+\alpha}^{T_j^{I,\alpha}+\alpha}(1-2\tanh^2 t)\cos\omega_0 t\,dt\\
&\qquad -\sin(\theta_0-\alpha\omega)\int_{T_{-j}^{I,\alpha}+\alpha}^{T_j^{I,\alpha}+\alpha}(1-2\tanh^2 t)\sin\omega_0 t\,dt\biggr).
\end{aligned}
$$

Since

$$
\lim_{j\to\pm\infty}\omega_0 T_j^{I,\alpha}=0 \mod 2\pi,
$$

we have

$$
\begin{aligned}
&\lim_{j\to\infty}\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}(1-2\tanh^2 t)\cos\omega_0 t\,dt\\
&=\lim_{j\to\infty}\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}2(1-\tanh^2 t)\ \cos\omega_0 t\,dt-\lim_{j\to\infty}\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}\cos\omega_0 t\,dt\\
&=2\pi\omega_0\operatorname{csch}\left(\frac{\pi\omega_0}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{j\to\infty}\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}(1-2\tanh^2 t)\sin\omega_0 t\,dt\\
&=\lim_{j\to\infty}\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}2(1-\tanh^2 t)\ \sin\omega_0 t\,dt-\lim_{j\to\infty}\int_{T_{-j}^{I,\alpha}}^{T_j^{I,\alpha}}\sin\omega_0 t\,dt=0.
\end{aligned}
$$

Hence, we obtain

$$
\bar{M}_1^I(\theta_0,\alpha)=-\pi\sqrt{8\omega_0^3 I}\ \operatorname{csch}\left(\frac{\pi\omega_0}{2}\right)(\cos\theta_0+\cos(\theta_0-\omega_0\alpha)).
$$

We see that $\bar{M}_k^I(\theta_0,\alpha)$, $k=1,2$, are not identically zero on any connected open set in $\mathbb{R}_+\times\mathbb{S}^1_{2\pi}\times\mathbb{R}$. Applying Theorem 4.5.5, we obtain the following.

Proposition 3.5.5. *The first integrals $F_2(x,I)$ and I do not persist near*

$$
\bar{\Gamma}^{\mathrm{h}}=\bar{\Gamma}^{\mathrm{h}}_{+,+}\cup\bar{\Gamma}^{\mathrm{h}}_{+,-}\cup\bar{\Gamma}^{\mathrm{h}}_{-,+}\cup\bar{\Gamma}^{\mathrm{h}}_{-,-}
$$

in (3.5.7) *for $\varepsilon>0$, where*

$$
\bar{\Gamma}^{\mathrm{h}}_{\pm,\pm}=\{(q^I_{\pm,\pm}(t;\alpha),I,\theta)\mid t\in\mathbb{R},I\in\mathbb{R}_+,\theta\in\mathbb{S}^1_{2\pi}\}.
$$

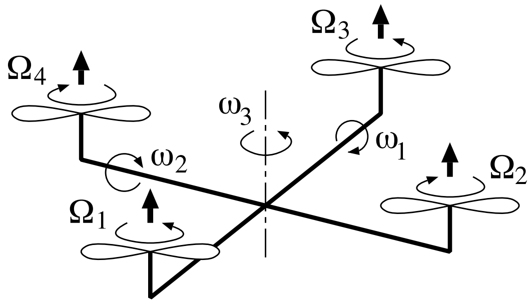

Figure 3.4: Mathematical model for a quadrotor helicopter.

Remark 3.5.6. *We have*

$$\det D\bar{M}^I(\theta,\alpha) = -4\pi^2\omega_0^2 I \ \mathrm{csch}^2\left(\frac{\pi\omega_0}{2}\right)\sin\theta_0\,\sin(\theta_0-\omega\alpha).$$

Hence, if $\bar{M}^I(\theta,\alpha)=0$, then $\det D\bar{M}^I(\theta,\alpha)\neq 0$. From Theorem 3.4.7 we see that the stable and unstable manifolds of the perturbed periodic orbit near $\gamma_I^{\mathrm{p}}=\{(x^I,I,\theta)\mid\theta\in\mathbb{S}^1_{2\pi}\}$ intersect transversely on its level set for $\varepsilon>0$ sufficiently small.

3.5.3 Periodically forced rigid body

We next consider a three-dimensional system

$$\begin{aligned}\dot\omega_1 &= \frac{I_2-I_3}{I_1}\omega_2\omega_3-\frac{I_0}{I_1}\Omega\omega_2+\frac{\ell b}{I_1}V_1,\\ \dot\omega_2 &= \frac{I_3-I_1}{I_2}\omega_3\omega_1+\frac{I_0}{I_2}\Omega\omega_1+\frac{\ell b}{I_2}V_2,\\ \dot\omega_3 &= \frac{I_1-I_2}{I_3}\omega_1\omega_2+\frac{\ell d}{I_3}V_3,\end{aligned}\tag{3.5.8}$$

which provides a mathematical model for a quadrotor helicopter shown in Fig. 3.4. In the model, $\omega_j\in\mathbb{R}$ and $I_j>0$, $j=1,2,3$, respectively, denote the angular velocities and moments of inertia about the quadrotor's principal axes, ℓ represents the length from the center of mass to the rotational axis of the rotor, and I_0, b and d represent the rotor's moment of inertia about the rotational axis, thrust factor and drag factor, respectively. Moreover,

$$\Omega=\Omega_2+\Omega_4-\Omega_1-\Omega_3$$

and

$$V_1=\Omega_4^2-\Omega_2^2,\quad V_2=\Omega_3^2-\Omega_1^2,\quad V_3=\Omega_2^2+\Omega_4^2-\Omega_1^2-\Omega_3^2,$$

where Ω_j is the angular velocity of the jth rotor for $j=$ 1-4. See [17,27] for the derivation of (3.5.8). In particular, the quadrotor can hover only if

$$\Omega_j=\Omega_0:=\frac{1}{2}\sqrt{\frac{m_0g}{b}},\quad j=\text{1-4},$$

where m_0 and g are, respectively, the quadrotor's mass and gravitational acceleration.

Let $T > 0$ be a constant, and let $\Omega_j = \Omega_0 + \varepsilon\Delta\Omega_j(t)$, where $\Delta\Omega_j(t)$ is a T-periodic function, for $j =$ 1-4. This corresponds to a situation in which the quadrotor is subjected to periodic perturbations when hovering. Let

$$\begin{aligned} &v_1(t) = \Delta\Omega_4(t) - \Delta\Omega_2(t), \quad v_2(t) = \Delta\Omega_3(t) - \Delta\Omega_1(t), \\ &v_3(t) = \Delta\Omega_4(t) + \Delta\Omega_2(t) - \Delta\Omega_3(t) - \Delta\Omega_1(t) \end{aligned}$$

and

$$\beta_0 = I_0\Omega_0, \quad \beta_1 = \beta_2 = 2\ell b\Omega_0^2, \quad \beta_3 = 2\ell d\Omega_0^2.$$

Equation (3.5.8) is written as

$$\begin{aligned} \dot\omega_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \varepsilon\left(-\frac{\beta_0}{I_1}v_3(t)\omega_2 + \frac{\beta_1}{I_1}v_1(t)\right) + O(\varepsilon^2), \\ \dot\omega_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \varepsilon\left(\frac{\beta_0}{I_2}v_3(t)\omega_1 + \frac{\beta_2}{I_2}v_2(t)\right) + O(\varepsilon^2), \\ \dot\omega_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \varepsilon\frac{\beta_3}{I_3}v_3(t) + O(\varepsilon^2), \end{aligned} \tag{3.5.9}$$

in which chaotic motions were discussed in [72] when $\beta_1 = 0$ and $v_2(t) = v_3(t) = \sin\nu t$ with $\nu > 0$ a constant. When $\varepsilon = 0$, Eq. (3.5.9) has a first integral

$$F(\omega) = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

and nonhyperbolic equilibria at

$$p_{1\pm}(c_1) = (\pm c_1, 0, 0), \quad p_{2\pm}(c_2) = (0, \pm c_2, 0), \quad p_{3\pm}(c) = (0, 0, \pm c_3)$$

on the level set $F(\omega) = c > 0$, where $c_j = \sqrt{2c/I_j}$, $j = 1, 2, 3$. The first integral $F(\omega)$ corresponds to the (Hamiltonian) energy of the unperturbed rigid body.

Let $X_\varepsilon(\omega, t) = X^0(\omega) + \varepsilon X^1(\omega, t) + O(\varepsilon^2)$ denote the non-autonomous vector field of (3.5.9) and define the corresponding autonomous vector field $\tilde X_\varepsilon(\omega, \theta) = \tilde X^0(\omega, \theta) + \varepsilon\tilde X^1(\omega, \theta) + O(\varepsilon^2)$ on $\mathbb{R}^3 \times \mathbb{S}^1_T$ like (4.5.3), where

$$\tilde X^0(\omega, \theta) = \begin{pmatrix} X^0(\omega) \\ 1 \end{pmatrix}, \quad \tilde X^1(\omega, \theta) = \begin{pmatrix} X^1(\omega, \theta) \\ 1 \end{pmatrix}.$$

The unperturbed vector field $\tilde X^0(\omega, \theta)$ has six one-parameter families of nonhyperbolic periodic orbits $\gamma_{j\pm, c_j}(t) = (p_{j\pm}(c_j), t)$, $j = 1, 2, 3$. We compute the integral (3.2.1) as

$$\mathscr{I}_{F, \gamma_{j\pm, c_j}} = \int_0^T dF(\tilde X^1)(p_j(c_j), t)dt = \pm c_j\beta_j \int_0^T v_j(t)dt, \quad j = 1, 2, 3,$$

and apply Theorems 3.2.1 and 3.2.2 to obtain the following.

Proposition 3.5.7. *For $j = 1, 2, 3$, if*

$$\beta_j \int_0^T v_j(t) \neq 0,$$

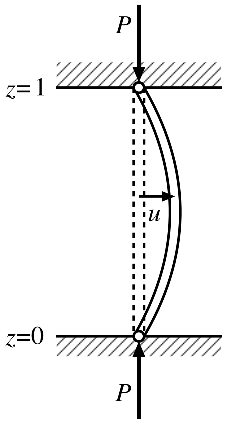

Figure 3.5: Buckeled beam. The variables u and P represent the deflection and compressive force, respectively. The length of the beam when $u \equiv 0$ is non-dimensionalized to the unity.

then the periodic orbit $\gamma_{j\pm,c_j}(t)$ does not persist for any $c_j > 0$ and the first integral $F(\omega)$ does not persist near

$$\{(p_{j+}(c_j),\theta) \in \mathbb{R} \mid c_j > 0, \theta \in \mathbb{S}^1_T\} \cup \{(p_{j-}(c_j),\theta) \in \mathbb{R} \mid c_j > 0, \theta \in \mathbb{S}^1_T\}$$

in (3.5.9)*.*

Remark 3.5.8.

(i) In [72], when $\beta_1 = 0$ and $v_2(t) = v_3(t) = \sin \nu t$ with $\nu > 0$ a constant, it was shown that the periodic orbits $\gamma_{2\pm,c_2}$ persist for $c_2 > 0$ if and only if $\beta_0 = 0$ or $\beta_3 = 0$ (see Proposition 2 of [72]).

(ii) The unperturbed vector field $X^0(\omega)$ has another first integral

$$\tilde{F}(\omega) = (I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2),$$

which corresponds to the angular momentum of the rigid body. We compute the integral (3.2.1) *as*

$$\mathscr{I}_{\tilde{F},\gamma} = \int_0^T d\tilde{F}(\hat{X}^1)(p_j(c_j),t)dt = \pm 2c_jI_j\beta_j \int_0^T v_j(t)dt, \quad j = 1,2,3,$$

so that the same statement as Proposition 4.6.6 holds for $\tilde{F}(\omega)$.

3.5.4 Three-mode truncation of a buckled beam

Finally, we consider a six-dimensional autonomous system

$$\begin{aligned} \dot{x}_1 &= x_4, \quad \dot{x}_4 = x_1 - (x_1^2 + \beta_1 x_2^2 + \beta_2 x_3^2)x_1, \\ \dot{x}_2 &= x_5, \quad \dot{x}_5 = -\omega_1^2 x_2 - \beta_1(x_1^2 + \beta_1 x_2^2 + \beta_2 x_3^2)x_2, \\ \dot{x}_3 &= x_6, \quad \dot{x}_6 = -\omega_2^2 x_3 - \beta_2(x_1^2 + \beta_1 x_2^2 + \beta_2 x_3^2)x_3, \end{aligned} \tag{3.5.10}$$

which represents a three-mode truncation of a buckled beam shown in Fig. 3.5, where $x_j \in \mathbb{R}$, $j = 1$-6, $\omega_j, \beta_j > 0$, $j = 1, 2$, are constants such that $\omega_1 < \omega_2$. See [67] for the details on the model. In (3.5.10) there is a saddle-center equilibrium at $(x_1, \dots, x_6) = (0, \dots, 0)$ and it has a homoclinic orbit. It was also shown in [71] that for almost all pairs of $\beta_1, \beta_2 > 0$ the system (3.5.10) exhibits chaotic motions and it is nonintegrable.

Let $x_j = \sqrt{\varepsilon} y_j$, $j = 1$-6, with the small parameter ε. We rewrite (3.5.10) as

$$\begin{aligned}
&\dot{y}_1 = y_4, \quad \dot{y}_4 = y_1 - \varepsilon(y_1^2 + \beta_1 y_2^2 + \beta_2 y_3^2) y_1,\\
&\dot{y}_2 = y_5, \quad \dot{y}_5 = -\omega_1^2 y_2 - \varepsilon\beta_1(y_1^2 + \beta_1 y_2^2 + \beta_2 y_3^2) y_2,\\
&\dot{y}_3 = y_6, \quad \dot{y}_6 = -\omega_2^2 y_3 - \varepsilon\beta_2(y_1^2 + \beta_1 y_2^2 + \beta_2 y_3^2) y_3,
\end{aligned} \tag{3.5.11}$$

which is regarded as as a perturbation of a linear system. When $\varepsilon = 0$, Eq. (3.5.11) has two one-parameter families of periodic orbits

$$\begin{aligned}
\gamma_{1,c}(t) &= (0, c\sin\omega_1 t, 0, 0, c\omega_1\cos\omega_1 t, 0),\\
\gamma_{2,c}(t) &= (0, 0, c\sin\omega_2 t, 0, 0, c\omega_2\cos\omega_2 t)
\end{aligned}$$

for $c > 0$, three first integrals

$$F_1(y) = -y_1^2 + y_4^2, \quad F_2(y) = \omega_1^2 y_2^2 + y_5^2, \quad F_3(y) = \omega_2^2 y_3^2 + y_6^2,$$

and six commutative vector fields

$$\begin{aligned}
&Z_1 = (y_1, 0, 0, y_4, 0, 0), \quad Z_2 = (y_4, 0, 0, y_1, 0, 0),\\
&Z_3 = (0, y_2, 0, 0, y_5, 0), \quad Z_4 = (0, y_5, 0, 0, -\omega_1^2 y_2, 0),\\
&Z_5 = (0, 0, y_3, 0, 0, y_6), \quad Z_6 = (0, 0, y_6, 0, 0, -\omega_2^2 y_3).
\end{aligned}$$

Moreover, the AVE of (3.5.11) with $\varepsilon = 0$ is given by

$$\begin{aligned}
&\dot{\eta}_1 = -\eta_4, \quad \dot{\eta}_2 = \omega_1^2 \eta_5, \quad \dot{\eta}_3 = \omega_2^2 \eta_6,\\
&\dot{\eta}_4 = -\eta_1, \quad \dot{\eta}_5 = -\hat{\eta}_2, \quad \dot{\eta}_6 = -\eta_3,
\end{aligned}$$

which has four linearly independent periodic solutions

$$\begin{aligned}
\tilde{\gamma}_1(t) &= (0, \omega_1\sin\omega_1 t, 0, 0, \cos\omega_1 t, 0),\\
\tilde{\gamma}_2(t) &= (0, \omega_1\cos\omega_1 t, 0, 0, -\sin\omega_1 t, 0),\\
\tilde{\gamma}_3(t) &= (0, 0, \omega_2\sin\omega_2 t, 0, 0, \cos\omega_2 t),\\
\tilde{\gamma}_4(t) &= (0, 0, \omega_2\cos\omega_2 t, 0, 0, -\sin\omega_2 t)
\end{aligned}$$

and two linearly independent unbounded solutions. We compute (3.2.1) and (3.3.9) as

$$\mathscr{I}_{F_j,\gamma_{\ell,c}} = \int_0^{2\pi/\omega_\ell} dF_j(X_1)(\gamma_{\ell,c}(t))dt = 0, \quad j = 1, 2, 3 \text{ and } \ell = 1, 2,$$

and

$$\begin{aligned}
\mathscr{J}_{\tilde{\gamma}_j, Z_k, \gamma_{\ell,c}} &= \int_0^{2\pi/\omega_\ell} \tilde{\gamma}_j(t)\cdot [X_1, Z_k]_{\gamma_{\ell,c}(t)} dt\\
&= \begin{cases} \dfrac{3}{2}\pi\beta_1^2 c^3 & \text{if } (j,k,\ell) = (2,3,1);\\ \dfrac{3}{2}\pi\beta_2^2 c^3 & \text{if } (j,k,\ell) = (4,5,2);\\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{3.5.12}$$

where X^1 represents the $O(\varepsilon)$-terms of the vector field in (3.5.11). In (3.5.12), the subscript j is allowed to take 1 or 2 for $\ell = 1$, and 3 or 4 for $\ell = 2$. Theorems 3.2.1, 3.2.2 and and 3.3.8 give no meaningful information on persistence of periodic orbits and first integrals, but application of Theorem 3.3.5 yields the following.

Proposition 3.5.9. *The commutative vector fields Z_3 and Z_5 do not persist near the (y_2, y_5)- and (y_3, y_6)-planes, respectively, in* (3.5.11). *Moreover, in* (3.5.10)*, near the origin, there is no commutative vector field which has the linear term*

$$\tilde{Z}_3 = (0, x_2, 0, 0, x_5, 0) \quad \text{or} \quad \tilde{Z}_5 = (0, 0, x_3, 0, 0, x_6).$$

Proof. The first part immediately follows from application of Theorem 3.3.5. The second part is easily proven since a vector field having such a linear term for (3.5.10) is transformed to $Z_3 + O(\varepsilon)$ or $Z_5 + O(\varepsilon)$ for (3.5.11). □

Remark 3.5.10.

(i) By the Lyapunov center theorem (e.g., Theorem 5.6.7 of [2]), there exist two families of periodic orbits in (3.5.10) *if $\omega_2/\omega_1, \omega_1/\omega_2 \notin \mathbb{Z}$. Hence, the periodic orbits $\gamma_{j,c}$, $j = 1, 2$, persist in* (3.5.11) *for such values of ω_j, $j = 1, 2$, at least.*

(ii) As shown in [71], the Hamiltonian system (3.5.10) *is nonintegrable for almost all pairs of β_j, $j = 1, 2$. Hence, the three first integrals $F_j(y)$, $j = 1, 2, 3$, do not persist in* (3.5.11) *for such values of β_j, $j = 1, 2$, at least.*

3.A Some auxiliary materials for Section 3.3

In this appendix, we provide some prerequisites for Section 3.3: Basic notions and facts on connections of vector bundles and linear differential equations. Similar materials are found in [22, 32, 47]. See, e.g., [16] for necessary information on vector bundles.

3.A.1 Connections and horizontal sections

We begin with connections of vector bundles and their horizontal sections. Henceforth M represents a C^1 m-dimensional manifold for $m \in \mathbb{N}$, and E represents a C^1 vector bundle of rank r over M with a projection $\pi : E \to M$ for some $m, r \in \mathbb{N}$. Let $C(M)$ be a set of all C^1 $\mathbb{R}$-valued functions on M and let $C(M, E)$ be a set of all C^1 sections of E. Let T^*M be the cotangent bundle of M. Note that $T^*M \otimes E$ is also a C^1 vector bundle. We first give basic definitions.

Definition 3.A.1. *An $\mathbb{R}$-linear map*

$$\nabla : C(M, E) \to C(M, T^*M \otimes E)$$

is called a connection *of the vector bundle E if*

$$\nabla(fs) = df \otimes s + f\nabla s \tag{3.A.1}$$

for any $f \in C(M)$ and $s \in C(M, E)$. A section $s \in C(M, E)$ is said to be horizontal *for the connection ∇ if $\nabla s = 0$.*

Let $U \subset M$ be an open neighborhood and let $\{e_j\}_{j=1}^r$ be a frame on U, so that any section $s \in C(M, E)$ is expressed as

$$s = \sum_{j=1}^{r} s^j e_j \tag{3.A.2}$$

on U for some $s^j \in C(M)$ for $j = 1, \ldots, r$.

Definition 3.A.2. *For each $i = 1, \ldots, r$ we can write*

$$\nabla e_i = \sum_j \theta_i^j \otimes e_j, \tag{3.A.3}$$

where $\theta_i^j : M \to T^*M$, $j = 1, \ldots, r$. The $r \times r$ matrix $\theta = (\theta_i^j)$ is called the connection form *of ∇ on U in the frame $\{e_j\}_{j=1}^r$.*

Let $s \in C(M, E)$. Using (3.A.1) and (3.A.3), we compute

$$\begin{aligned}
\nabla s &= \sum_{j=1}^{r} \nabla(s^j e_j) = \sum_{j=1}^{r} (ds^j \otimes e_j + s^j \nabla e_j) \\
&= \sum_{i=1}^{r} ds^i \otimes e_i + \sum_{i=1}^{r} s^i \left(\sum_{j=1}^{r} \theta_i^j \otimes e_j \right) = \sum_{i=1}^{r} \left(ds^i + \sum_{j=1}^{r} s^j \theta_j^i \right) \otimes e_i.
\end{aligned}$$

Hence, the condition for the section s to be horizontal, $\nabla s = 0$, is equivalent to

$$ds^i + \sum_{j=1}^{r} s^j \theta_j^i = 0, \quad i = 1, ..., r, \tag{3.A.4}$$

on U.

Definition 3.A.3. *Let E^* be the dual bundle of E. A connection ∇^* of E^* given by*

$$d\langle s, \alpha \rangle = \langle \nabla s, \alpha \rangle + \langle s, \nabla^* \alpha \rangle \tag{3.A.5}$$

for any $s \in C(M, E)$ and $\alpha \in C(M, E^*)$ is called a dual connection *of ∇.*

Let $\{e^j\}_{j=1}^r$ be the dual frame for the frame $\{e_j\}_{j=1}^r$, i.e.,

$$\langle e_i, e^j \rangle = \delta_{ij}, \quad i, j = 1, \ldots, r, \tag{3.A.6}$$

where δ_{ij} is Kronecker's delta. We have the following relation between connections and their dual connections.

Proposition 3.A.4. *Let $\theta = (\theta_i^j)$ be the connection form of a connection ∇ on U. Then the connection form θ^* of the dual connection ∇^* is given by $\theta_i^{*j} = -\theta_j^i$ on U.*

Proof. Using (3.A.5) and (3.A.6), we compute

$$0 = d\langle e_i, e^j \rangle = \langle \nabla e_i, e^j \rangle + \langle e_i, \nabla^* e^j \rangle.$$

Since by (3.A.3) and (3.A.6)

$$\langle \nabla e_i, e^j \rangle = \left\langle \sum_{k=1}^{r} \theta_i^k \otimes e_k, e^j \right\rangle = \theta_i^j = \left\langle e_i, \sum_{k=1}^{r} \theta_k^j \otimes e^k \right\rangle,$$

we obtain

$$\left\langle e_i, \sum_{k=1}^{r} \theta_k^j \otimes e^k + \nabla^* e^j \right\rangle = 0, \quad i, j = 1, \ldots, r.$$

Hence,

$$\nabla^* e^j = \sum_{k=1}^{r} -\theta_k^j \otimes e^k$$

for $j = 1, \ldots, r$. □

3.A.2 Connections and linear differential equations

Let $m = 1$ and assume that the one-dimensional manifold M is paracompact and connected. We will see below that a connection of the vector bundle E defines a linear differential equation and horizontal sections of the connection correspond to solutions to the differential equations.

Take an open neighborhood $U \subset M$ and its local coordinate $t \in \mathbb{R}$. Let ∇ be a connection and let $s \in C(M, E)$ be a horizontal section of ∇ given by (3.A.2). We write the connection form $\theta = (\theta_i^j)$ as

$$\theta_i^j = a_{ij}(t) dt$$

for some $a_{ij}(t) \in C(M)$. Then Eq. (3.A.4) is expressed as

$$ds^i + \sum_{j=1}^{r} a_{ji}(t) s^j dt = 0, \quad i = 1, ..., r. \tag{3.A.7}$$

Let $A(t) = (A_{ij}(t))$ be an $r \times r$ matrix with $A_{ij}(t) := -a_{ji}(t)$ and let $\hat{s}(t) = (s^1(t), ..., s^r(t))^{\mathrm{T}}$. From (3.A.7) we obtain a linear differential equation

$$\frac{d}{dt}\hat{s}(t) = A(t)\hat{s}(t). \tag{3.A.8}$$

Thus, the relation $\nabla s = 0$ is locally represented by a linear differential equation. Below we apply the above argument to the VE (3.3.2) and AVE (3.3.5) to derive (3.3.3) and (3.3.6), respectively.

Derivation of (3.3.3)

We consider the VE (3.3.2) and set $M = \Gamma$ and $E = T_\Gamma$ with $r = n$. Choose the frame $\left(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}\right)$ and write

$$X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}$$

locally. We compute

$$\begin{aligned}\nabla \frac{\partial}{\partial x_i} =& dt \otimes \mathcal{L}_X \left(\frac{\partial}{\partial x_i}\right)\Big|_\Gamma = dt \otimes \left[\sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right]_\Gamma \\ =& dt \otimes \left(-\sum_{j=1}^{n} \frac{\partial X_j}{\partial x_i}\frac{\partial}{\partial x_j}\right)\Big|_\Gamma = -\sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j}(\phi(t)) dt \otimes \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n,\end{aligned}$$

so that

$$\theta_i^j = -\frac{\partial X_j}{\partial x_i}(\phi(t)) dt, \quad \text{i.e.,} \quad A_{ij}(t) = \frac{\partial X_i}{\partial x_j}(\phi(t)), \quad i, j = 1, \dots, n. \tag{3.A.9}$$

This yields (3.3.3) along with (3.A.8).

Derivation of (3.3.6)

We next consider the AVE (3.3.5).Choose the frame $(dx_1, \dots, dx_n)$. Using Proposition 3.A.4 and (3.A.9), we obtain

$$\theta_i^{*j} = \frac{\partial X_i}{\partial x_j}(\phi(t)) \quad \text{i.e.,} \quad A_{ij}(t) = -\frac{\partial X_j}{\partial x_i}(\phi(t)), \quad i, j = 1, \dots, n.$$

This yields (3.3.6) along with (3.A.8).

Chapter 4

Nonintegrability of Nearly Integrable Systems

4.1 Introduction

In his famous memoir [51], which was related to a prize competition celebrating the 60th birthday of King Oscar II, Henri Poincaré studied two-degree-of-freedom Hamiltonian systems depending on a small parameter, say ε here although he used the letter μ instead, such that they are integrable when $\varepsilon = 0$, and showed the nonexistsnce of first integrals which are analytic in the state variables and parameter ε and functionally independent of Hamiltonians, under some nondegenerate conditions. If there exists such a first integral, then the Hamiltonian systems are integrable for $|\varepsilon| \geq 0$ sufficiently small in the sense of Liouville [9, 43]. The result was improved significantly in the first volume of his masterpieces [52] published two years later, so that more-degree-of-freedom Hamiltonian systems can be treated. Using these results, he discussed the nonexistence of such first integrals in the restricted planar and spacial three-body problem there. See also [14] for an account of his work from a mathematical and historical perspectives. Subsequently, his results were sophisticated and generalized to non-Hamiltonian systems [35, 36]. In particular, Kozlov [36] treated multi-dimensional systems of the form

$$\dot{I} = \varepsilon h(I, \theta; \varepsilon), \quad \dot{\theta} = \omega(I) + \varepsilon g(I, \theta; \varepsilon), \quad (I, \theta) \in \mathbb{R}^\ell \times \mathbb{T}^m, \tag{4.1.1}$$

where ε is a small parameter such that $|\varepsilon| \ll 1$, $\mathbb{T}^m = \prod_{j=1}^m \mathbb{S}^1$ with $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is an m-dimensional torus and $h(I, \theta; \varepsilon)$, $\omega(I)$ and $g(I, \theta; \varepsilon)$ are analytic in (I, θ, ε). Note that the system (4.1.1) is Hamiltonian if $\ell = m$ as well as $\varepsilon = 0$ or

$$\mathrm{D}_I h(I, \theta; \varepsilon) = -\mathrm{D}_\theta g(I, \theta; \varepsilon),$$

and non-Hamiltonian if not. When $\varepsilon = 0$, Eq. (4.1.1) becomes

$$\dot{I} = 0, \quad \dot{\theta} = \omega(I) \tag{4.1.2}$$

which we refer to as the *unperturbed system* for (4.1.1). We often use this terminology for other systems below. Here we state some details of his result.

We expand $h(I, \theta; 0)$ in Fourier series as

$$h(I, \theta; 0) = \sum_{r \in \mathbb{Z}^m} \hat{h}_r(I) \exp(ir \cdot \theta), \tag{4.1.3}$$

where $\hat{h}_r(I)$, $r \in \mathbb{Z}^m$, are the Fourier coefficients and "$\cdot$" represents the inner product. We assume the following for (4.1.1):

(K1) The system (4.1.1) has s first integrals $F_j(I, \theta; \varepsilon)$, $j = 1, \ldots, s$, which are analytic in (I, θ, ε);

(K2) If $r \in \mathbb{Z}^m$ and $r \cdot \omega(I) = 0$ for any $I \in \mathbb{R}^\ell$, then $r = 0$.

If assumption (K2) holds, then we say that the unperturbed system (4.1.2) is *nondegenerate*. Under (K1) and (K2) we can show that $F_j(I, \theta; 0)$, $j = 1, \ldots, s$, are independent of θ (see Lemma 1 in Section 1 of Chapter IV of [36]), and write $F_{j0}(I) = F_j(I, \theta; 0)$ and $F_0(I) = (F_{10}(I), \ldots, F_{s0}(I))$. We refer to $\mathscr{P}_s \subset \mathbb{R}^\ell$ as a *Poincaré set* if for each $I \in \mathscr{P}_s$ there exists linearly independent vectors $r_j \in \mathbb{Z}^m$, $j = 1, \ldots, \ell - s$, such that

(i) $r_j \cdot \omega(I) = 0$, $j = 1, \ldots, \ell - s$;

(ii) $\hat{h}_{r_j}(I)$, $j = 1, \ldots, \ell - s$, are linearly independent.

Let U be a domain in $\mathbb{R}^\ell$. A set $\Delta \subset U$ is called a *key set* (or *uniqueness set*) for $C^\omega(U)$ if any analytic function vanishing on Δ vanishes on U. For example, any dense set in U is a key set for $C^\omega(U)$. In this situation, we have the following theorem (see Section 1 of Chapter IV of [36] for its proof).

Theorem 4.1.1 (Kozlov)**.** *Suppose that assumptions (K1) and (K2) hold, the Jacobian matrix* $\mathrm{D}F_0(I)$ *has the maximum rank at a point* $I_0 \in \mathbb{R}^\ell$ *and a Poincaré set* $\mathscr{P}_s \subset U$ *is a key set for* $C^\omega(U)$*, where* U *is a neighborhood of* I_0*. Then the system* (4.1.1) *has no first integral which is real-analytic in* (I, θ, ε) *and functionally independent of* $F_j(I, \theta; \varepsilon)$*,* $j = 1, \ldots, s$*, near* $\varepsilon = 0$*.*

A version of Theorem 4.1.1 for the Hamiltonian case $\ell = m$ was given in [35] (see also Theorem 7.1 of [10]). The Hamiltonian case of $s = 1$ with a dense Poincaré set in Theorem 4.1.1 was treated by Poincaré for $\ell = m \geq 2$ in [52]. When $s = 0$ in (K1), Theorem 4.1.1 means that under its hypotheses there exists no first integral which is analytic in (I, θ, ε). When $s = 1$ in (K1), which always occurs if the system (4.1.1) is Hamiltonian, it means that under its hypotheses there exists no first integral which is analytic in $(I, \theta) \in U \times \mathbb{T}^m$ and ε near $\varepsilon = 0$ and functionally independent of $F_1(I, \theta, \varepsilon)$. When $s = m - 1$ in (K1), if besides (K1) and (K2) there exists a key set Δ for $C^\omega(U)$ such that $r \cdot \omega(I) = 0$ and $h_r(I) \neq 0$ on Δ for some $r \in \mathbb{Z}^m \setminus \{0\}$, there exists no additional first integral which is analytic in (I, θ, ε). For Hamiltonian systems, if they are not Liouville-integrable for any $|\varepsilon| > 0$ sufficiently small, then there does not exist such a first integral. For non-Hamiltonian systems this may be true in an appropriate meaning but some additional ingredients are needed as seen below.

In this chapter we study more general dynamical systems of the form

$$\dot{x} = X_\varepsilon(x), \quad x \in \mathscr{M}, \tag{4.1.4}$$

where ε is a small parameter such that $|\varepsilon| \ll 1$, $\mathscr{M}$ is an n-dimensional analytic manifold for $n \geq 2$ and the vector field X_ε is analytic in x and ε. Let $X_\varepsilon = X^0 + \varepsilon X^1 + O(\varepsilon^2)$ for $|\varepsilon| > 0$ sufficiently small. When $\varepsilon = 0$, the system (4.1.4) becomes

$$\dot{x} = X^0(x), \tag{4.1.5}$$

which is assumed to be *analytically* $(q, n-q)$*-integrable* in the following meaning of Bogoyavlenskij [15] for some positive integer $q \leq n$.

Definition 4.1.2 (Bogoyavlenskij)**.** *The system* (4.1.5) *is called* $(q, n-q)$-integrable *or simply* integrable *if there exist q vector fields* $Y_1(x)(:= X^0(x)), Y_2(x), \ldots, Y_q(x)$ *and* $n-q$ *scalar-valued functions* $F_1(x), \ldots, F_{n-q}(x)$ *such that the following two conditions hold:*

(i) $Y_1(x), \ldots, Y_q(x)$ *are linearly independent almost everywhere and commute with each other, i.e.,* $[Y_j, Y_k](x) \equiv 0$ *for* $j, k = 1, \ldots, q$*, where* $[\cdot, \cdot]$ *denotes the Lie bracket;*

(ii) $F_1(x), \ldots, F_{n-q}(x)$ *are functionally independent, i.e.,* $dF_1(x), \ldots, dF_{n-q}(x)$ *are linearly independent almost everywhere, and* $F_1(x), \ldots, F_{n-q}(x)$ *are first integrals of* $Y_1, \ldots, Y_q$*, i.e.,* $dF_k(Y_j) = 0$ *for* $j = 1, \ldots, q$ *and* $k = 1, \ldots, n-q$.

If $Y_1, Y_2, \ldots, Y_q$ *and* $F_1, \ldots, F_{n-q}$ *are analytic (resp. meromorphic), then Eq.* (4.1.5) *is said to be* analytically *(resp.* meromorphic*)* integrable.

Definition 4.1.2 is considered as a generalization of Liouville-integrability for Hamiltonian systems [9,43] since an m-degree-of-freedom Liouville-integrable Hamiltonian system with $m \geq 1$ has not only m functionally independent first integrals but also m linearly independent commutative (Hamiltonian) vector fields generated by the first integrals. The $(\ell + m)$-dimensional system (4.1.2) is (m, ℓ)-integrable in the Bogoyavlenskij sense.

The system (4.1.4) is regarded as a perturbation of the analytically integrable system (4.1.5). If there exist q' linearly independent, analytic commutative vector fields and $n - q'$ functionally independent, analytic first integrals depending on ε analytically near $\varepsilon = 0$ for some positive integer $q' \leq n$, then the perturbed system (4.1.4) is analytically $(q', n-q')$-integrable for $|\varepsilon| \geq 0$ sufficiently small. We especially assume that there exists a q-parameter family of periodic orbits on a regular level set of the $n-q$ first integrals having a connected and compact component (see Section 4.2 for our precise definitions containing this one) and give sufficient conditions for nonexistence of such $n-q$ real-analytic first integrals and for real-analytic nonintegrability of the perturbed system (4.1.4) near the regular level set such that the first integrals and commutative vector fields depend analytically on ε near $\varepsilon = 0$. The persistence of such first integrals and commutative vector fields in the perturbed system (4.1.4) along with that of periodic and homoclinic orbits was previously discussed in Chapter 3. Our approach is different from those of Poincaré [51,52] and Kozlov [35,36] and based on the technique of Chapter 3. Recently, another sufficient condition for nonintegrability of nearly integrable dynamical systems near resonant periodic orbits was also obtained using a different approach in [73] and the theory was applied to prove the nonintegrability of the restricted three-body problem in [74], when the independent and state variables are extended to complex ones.

The unperturbed system (4.1.5) can be transformed to (4.1.2) under our assumptions, as stated in Proposition 4.2.1 below. However, Theorem 4.1.1 only says in our setting for (4.1.1) that there exists no such additional first integral even if its hypotheses hold. In particular, it does not allow us to determine the nonintegrability of (4.1.4) in the Bogoyavlenskij sense when it is non-Hamiltonian. We also describe a consequence of our results to (4.1.1) and show how it improves the results of Poincaré [52] and Kozlov [35,36].

Moreover, we discuss a relationship between our results and the subharmonic and homoclinic Melnikov methods [26,62,65] for time-periodic perturbations of single-degree-of-freedom analytic Hamiltonian systems, which can be transformed to the form of (4.1.4) with $\ell = 1$ and $m = 2$, i.e, $(2, 1)$-integrable, and have a one-parameter family of periodic orbits when $\varepsilon = 0$. As well known, if the subharmonic Melnikov functions have a simple zero, then the corresponding unperturbed periodic orbits persist in the perturbed systems. Morales-Ruiz [44] studied the Hamiltonian perturbation case in which the unperturbed systems have homoclinic orbits, and showed a relationship between their nonintegrability and a version due to Ziglin [78] of the homoclinic Melnikov method when the independent and state variables are extended to complex ones and the small parameter ε is also regarded as a state variable. More concretely, under some restrictive conditions, based on the generalized version due to Ayoul and Zung [12] of the Morales-Ramis theory [43, 47], which provides a sufficient condition for nonintegrability of autonomous dynamical systems, he essentially proved that they are meromorphically nonintegrable if the homoclinc Melnikov functions which are obtained as integrals along closed loops on the complex plane are not identically zero. Here the version of the Melnikov method enables us to detect transversal self-intersection of complex separatrices of periodic orbits and to prove its analytic nonintegrability, unlike the standerd version [26, 42, 62]. Here we prove two variants of Morales-Ruiz [44] for periodic orbits: if the subharmonic Melnikov functions for a dense set of the unperturbed periodic orbits are not identically zero, then there exists no first integral depending analytically on ε near $\varepsilon = 0$; and if the 'standard' homoclinic Melnikov functions [26, 42, 62] are not identically zero, then the perturbed systems are not Bogoyavlenskij-integarble such that the commutative vector fields and first integrals depend analytically on ε near $\varepsilon = 0$.

We illustrate our theory for three examples: Simple pendulum with a constant torque, second-order coupled oscillators and the periodically forced Duffing oscillator [26, 31, 62]. Real-analytic nonintegrability is discussed in special cases of the second and third examples while existence of real-analytic first integrals in the rest. In particular, the special case of the third one is shown to be nonintegrable in the above meaning even if it does not have transverse homoclinic orbits to a periodic orbit.

This chapter is organized as follows: In Section 4.2, we state our precise assumptions and main theorems. In Section 4.3, we give proofs of the main theorems. We describe a consequence of our results to (4.1.1) in Section 4.4 and discuss a relationship between our results and the subharmonic and homoclinic Melnikov methods for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems in Section 4.5. Finally, we give the three examples in Section 4.6.

4.2 Main Results

In this section, we state our main results for (4.1.4). We first make the following assumptions on the unperturbed system (4.1.5):

(A1) For some positive integer $q < n$, the system (4.1.5) is analytically $(q, n-q)$ Bogoyavlenskij-integrable, i.e., there exist q analytic vector fields $Y_1(x)(:= X^0(x)), \ldots, Y_q(x)$ and $n - q$ analytic scalar-valued functions $F_1(x), \ldots, F_{n-q}(x)$ such that conditions (i) and (ii) of Definition 4.1.2 hold.

(A2) Let $F(x) = (F_1(x), \ldots, F_{n-q}(x))$. There exists a regular value $c \in \mathbb{R}^{n-q}$ of F, i.e., $\operatorname{rank} dF(x) = n - q$ when $F(x) = c$, such that $F^{-1}(c)$ has a connected and compact component and $Y_1(x), \ldots, Y_q(x)$ are linearly independent on $F^{-1}(c)$.

Henceforth we assume without loss of generality that $F^{-1}(c)$ is connected and compact itself in (A2), by reducing the domain of F if necessary. Under (A1) and (A2) we have the following result like a well-known theorem for Hamiltonian systems [9] (see [15,80,81] for the details).

Proposition 4.2.1 (Liouville-Miuner-Arnold-Jost). *Suppose that assumptions (A1) and (A2) hold. Then we have the following:*

(i) *The level set $F^{-1}(c)$ is analytically diffeomorphic to the q-dimensional torus $\mathbb{T}^q$;*

(ii) There exists an analytic diffeomorphism $\varphi : U \times \mathbb{T}^q \to \mathcal{U}$, where U and $\mathcal{U}$ are, respectively, neighborhoods of $I = I_0$ in $\mathbb{R}^{n-q}$ and of $F^{-1}(c)$ in $\mathscr{M}$ for some $I_0 \in \mathbb{R}^{n-q}$, such that

(iia) $\varphi(\{I_0\} \times \mathbb{T}) = F^{-1}(c)$;

(iib) $F \circ \varphi(I, \theta)$ *depends only on* I, *where* $(I, \theta) \in U \times \mathbb{T}^q$;

(iic) The flow of X^0 on $\mathcal{U}$ is analytically conjugate to that of (4.1.2) *with $\ell = n - q$ and $m = q$ on $U \times \mathbb{T}^q$.*

The variables I and θ are called the *action* and *angle variables* as in Hamiltonian systems, and $\omega(I)$ is referred to as the *angular frequency vector*. Let $\omega_j(I)$ be the jth component of $\omega(I)$ for $j = 1, \dots, q$. Henceforth U denotes the neighborhood of $I = I_0$ in Proposition 4.2.1. Moreover, we assume the following on (4.1.2) along with (K2):

(A3) There exists a key set D_{R} for $C^\omega(U)$ such that for $I \in D_{\mathrm{R}}$ a resonance of multiplicity $q - 1$,

$$\dim_{\mathbb{Q}} \langle \omega_1(I), \dots, \omega_q(I) \rangle = 1,$$

occurs with $\omega(I) \neq 0$, i.e., there exists a positive constant $\omega_0(I)$ depending on I such that

$$\frac{\omega(I)}{\omega_0(I)} \in \mathbb{Z}^q \setminus \{0\}.$$

We choose $\omega_0(I)$ as large as possible below.

We easily see that if $\operatorname{rank} \mathrm{D}\omega(I^*) = q$ for some $I^* \in U$, then both (K2) and (A3) hold. Assumption (A3) also implies that if $I \in D_{\mathrm{R}}$, then the system (4.1.2) has a q-parameter family of periodic orbits

$$(I, \theta) = (I, \omega(I)t + \tau), \quad \tau \in \mathbb{T}^q, \tag{4.2.1}$$

with the period $T^I = 2\pi/\omega_0(I)$. Note that the periodic orbits (4.2.1) is parameterized by $(q-1)$ parameters essentially since two periodic orbits $(I, \omega(I)t + \tau)$ and $(I, \omega(I)t + \tau_0)$ represent the same orbit if $\tau - \tau_0 = \omega(I)t_0$ for some $t_0 \in \mathbb{R}$. We also have a q-parameter (but essentially $(q-1)$-parameter) family of periodic orbits

$$\gamma_\tau^I(t) = \varphi(I, \omega(I)t + \tau), \quad (I, \tau) \in D_{\mathrm{R}} \times \mathbb{T}^q,$$

with the period T^I in the unperturbed system (4.1.5). Define the integrals

$$\mathscr{I}_{F_k}^I(\tau) := \int_0^{T^I} dF_j(X^1)_{\gamma_\tau^I(t)} dt, \quad k = 1, \dots, n - q, \tag{4.2.2}$$

for $I \in D_{\mathrm{R}}$ and set $\mathscr{I}_F^I(\tau) := (\mathscr{I}_{F_1}^I(\tau), \ldots, \mathscr{I}_{F_{n-q}}^I(\tau))$. Note that

$$\mathscr{I}_{F_k}^I(\tau + \omega(I)t) = \mathscr{I}_{F_k}^I(\tau)$$

for $\tau \in \mathbb{T}^q$ and $t \in \mathbb{R}$. We state the first of our main results as follow.

Theorem 4.2.2. *Suppose that assumptions (A1)-(A3) and (K2) hold. If there exists a key set $D \subset D_{\mathrm{R}}$ for $C^\omega(U)$ such that $\mathscr{I}_F^I(\tau)$ is not identically zero for any $I \in D$, then the perturbed system* (4.1.4) *does not have $n - q$ real-analytic first integrals in a neighborhood of $F^{-1}(c)$ near $\varepsilon = 0$ such that they are functionally independent for $|\varepsilon| \neq 0$ and depend analytically on ε.*

We prove Theorem 4.2.2 in Section 4.3.1.

We next consider a special case in which Eq. (4.1.4) is a two- or more-degree-of-freedom Hamiltonian system. For an integer $m \geq 2$, let $(\mathscr{M}, \Omega)$ be a $2m$-dimensional analytic symplectic manifold with a symplectic form Ω, and let $H_\varepsilon(x) = H^0(x) + \varepsilon H^1(x) + O(\varepsilon^2)$ be an analytic Hamiltonian for (4.1.4) and depend analytically on ε near $\varepsilon = 0$. We assume the following:

(A1') The unperturbed Hamiltonian system (4.1.5) with the Hamiltonian $H^0(x)$ is real-analytically Liouville-integrable, i.e., there exist m functionally independent analytic first integrals $F_1(x)(:= H^0(x)), F_2(x), \ldots, F_m(x)$ such that they are pairwise Poisson commutative, i.e., $\{F_j, F_k\} = 0$ for $j, k = 1, \ldots, m$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket for the symplectic form Ω.

Assumption (A1') means (A1) with $q = m$ and $Y_j = X_{F_j}$ for $j = 1, \ldots, m$, where X_{F_j} denotes the Hamiltonian vector field for the Hamiltonian $F_j(x)$. So we immediately obtain the following from Theorem 4.2.2.

Corollary 4.2.3. *Suppose that (A1'), (A2), (A3) and (K2) hold. If there exists a key set $D \subset D_{\mathrm{R}}$ for $C^\omega(U)$ such that $\mathscr{I}_F^I(\tau)$ with $X_1 = X_{H^1}$ is not identically zero for any $I \in D$, then the perturbed Hamiltonian system for the Hamiltonian $H_\varepsilon(x)$ is not real-analytically Liouville-integrable in a neighborhood of $F^{-1}(c)$ near $\varepsilon = 0$ such that $m - 1$ additional first integrals also depend analytically on ε.*

For integrability of non-Hamiltonian systems (4.1.4), we have to consider not only first integrals but also commutative vector fields. So we need the following assumption additionally:

(A4) For some $I^* \in U$, the Jacobian matrix $\mathrm{D}\omega(I^*)$ is injective, i.e.,

$$\operatorname{rank} \mathrm{D}\omega(I^*) = n - q.$$

Note that (A4) holds only when $n - q \leq q$. Finally, we state the second main result as follows.

Theorem 4.2.4. *Suppose that assumptions (A1)-(A4) and (K2) hold. If there exists a key set $D \subset D_{\mathrm{R}}$ for $C^\omega(U)$ such that $\mathscr{I}_F^I(\tau)$ is not constant for any $I \in D$, then for $|\varepsilon| \neq 0$ sufficiently small the perturbed system* (4.1.4) *is not real-analytically integrable in the Bogoyavlenskij sense near $F^{-1}(c)$ such that the first integrals and commutative vector fields depend analytically on ε near $\varepsilon = 0$: In particular there exists only such $n - q - 1$ first integrals and such $q - 1$ commutative vector fields at most.*

Remark 4.2.5.

(i) *Theorems 4.2.2 and 4.2.4 exclude even the possibility that the first integrals are functionally independent for* $|\varepsilon| \neq 0$ *sufficiently small but not for* $\varepsilon = 0$.

(ii) *Theorem 4.2.4 does not exclude the possibility that the system* (4.1.4) *is real-analytically integrable in the Bogoyavlenskij sense near* $F^{-1}(c)$ *such that in a punctured neighborhood of* $\varepsilon = 0$ *there exist such* $n - q'(\geq 0)$ *first integrals and* q' *commutative vector fields for some* $q' > q$.

We prove Theorem 4.2.4 in Section 4.3.2.

4.3 Proofs of the main theorems

In this section, we prove Theorems 4.2.2 and 4.2.4. The proofs are based on the results of Chapter 3. Henceforth we mean real-analyticity when saying analyticity.

4.3.1 Proof of Theorem 4.2.2

We first prove Theorem 4.2.2. Henceforth we assume that conditions (A1)-(A3) and (K2) hold. We begin with the following lemma.

Lemma 4.3.1. *Suppose that* $G_1(x), \dots, G_k(x)$ *are analytic first integrals of the unperturbed system* (4.1.5) *near* $F^{-1}(c)$ *for* $k \geq 1$ *and they may be functionally dependent. Let* $G(x) = (G_1(x), \dots, G_k(x))$ *and let* ϕ_t *denote the flow generated by* (4.1.5). *Then the following statements hold:*

(i) There exists an analytic map $\psi : F(\mathcal{U}) \to \mathbb{R}^k$ *satisfying* $G = \psi \circ F$ *in a neighborhood* $\mathcal{U}$ *of* $F^{-1}(c)$ *in* $\mathscr{M}$.

(ii) If $dG_1, \dots, dG_k$ *are linearly independent at* $x \in \mathscr{M}$, *then so are they at* $\phi_t(x)$.

Proof. For the proof of part (i) we first transform (4.1.5) to (4.1.2) with $\ell = n - q$ and $m = q$, based on Proposition 4.2.1. In particular, by Proposition 4.2.1(iia) $F \circ \varphi(I, \theta)$ depends only on I. Let $\tilde{F}(I) = F \circ \varphi(I, \theta)$. On the other hand, we use Lemma 1 in Section 1 of Chapter IV of [36] to show that $G(\varphi(I, \theta))$ depends only on I. Let $\tilde{G}(I) = G(\varphi(I, \theta))$. So we have

$$G = \tilde{G} \circ \varphi^{-1} = \tilde{G} \circ \tilde{F}^{-1} \circ \tilde{F} \circ \varphi^{-1}$$

and take $\psi = \tilde{G} \circ \tilde{F}^{-1}$ to obtain part (i).

We turn to the proof of part (ii). Since the pull-back of G_j by ϕ_t satisfies $\phi_t^* G_j = G_j$ for $j = 1, \dots, k$, we have

$$\begin{aligned}
\phi_t^*(dG_1 \wedge \cdots \wedge dG_k)_x &= (\phi_t^* dG_1 \wedge \cdots \wedge \phi_t^* dG_k)_x \\
&= (d(\phi_t^* G_1) \wedge \cdots \wedge d(\phi_t^* G_k))_x = (dG_1 \wedge \cdots \wedge dG_k)_x \neq 0,
\end{aligned}$$

where '$\wedge$' represents the wedge product. On the other hand, by the definition of pull-back

$$\phi_t^*(dG_1 \wedge \ldots \wedge dG_k)_x = (dG_1 \wedge \ldots \wedge dG_k)_{\phi_t(x)} \circ (d\phi_t)_x.$$

Since ϕ^t is a diffeomorphism, we see that $(d\phi_t)_x$ is regular. Hence,

$$(dG_1 \wedge \ldots \wedge dG_k)_{\phi_t(x)} \neq 0,$$

which yields part (ii). $\square$

Using Theorem 3.2.2 on persistence of first integrals and Lemma 4.3.1, we obtain the following result.

Lemma 4.3.2. *Let $I \in D_{\mathrm{R}}$. Suppose that near $\varepsilon = 0$ the perturbed system* (4.1.4) *has $n-q$ analytic first integrals $G_1^\varepsilon, \ldots, G_{n-q}^\varepsilon$ near $\mathscr{T}_I = \{\gamma_\tau^I \mid \tau \in \mathbb{T}^q\}$ in $\mathscr{M}$ such that they are functionally independent on $\mathscr{T}_I$ and depend analytically on ε. Then $\mathscr{I}_F^I(\tau)$ must be identically zero.*

Proof. Assume that the hypothesis of the lemma holds. Theorem 3.2.2, we have

$$\mathscr{I}_{G^0}^I(\tau) = \int_0^{T^I} dG^0(X^1)_{\gamma_\tau^I(t)} dt = 0.$$

On the other hand, by Lemma 4.3.1, $G^0 := (G_1^0, \ldots, G_{n-q}^0)$ is expressed as $G^0 = \psi \circ F$ for some analytic map $\psi : F(\mathcal{U}) \to \mathbb{R}^{n-q}$, where $\mathcal{U}$ is a neighborhood of $F^{-1}(c)$. Since F is constant along the periodic orbit $\gamma_\tau^I(t)$, we have $dG^0_{\gamma_\tau^I(t)} = d\psi_{F(\gamma_\tau^I(0))} dF_{\gamma_\tau^I(t)}$, so that

$$\mathscr{I}_{G^0}^I(\tau) = \int_0^{T^I} d\psi_{F(\gamma_\tau^I(0))} dF_{\gamma_\tau(t)}(X_{\gamma_\tau(t)}) dt = d\psi_{F(\gamma_\tau^I(0))} \mathscr{I}_F^I(\tau).$$

Since dG^0 and dF have the maximum rank on $\gamma_\tau^I(t)$ so that $\operatorname{rank} d\psi_{F(\gamma_\tau^I(0))} = n-q$, we obtain $\mathscr{I}_F^I(\tau) = 0$ for any $\tau \in \mathbb{T}^q$. $\square$

For the proof of Theorem 4.2.2 we also need the following result (see Appendix 4.A for its proof).

Proposition 4.3.3. *Let $k \leq n-q$ be a positive integer. Suppose that in a neighborhood of $F^{-1}(c)$ the perturbed system* (4.1.4) *has k first integrals that are analytic in (x, ε) near $\varepsilon = 0$. If they are functionally independent for $\varepsilon \neq 0$, then in a neighborhood of $F^{-1}(c)$ there exist k first integrals that are analytic in (x, ε) and functionally independent near $\varepsilon = 0$.*

Remark 4.3.4. *From Proposition 4.3.3 we also see that the condition for $\mathrm{D}F_0(I)$ to have a maximum rank at a point $I_0 \in \mathbb{R}^\ell$ was unnecessary in Theorem 4.1.1.*

Proof of Theorem 4.2.2. Suppose that $\mathscr{I}_F^I(\tau)$ is not identically zero for any $I \in D$. Using Lemma 4.3.2 for each $I \in D$, we see that the perturbed system (4.1.4) does not have $n-q$ analytic first integrals near $\mathscr{T}_I$ such that they are functionally independent on $\mathscr{T}_I$ and depend analytically on ε near $\varepsilon = 0$.

Additionally, suppose that there are $n-q$ analytic first integrals such that they are functionally independent for $|\varepsilon| \neq 0$ sufficiently small but not at $\varepsilon = 0$ and depend analytically on ε near $\varepsilon = 0$. Then by Proposition 4.3.3 there exist $n-q$ analytic first integrals $G_1^\varepsilon, \ldots, G_{n-q}^\varepsilon$ which are functionally independent and depend analytically on ε near $\varepsilon = 0$. Hence, $dG_1^0, \ldots, dG_{n-q}^0$ are linearly dependent on $\mathscr{T}_I$ for $I \in D$. As in the proof of Lemma 4.3.1 (i), we consider the transformed system (4.1.2) and write $\tilde{G}_j(I) = G_j(\varphi(I, \theta))$ for $j = 1, \ldots n-q$. Let $\tilde{G}(I) = (\tilde{G}_1(I), \ldots, \tilde{G}_{n-q}(I))$. We see that the determinant of the Jacobi matrix of $\tilde{G}(I)$ is zero for $I \in D$, so that it is identically zero on U since D is a key set for $C^\omega(U)$. This yields a contradiction. $\square$

4.3.2 Proof of Theorem 4.2.4

We turn to the proof of Theorem 4.2.4. Henceforth, we assume that conditions (A1)-(A4) and (K2) hold.

We begin with the following lemma. Recall that $Y_l(x)$, $l=1,\dots,q$, are commutative vector fields for the unperturbed system (4.1.5).

Lemma 4.3.5. *An analytic vector field $Z(x)$ commutes with the vector field $X^0(x)$ if and only if it can be written as $Z(x)=\sum_{l=1}^{q}\rho_l(F(x))Y_l(x)$, where $\rho_l:\mathbb{R}^{n-q}\to\mathbb{R}$, $l=1,\dots,n-q$, are analytic. In particular, the vector field $X^0(x)$ has only q commutative vector fields which are linearly independent almost everywhere.*

Proof. By Proposition 4.2.1, we may consider (4.1.2). So we prove that it is necessary and sufficient for an analytic vector field $Z(I,\theta)$ to commute with the vector field of (4.1.2) that it can be written as

$$Z(I,\theta)=\sum_{l=1}^{q}\rho_l(I)\frac{\partial}{\partial\theta_l}. \tag{4.3.1}$$

The sufficiency is obvious. The necessity follows from assumptions (K2) and (A4) by Lemma 1 in Section 3 of Chapter IV of [36]. □

Using Theorem 3.3.5 on persistence of commutative vector fields and Lemma 4.3.5, we obtain the following.

Lemma 4.3.6. *Let $I\in D_{\mathrm{R}}$. Suppose that near $\varepsilon=0$ the perturbed system (4.1.4) has q analytic commutative vector fields $Z_1^\varepsilon(x),\dots,Z_q^\varepsilon(x)$ near $\mathscr{T}_I=\{\gamma_\tau^I\mid\tau\in\mathbb{T}^q\}$ in $\mathscr{M}$ such that they are linearly independent on $\mathscr{T}_I$ and depend analytically on ε. Then $\mathscr{I}_F^I(\tau)$ must be constant.*

Proof. We first transform (4.1.4) to (4.1.1) with $\ell=n-q$ and $m=q$, as in the proof of Lemma 4.3.1. Assume that the hypothesis of the lemma holds. Then by Lemma 4.3.5, $Z_j^0(I,\theta)$ has the form (4.3.1) with $\rho_l(I)=\rho_{jl}(I)$, $l=1,\dots,q$, for $j=1,\dots q$. Let

$$\tilde{X}^1(I,\theta)=\sum_{j=1}^{\ell}h_j(I,\theta;0)\frac{\partial}{\partial I_j}+\sum_{k=1}^{m}g_k(I,\theta;0)\frac{\partial}{\partial\theta_k}$$

where $h_j(I,\theta;\varepsilon)$ (resp. $g_k(I,\theta;\varepsilon)$) is jth (resp. kth) component of $h(I,\theta;\varepsilon)$ (resp. $g(I,\theta;\varepsilon)$) for $j=1,\dots,\ell$ (for $k=1,\dots,m$). Using Theorem 3.3.5, we have

$$\begin{aligned}0&=\int_0^{T^I}dI\left(\left[Z_j^0(I,\theta),\tilde{X}^1(I,\theta)\right]\right)_{\theta=\omega(I)t+\tau}dt\\&=\sum_{l=1}^{q}\rho_{jl}(I)\left(\int_0^{T^I}\frac{\partial h}{\partial\theta_l}(I,\omega(I)t+\tau;0)dt\right)=\sum_{l=1}^{q}\rho_{jl}(I)\frac{d\mathscr{I}_I^I}{d\tau_l}(\tau).\end{aligned}$$

Since $Z_1^0,\dots,Z_q^0$ are linearly independent on $\mathscr{T}_I$, the $q\times q$ matrix $(\rho_{jl})_{j,l=,1,\dots,q}$ is invertible, so that by $n-q\le q$, $(d\mathscr{I}_I^I/d\tau)(\tau)=O$, i.e., $\mathscr{I}_I^I(\tau)$ is constant.

On the other hand, by Lemma 4.3.1, there exists an analytic map $\psi:F(\mathcal{U})\to\mathbb{R}^k$ such that $I=\psi\circ F(x)$. As in the proof of Lemma 4.3.2, we show that $\operatorname{rank}d\psi_{F(\gamma^I(0))}=n-q$ and $\mathscr{I}_I^I(\tau)=d\psi_{F(\gamma^I(0))}\mathscr{I}_F^I(\tau)$. Hence, $\mathscr{I}_F^I(\tau)$ is also constant. □

Proof of Theorem 4.2.4. Suppose that $\mathscr{I}_F^I(\tau)$ is not constant for any $I \in D$. From Theorem 4.2.2 we see that there exist only $n-q-1$ first integrals at most such that they are functionally independent and depend analytically on ε near $\varepsilon = 0$.

Assume that there exist q analytic commutative vector fields $Z_1^\varepsilon, \dots, Z_q^\varepsilon$ such that for $\varepsilon = 0$ they are linearly independent almost everywhere and depend analytically on ε. Applying Lemma 4.3.6 for each $I \in D$, we see that the perturbed system (4.1.4) does not have q analytic commutative vector fields near $\mathscr{T}_I$ such that they are linearly independent on $\mathscr{T}_I$ and depend analytically on ε near $\varepsilon = 0$. Hence, $Z_1^\varepsilon, \dots, Z_q^\varepsilon$ are linearly dependent on $\mathscr{T}_I$ for $I \in D$ at $\varepsilon = 0$ if they depend analytically on ε near $\varepsilon = 0$. As in the proof of Lemma 4.3.6, we consider the transformed system (4.1.2) and use Lemma 4.3.5 to write $Z_j^0(I, \theta)$ in the form (4.3.1) with $\rho_l(I) = \rho_{jl}(I)$, $l = 1, \dots, q$, for $j = 1, \dots q$. We see that the determinant of the matrix $(\rho_{jl}(I))_{j,l=,1,\dots,q}$ is zero for $I \in D$, so that it is identically zero on U since D is a key set for $C^\omega(U)$. This yields a contradiction. So we obtain the desired result. □

4.4 Consequences of the Theory to (4.1.1)

In this section, we consider nearly integrable systems of the form (4.1.1) written in the action-angle coordinates and describe consequences of Theorems 4.2.2 and 4.2.4 to it. The unperturbed system (4.1.2) is (m, ℓ)-integrable in the Bogoyavlenskij sense and has ℓ first integrals $I_1, \dots, I_\ell$ and m commutative vector fields $\omega(I)\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \dots, \frac{\partial}{\partial\theta_m}$. Thus, conditions (A1) and (A2) with $n = \ell + m$ and $q = m$ already hold. In particular, the level set of $I = c$ given by $\{c\} \times \mathbb{T}^m$ is connected and compact. Take some $I_0 \in \mathbb{R}^\ell$ and let U be its neighborhood in $\mathbb{R}^\ell$, as in the preceding sections.

We first discuss consequences of Theorem 4.2.2 to (4.1.1) and assume that conditions (K2) and (A3) with $n = \ell + m$ and $q = m$ hold. For $I \in D_\mathrm{R}$ the unperturbed system (4.1.2) has an m-parameter family of periodic orbits given by (4.2.1) with $q = m$. The integrals given by (4.2.2) for the ℓ first integrals $I = (I_1, \dots, I_\ell)$ become

$$\mathscr{I}_I^I(\tau) = \int_0^{T^{\bar{I}}} h(I, \omega(I)t + \tau; 0)dt,$$

where $\tau \in \mathbb{T}^m$.

Assume that $m > 1$. Using the Fourier expansion of $h(I, \theta; 0)$ given in (4.1.3), we rewrite the above integral as

$$\mathscr{I}_I^I(\tau) = \int_0^{T^{\bar{I}}} \sum_{r \in \mathbb{Z}^m} \hat{h}_r(I) \exp(ir \cdot (\omega(I)t + \tau))dt = T^I \sum_{r \in \Lambda_I} \hat{h}_r(I) e^{ir\cdot\tau}, \tag{4.4.1}$$

where $\Lambda_I = \{r \in \mathbb{Z}^m \mid r \cdot \omega(I) = 0\}$. Applying Theorem 4.2.2, we obtain the following result for (4.1.1). Recall that $\hat{h}_r(I)$, $r \in \mathbb{Z}^m$, represent the Fourier coefficients of $h(I, \theta; 0)$ (see Eq. (4.1.3)).

Theorem 4.4.1. *Let $m > 1$, and suppose that assumptions (K2) and (A3) with $n = \ell + m$ and $q = m$ hold. If there exists a key set $D \subset D_\mathrm{R}$ for $C^\omega(U)$ such that $\hat{h}_r(I) \neq 0$ for some $r \in \Lambda_I$ with $I \in D$, then the perturbed system (4.1.1) does not have ℓ real-analytic first integrals in a neighborhood of the level set $\{c\} \times \mathbb{T}^m$ near $\varepsilon = 0$ such that they are functionally independent for $|\varepsilon| \neq 0$ and depend analytically on ε.*

Remark 4.4.2.

(i) *From the proof given in [36] we see that the conclusion of Theorem 4.1.1 also holds even if the zero vector is taken as one of* $r_j \in \mathbb{Z}^m$, $j = 1, \ldots, \ell - s$, *in the definition of a Poincaré set. This fact was overlooked in [36].*

(ii) *If a Poincaré set* $\mathscr{P}_{\ell-1} \subset U$ *modified as stated in part (i) is a key set for* $C^\omega(U)$, *then condition (A3) holds. Moreover, there exists a key set* $D \subset D_{\mathrm{R}}$ *for* $C^\omega(U)$ *such that* $\hat{h}_r(I) \neq 0$ *with some* $r \in \Lambda_I$ *for* $I \in D$ *if and only if such a Poincaré set* $\mathscr{P}_{\ell-1} \subset U$ *is a key set for* $C^\omega(U)$.

(iii) *The hypothesis of Theorem 4.4.1 holds if both of* $\omega(I)$ *and* $\hat{h}_0(I)$ *are not identically zero in* U.

(iv) *If the system* (4.1.1) *is Hamiltonian, then* $\hat{h}_0(I) \equiv 0$.

(v) *From Theorem 3.2.2 and Eq.* (4.4.1) *we see that the first integrals* $I_1, \ldots, I_m$ *do not persist in* (4.1.1) *near the resonant torus* $\{I\} \times \mathbb{T}^m$ *if* $\hat{h}_r(I) \neq 0$ *for some* $r \in \Lambda_I$.

Let $m = 1$ and assume that $\omega(I) \neq 0$. Then the integral (4.4.1) becomes

$$\mathscr{I}_I^I(\tau) = \int_0^{2\pi/\omega(I)} h\,(I, \omega(I)t + \tau; 0)\, dt = \frac{2\pi \hat{h}_0(I)}{\omega(I)} \tag{4.4.2}$$

since $\Lambda_I = \{0\} \subset \mathbb{Z}$. Noting that assumptions (K2) and (A3) hold if $\omega(I) \neq 0$ for some $I \in U$, we obtain the following.

Theorem 4.4.3. *Let* $m = 1$. *If* $\omega(I) \neq 0$ *and* $\hat{h}_0(I) \neq 0$ *for some* $I \in U$, *then the perturbed system* (4.1.1) *does not have* ℓ *real-analytic first integrals in a neighborhood of* $\{c\} \times \mathbb{S}^1$ *near* $\varepsilon = 0$ *such that they are functionally independent for* $|\varepsilon| \neq 0$ *and depend analytically on* ε.

Assuming the existence of $\ell - 1$ functionally independent first integrals in the perturbed system (4.1.1) and taking Remarks 4.4.2(i) and (ii) into account, we obtain the same result as Theorems 4.4.1 and 4.4.3 from Theorem 4.1.1 (see also Remark 4.3.4). Moreover, when the existence of such only s $(< \ell - 1)$ first integrals is assumed, Theorem 4.1.1 guarantees the nonexistence of no additional first integral if a Poincaré set $\mathscr{P}_s$ modified in Remark 4.4.2(i) is a key set for $C^\omega(U)$, in particular $\hat{h}_{r_j}(I)$, $j = 1, \ldots, \ell - s$ (> 1), are linearly independent. Note that such a Poincaré set $\mathscr{P}_s$ does not exist when $m = 1$ and $s < \ell - 1$.

We next apply Theorem 4.2.4 to (4.1.1). When $m = 1$, the integral $\mathscr{I}_I^I(\tau)$ is constant by (4.4.2), so that Theorem 4.2.4 does not apply. Thus, we obtain the following result.

Theorem 4.4.4. *Let* $m > 1$, *and suppose that assumptions (K2), (A3) and (A4) hold. If there exists a key set* $D \subset D_{\mathrm{R}}$ *for* $C^\omega(U)$ *such that* $\hat{h}_r(I) \neq 0$ *for some* $r \in \Lambda_I \setminus \{0\}$ *with* $I \in D$, *then for* $|\varepsilon| \neq 0$ *sufficiently small the perturbed system* (4.1.1) *is not real-analytically integrable in the meaning of Theorem 4.2.4 near* $\{c\} \times \mathbb{T}^m$.

4.5 Relationships with the Melnikov Methods

In this section, we discuss relationships of our main results in Section 4.2 with the subharmonic and homoclinic Melnikov methods for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems. See [26, 42, 62, 65] for the details of the Melnikov methods. A concise review of the methods was also given in Section 3.4.1.

Consider systems of the form

$$\dot{x} = J\mathrm{D}H(x) + \varepsilon u(x, \nu t), \quad x \in \mathbb{R}^2, \tag{4.5.1}$$

where ε is a small parameter as in the preceding sections, $\nu > 0$ is a constant, $H : \mathbb{R}^2 \to \mathbb{R}$ and $u : \mathbb{R}^2 \times \mathbb{S} \to \mathbb{R}^2$ are analytic, and J is the 2×2 symplectic matrix,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Equation (4.5.1) represents a time-periodic perturbation of the single-degree-of-freedom Hamiltonian system

$$\dot{x} = J\mathrm{D}H(x), \tag{4.5.2}$$

with the Hamiltonian $H(x)$. Letting $\phi = \nu t \mod 2\pi$ such that $\phi \in \mathbb{S}^1$, we rewrite (4.5.1) as an autonomous system,

$$\dot{x} = J\mathrm{D}H(x) + \varepsilon u(x, \phi), \quad \dot{\phi} = \nu. \tag{4.5.3}$$

We easily see that assumptions (A1) and (A2) hold in (4.5.3) with $\varepsilon = 0$: $H(x)$ is a first integral and $(0, 1) \in \mathbb{R}^2 \times \mathbb{R}$ is a commutative vector field. We make the following assumptions on the unperturbed system (4.5.2):

(M1) There exists a one-parameter family of periodic orbits $x^\alpha(t)$ with period $\hat{T}^\alpha > 0$, $\alpha \in (\alpha_1, \alpha_2)$, for some $\alpha_1 < \alpha_2$. Moreover, $\hat{T}^\alpha$ is not constant as a function of α.

(M2) $x^\alpha(t)$ is analytic with respect to $\alpha \in (\alpha_1, \alpha_2)$.

Note that in (M1) $x^\alpha(t)$ is automatically analytic with respect to t since the vector field of (4.5.2) is analytic.

We assume that at $\alpha = \alpha^{l/n}$

$$\frac{2\pi}{\hat{T}^\alpha} = \frac{n}{l}\nu, \tag{4.5.4}$$

where l and n are relatively prime integers. We define the *subharmonic Melnikov function* as

$$M^{l/n}(\phi) = \int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t)) \cdot u(x^\alpha(t), \nu t + \phi)\mathrm{d}t, \tag{4.5.5}$$

where $\alpha = \alpha^{l/n}$. Let $T^\alpha = n\hat{T}^\alpha = 2\pi l/\nu$ for $\alpha = \alpha^{l/n}$. If $M^{l/n}(\phi)$ has a simple zero at $\phi = \phi_0$ and $d\hat{T}^\alpha/d\alpha \neq 0$ at $\alpha = \alpha^{l/n}$, then for $|\varepsilon| > 0$ sufficiently small there exists a T^α-periodic orbit near $(x, \phi) = (x^\alpha(t), \nu t + \phi_0)$ in (4.5.3). See Theorem 3.1 of [65]. A similar result is also found in [26, 62]. The stability of the periodic orbit can also be determined easily [65]. Moreover, several bifurcations of periodic orbits when $d\hat{T}^\alpha/d\alpha \neq 0$ or not were discussed in [65, 68, 69].

On the other hand, since it is a single-degree-of-freedom Hamiltonian system, the unperturbed system (4.5.2) is integrable, so that it can be transformed into the form (4.1.2) with $\ell, m = 1$. So the perturbed system (4.5.3) is transformed into the form (4.1.1) with $\ell = 1$ and $m = 2$. Here we take $I = \alpha$ unlike [65, 73], and have $\omega(I) = (\Omega(I), \nu)$, where

$$\Omega(\alpha) = \frac{2\pi}{\hat{T}^\alpha}.$$

We remark that the transformed system is not Hamiltonian even when $\varepsilon = 0$, unlike [65, 73]. Choose a point $\alpha = \alpha_0 \in (\alpha_1, \alpha_2)$ such that $d\hat{T}^\alpha/d\alpha \neq 0$, and let U be a neighborhood of α_0. We see that assumptions (K2) and (A3) hold for

$$D_\mathrm{R} = \{\alpha^{l/n} \mid l, n \in \mathbb{N}\} \cap U.$$

Let $\alpha = \alpha^{l/n}$ and let $\gamma_\tau^\alpha(t) = (x^\alpha(t + \tau_1), \nu(t + \tau_1) + \tau_2)$. We see that $\gamma_\tau^\alpha(t)$ is a T^α-periodic orbit in (4.5.3) with $\varepsilon = 0$. Note that $\gamma_\tau^\alpha(t)$ is essentially parameterized by a single parameter, say $\phi := \nu\tau_1 + \tau_2$. So we write $\gamma_\phi^\alpha(t) = (x^\alpha(t), \nu t + \phi)$. The integral (4.2.2) for $H(x)$ along $\gamma_\phi^\alpha(t)$ becomes

$$\mathscr{I}_H^\alpha(\phi) = \int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t)) \cdot u(x^\alpha(t), \nu t + \phi) dt = M^{l/n}(\phi) \tag{4.5.6}$$

by (4.5.5). As stated above, if $M^{l/n}(\phi)$ has a simple zero at $\phi = \phi_0$, then there exists a T^α-periodic orbit near $\gamma_{\phi_0}^\alpha(t)$. Applying Theorems 4.2.2 and 4.2.4, we have the following two results.

Theorem 4.5.1. *Suppose that there exists a key set $D \subset D_\mathrm{R}$ for $C^\omega(U)$ such that $M^{l/n}(\phi)$ is not identically zero for $\alpha^{l/n} \in D$. Then for $|\varepsilon| \neq 0$ sufficiently small the system* (4.5.3) *has no real-analytic first integral in a neighborhood of $\{x^{\alpha_0}(t) \mid t \in [0, \hat{T}^{\alpha_0})\} \times \mathbb{S}^1$ such that it depends analytically on ε near $\varepsilon = 0$.*

Theorem 4.5.2. *Suppose that there exists a key set $D \subset D_\mathrm{R}$ for $C^\omega(U)$ such that $M^{l/n}(\phi)$ is not constant for $\alpha^{l/n} \in D$. Then for $|\varepsilon| \neq 0$ sufficiently small the system* (4.5.3) *is not real-analytically integrable in the meaning of Theorem 4.2.4 in a neighborhood of $\{x^{\alpha_0}(t) \mid t \in [0, \hat{T}^{\alpha_0})\} \times \mathbb{S}^1$.*

Remark 4.5.3.

(i) *If D has an accumulation point, then it becomes a key set for $C^\omega(U)$.*

(ii) *If the system* (4.5.1) *is Hamiltonian, then the hypotheses of Theorems 4.5.1 and 4.5.2 are equivalent to the condition that $M^{l/n}(\phi)$ has a simple zero. Actually, letting*

$$u(x, \phi) = J\mathrm{D}_x H^1(x, \phi) = J \sum_{r \in \mathbb{Z}} \mathrm{D}\hat{H}_r^1(x) e^{ir\phi},$$

we have

$$\begin{aligned} M^{l/n}(\phi) &= \int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t)) \cdot J\mathrm{D}_x H^1(x^\alpha(t), \nu t + \phi) dt \\ &= \sum_{r \in \mathbb{Z}} e^{ir\phi} \int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t)) \cdot J\mathrm{D}\hat{H}_r^1(x^\alpha(t)) e^{ir\nu t} dt \end{aligned}$$

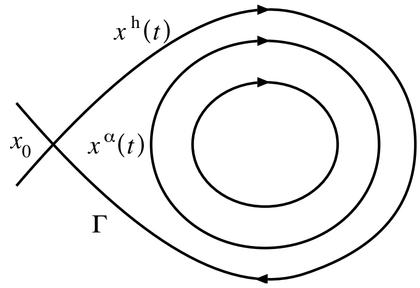

Figure 4.1: Assumption (M3).

and

$$\begin{aligned}&\int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t))\cdot J\mathrm{D}\hat{H}_0^1(x^\alpha(t))dt\\&=-\int_0^{2\pi l/\nu} \mathrm{D}\hat{H}_0^1(x^\alpha(t))\cdot J\mathrm{D}H(x^\alpha(t))dt\\&=-\int_0^{2\pi l/\nu} \mathrm{D}\hat{H}_0^1(x^\alpha(t))\cdot \dot{x}^\alpha(t)dt=0,\end{aligned}$$

where $\hat{H}_r^1(x)$, $r\in\mathbb{Z}$, represent the Fourier coefficients of $H^1(x,\phi)$. Thus, we obtain the claim.

We additionally assume the following on the unperturbed system (4.5.2):

(M3) There exists a hyperbolic saddle x_0 with a homoclinic orbit $x^{\mathrm{h}}(t)$ such that

$$\lim_{\alpha\to\alpha_2}\sup_{t\in\mathbb{R}} d(x^\alpha(t),\Gamma)=0,$$

where $\Gamma=\{x^{\mathrm{h}}(t)\mid t\in\mathbb{R}\}\cup\{x_0\}$ and $d(x,\Gamma)=\inf_{y\in\Gamma}|x-y|$. See Fig. 4.1.

We define the *homoclinic Melnikov function* as

$$M(\phi)=\int_{-\infty}^{\infty}\mathrm{D}H(x^{\mathrm{h}}(t))\cdot u(x^{\mathrm{h}}(t),t+\phi)dt. \tag{4.5.7}$$

If $M(\phi)$ has a simple zero, then for $|\varepsilon|>0$ sufficiently small there exist transverse homoclinic orbits to a periodic orbit near $\{x_0\}\times\mathbb{S}^1$ in (4.5.3) [26, 42, 62]. The existence of such transverse homoclinic orbits implies that the system (4.5.3) exhibits chaotic motions by the Smale-Birkhoff theorem [26, 62] and has no resl-analytic (additional) first integral (see, e.g., Chapter III of [49]). We easily show that

$$\lim_{l\to\infty} M^{l/1}(\phi)=M(\phi) \tag{4.5.8}$$

for each $\phi\in\mathbb{S}^1$ (see Theorem 4.6.4 of [26]). Let U be a neighborhood of $\alpha=\alpha_2$. It follows from (4.5.8) that if $M(\phi)$ is not identically zero or constant, then for $l>0$ sufficiently large neither is $M^{l/1}(\phi)$. Let $\hat{U}\subset\mathbb{R}^2$ be a region such that $\partial\hat{U}\supset\Gamma$ and $\hat{U}\supset\{x^\alpha(t)\mid t\in[0,\hat{T}^\alpha)\}$ for some $\alpha\in(\alpha_1,\alpha_2)$. We obtain the following from Theorems 4.5.1 and 4.5.2.

Theorem 4.5.4. *Suppose that $M(\phi)$ is not identically zero Then for $|\varepsilon| \neq 0$ sufficiently small the system* (4.5.3) *has no real-analytic first integral in $\hat{U} \times \mathbb{S}^1$such that it depends analytically on ε near $\varepsilon = 0$.*

Theorem 4.5.5. *Suppose that $M(\phi)$ is not constant. Then for $|\varepsilon| \neq 0$ sufficiently small the system* (4.5.3) *is not real-analytically integrable in the meaning of Theorem 4.2.4 in $\hat{U} \times \mathbb{S}^1$.*

Remark 4.5.6.

(i) Theorems 4.5.4 and 4.5.5, respectively, mean that the system (4.5.3) *has no first integral and is nonintegrable even if the Melnikov function $M(\phi)$ does not have a simple zero, i.e., there may exist no transverse homoclinic orbit to the periodic orbit in* (4.5.3)*, but it is not identically zero and constant. See Section 4.6.3.*

(ii) As in Remark 4.5.3(ii), if the system (4.5.1) *is Hamiltonian, then the hypotheses of Theorems 4.5.4 and 4.5.5 are equivalent to the condition that $M^{l/n}(\phi)$ has a simple zero.*

(iii) In the statements of Theorems 4.5.4 and 4.5.5, the region $\hat{U} \times \mathbb{S}^1$ may be replaced with a neighborhood of $\Gamma \times \mathbb{S}^1$ although they are weakened.

4.6 Examples

We now illustrate the above theory for four examples: Simple pendulum with a constant torque, second-order coupled oscillators, the periodically forced Duffing oscillator [26, 31, 62].

4.6.1 Simple pendulum with a constant torque

Consider a simple pendulum with a constant torque:

$$\dot{I} = \varepsilon(\beta \sin\theta + 1), \quad \dot{\theta} = I, \quad (I, \theta) \in \mathbb{R} \times \mathbb{T} \tag{4.6.1}$$

where $\beta \in \mathbb{R}$ is a constant. Equation (4.6.1) is of the form (4.1.1) with $m = \ell = 1$. Using Theorem 4.4.3, we obtain the following.

Proposition 4.6.1. *The system* (4.6.1) *has no real-analytic first integral depending analytically on ε near $\varepsilon = 0$.*

Remark 4.6.2.

(i) The system (4.6.1) *has the first integral*

$$F(I, \theta; \varepsilon) = \frac{1}{2} I^2 + \varepsilon(\beta \cos\theta - \theta)$$

and is $(1,1)$-integrable as a system on $\mathbb{R} \times \mathbb{R}$, although $F(I, \theta; \varepsilon)$ is not even a function on $\mathbb{R} \times \mathbb{S}^1$.

(ii) Let $\beta = 0$. Then the system (4.6.1) *is $(2,0)$-integrable when $\varepsilon \neq 0$, where the vector fields $\varepsilon \frac{\partial}{\partial I} + I \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \theta}$ are commutative and linearly independent. However, when $\varepsilon = 0$, the two vector fields are linearly dependent and Eq.* (4.6.1) *is not $(2,0)$-integrable. See also Remark 4.2.5(ii).*

4.6.2 Second-order coupled oscillators

Consider

$$\begin{aligned}\dot{I}_j =&\varepsilon\left(-\delta I_j+\Omega_j+\sum_{i=1}^{\ell}\sum_{k\in\mathbb{N}^2}a_k\sin(k_1\theta_j-k_2\theta_i)\right),\\ \dot{\theta}_j =&I_j,\quad j=1,\dots,\ell,\end{aligned}\tag{4.6.2}$$

where $\delta,\Omega_j\geq 0$, $j=1,\dots,\ell$, and a_k, $k=(k_1,k_2)\in\mathbb{N}^2$, are constants such that $|a_k|\leq Me^{-(k_1+k_2)\delta}$ for some $M,\delta>0$. We see by the remark after Lemma 2 in Section 12 of Chapter 3 in [8] that the vector field of (4.6.2) is analytic. Equation (4.6.2) has the form (4.1.1) with $m=\ell$ and is rewritten in a system of second-order differential equations as

$$\ddot{\theta}_j+\varepsilon\delta\dot{\theta}=\varepsilon\left(\Omega_j+\sum_{i=1}^{\ell}\sum_{k\in\mathbb{N}^2}a_k\sin(k_1\theta_j-k_2\theta_i)\right),\quad j=1,\dots,\ell,$$

which reduces to the *second-order Kuramoto model* [54] when $a_k\neq 0$ for $k=(1,1)$ and $a_k=0$ for $k\neq(1,1)$. Obviously, assumptions (K2), (A3) and (A4) hold. Using Theorems 4.4.1 and 4.4.4, we obtain the following.

Proposition 4.6.3. *The following statements hold for* (4.6.2)*:*

(i) If one of δ and Ω_j, $j=1,\dots,\ell$, is nonzero at least, then the system (4.6.2) *does not have ℓ real-analytic first integrals near $\varepsilon=0$ such that they are functionally independent for $|\varepsilon|\neq 0$ and depend analytically on ε;*

(ii) If $K_1=\{k_1/k_2\mid a_k,k_2\neq 0\}$ or $K_2=\{k_2/k_1\mid a_k,k_1\neq 0\}$ has an accumulation point, then for $|\varepsilon|\neq 0$ sufficiently small the system (4.6.2) *is not real-analytically integrable in the meaning of Theorem 4.2.4.*

Proof. Part (i) immediately follows from Theorem 4.4.1 and Remark 4.4.2(iii) since $\hat{h}_0(I)$ is not identically zero if one of δ and Ω_j, $j=1,\dots,\ell$, is nonzero at least.

We turn to the proof of part (ii). Let $D_1=\{I\in\mathbb{R}^\ell\mid k_1I_j-k_2I_i=0,\ i,j=1,\dots,\ell,\ k_1/k_2\in K_1\}$ and $D_2=\{I\in\mathbb{R}^\ell\mid k_1I_j-k_2I_i=0,\ i,j=1,\dots,\ell,\ k_2/k_1\in K_2\}$. If K_1 (resp. K_2) has an accumulation point, then D_1 (resp. D_2) is a key set for $C^\omega(\mathbb{R}^\ell)$. Their claim is shown as follows. Assume that K_1 has an accumulation point. Let $f(I)\in C^\omega(\mathbb{R}^\ell)$ be an analytic function which vanishes on D_1, and take a line $L_b:=\{I\in\mathbb{R}^\ell\mid(I_1,\dots,I_{\ell-1})=b\}$ for $b=(b_1,\dots,b_{\ell-1})\in\mathbb{R}^{\ell-1}$ fixed . Then $(b_1,\dots,b_{\ell-1},k_1b_1/k_2)\in L_b\cap D_1$ for all $k_1/k_2\in K_1$, so that $f(I)$ is identically zero on L_b. This means that $f(I)$ is identically zero in $\mathbb{R}^\ell$, and consequently D_1 is a key set for $C^\omega(\mathbb{R}^\ell)$. Similarly, we see that the claim is true for K_2 and D_2. Applying Theorem 4.4.4, we obtain the desired result. □

4.6.3 Periodically forced Duffing oscillator

Consider the periodically forced Duffing oscillator

$$\dot{x}_1=x_2,\quad \dot{x}_2=ax_1-x_1^3+\varepsilon(\beta\cos\nu t-\delta x_2),\tag{4.6.3}$$

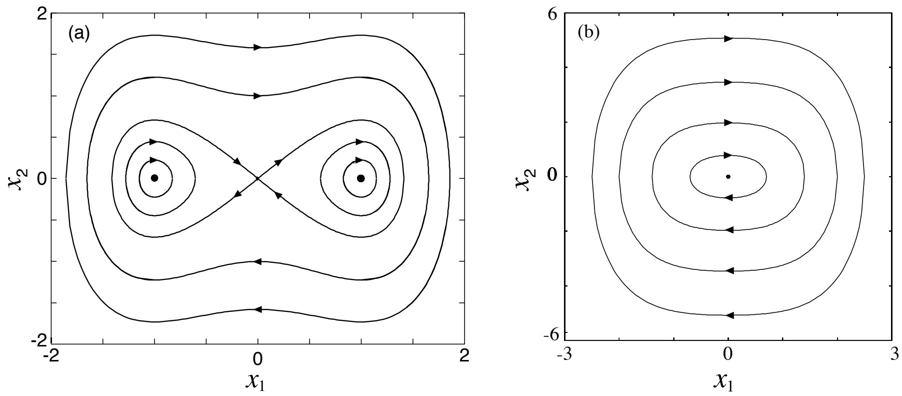

Figure 4.2: Phase portraits of (4.6.3) with $\varepsilon = 0$: (a) $a = 1$; (b) $a = -1$.

where $\nu > 0$ and $\beta, \delta \geq 0$ are constants, and $a = -1$ or 1. The system (4.6.3) has the form (4.5.1) with

$$H = -\frac{1}{2}ax_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

and the autonomous system (4.5.3) becomes

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = ax_1 - x_1^3 + \varepsilon(\beta\cos\theta - \delta x_2), \quad \dot{\phi} = \nu, \tag{4.6.4}$$

where $(x, \phi) \in \mathbb{R}^2 \times \mathbb{S}$. See Fig. 4.2 for the phase portraits of (4.6.3) with $\varepsilon = 0$.

We begin with the case of $a = 1$. When $\varepsilon = 0$, in the phase plane there exist a pair of homoclinic orbits

$$x_\pm^{\mathrm{h}}(t) = (\pm\sqrt{2}\,\mathrm{sech}\,t, \mp\sqrt{2}\,\mathrm{sech}\,t\,\tanh t),$$

a pair of one-parameter families of periodic orbits

$$\begin{aligned} x_\pm^k(t) = &\left(\pm\frac{\sqrt{2}}{\sqrt{2-k^2}}\,\mathrm{dn}\left(\frac{t}{\sqrt{2-k^2}}\right),\right.\\ &\left.\mp\frac{\sqrt{2}k^2}{2-k^2}\,\mathrm{sn}\left(\frac{t}{\sqrt{2-k^2}}\right)\mathrm{cn}\left(\frac{t}{\sqrt{2-k^2}}\right)\right), \quad k \in (0,1), \end{aligned}$$

inside each of them, and a one-parameter periodic orbits

$$\begin{aligned} \tilde{x}^k(t) = &\left(\frac{\sqrt{2}k}{\sqrt{2k^2-1}}\,\mathrm{cn}\left(\frac{t}{\sqrt{2k^2-1}}\right),\right.\\ &\left.-\frac{\sqrt{2}k}{2k^2-1}\,\mathrm{sn}\left(\frac{t}{\sqrt{2k^2-1}}\right)\mathrm{dn}\left(\frac{t}{\sqrt{2k^2-1}}\right)\right), \quad k \in \left(1/\sqrt{2}, 1\right), \end{aligned}$$

outside of them, as shown in Fig. 4.2(a), where sn, cn and dn represent the Jacobi elliptic functions with the elliptic modulus k. The periods of $x_\pm^k(t)$ and $\tilde{x}^k(t)$ are given by $\hat{T}^k = 2K(k)\sqrt{2-k^2}$ and $\tilde{T}^k = 4K(k)\sqrt{2k^2-1}$, respectively, where $K(k)$ is the complete elliptic integral of the first kind. Note that $x_\pm^k(t)$ approaches $x_\pm^{\mathrm{h}}(t)$ as $k \to 1$. See [26, 62]. See also [20] for general information on elliptic functions.

Assume that the resonance conditions

$$l\hat{T}^k = \frac{2\pi n}{\nu}, \quad \text{i.e.,} \quad \nu = \frac{2\pi n}{2lK(k)\sqrt{2-k^2}}, \tag{4.6.5}$$

and

$$l\tilde{T}^k = \frac{2\pi n}{\nu}, \quad \text{i.e.,} \quad \nu = \frac{2\pi n}{4lK(k)\sqrt{2k^2-1}}, \tag{4.6.6}$$

hold for $x_\pm^k(t)$ and $\tilde{x}^k(t)$, respectively, with $l, n > 0$ relatively prime integers. Then the subharmonic Melnikov function (4.5.5) for $x_\pm^k(t)$ and $\tilde{x}^k(t)$ are

$$M_\pm^{n/l}(\tau) = -\delta J_1(k,l) \pm \beta J_2(k,n,l)\sin\tau$$

and

$$\tilde{M}^{n/l}(\tau) = -\delta \tilde{J}_1(k,l) + \beta \tilde{J}_2(k,n,l)\sin\tau,$$

respectively, where

$$\begin{aligned}
J_1(k,l) &= \frac{4l[(2-k^2)E(k) - 2k'^2K(k)]}{3(2-k^2)^{3/2}},\\
J_2(k,n,l) &= \begin{cases} \sqrt{2}\pi\nu \operatorname{sech}\left(\dfrac{n\pi K(k')}{K(k)}\right) & (\text{for } l=1);\\ 0 & (\text{for } l \neq 1),\end{cases}\\
\tilde{J}_1(k,l) &= \frac{8l[(2k^2-1)E(k) + k'^2K(k)]}{3(2k^2-1)^{3/2}},\\
\tilde{J}_2(k,n,l) &= \begin{cases} 2\sqrt{2}\pi\nu \operatorname{sech}\left(\dfrac{n\pi K(k')}{2K(k)}\right) & (\text{for } l=1 \text{ and } n \text{ odd});\\ 0 & (\text{for } l \neq 1 \text{ or } n \text{ even}).\end{cases}
\end{aligned}$$

Here $E(k)$ is the complete elliptic integral of the second kind and $k' = \sqrt{1-k^2}$ is the complimentary elliptic modulus. When $\delta \neq 0$, the subharmonic Melnikov functions $M_\pm^{n/l}(\tau)$ and $\tilde{M}^{n/l}(\tau)$ are not identically zero for any relatively prime integers $n, l > 0$ since $J_1(k,l)$ and $\tilde{J}_1(k,l)$ are not zero. Moreover, the homoclinic Melnikov function (4.5.7) for $x_\pm^{\mathrm{h}}(t)$ is

$$M_\pm(\tau) = -\frac{4}{3}\delta \pm \sqrt{2}\pi\nu\beta \operatorname{csch}\left(\frac{\pi\nu}{2}\right)\sin\tau,$$

which is not identically zero for $\beta \neq 0$. See [26, 62] for the computations of the Melnikov functions.

Let

$$\begin{aligned}
R &= \{k \in (0,1) \mid k \text{ satisfies (4.6.5) for } n, l \in \mathbb{N}\},\\
\tilde{R} &= \left\{k \in \left(1/\sqrt{2}, 1\right) \mid k \text{ satisfies (4.6.6) for } n, l \in \mathbb{N}\right\},
\end{aligned}$$

and let

$$\begin{aligned}
S_\pm^k &= \{(x_\pm^k(t), \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \mid t \in [0, \hat{T}^k), \theta \in \mathbb{S}^1\},\\
\tilde{S}^k &= \{(\tilde{x}^k(t), \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \mid t \in [0, \tilde{T}^k), \theta \in \mathbb{S}^1\},\\
\Gamma_\pm &= \{x_\pm^{\mathrm{h}}(t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\} \cup \{0\}.
\end{aligned}$$

Noting that

$$\lim_{n\to\infty} \tilde{M}^{2n+1/1}(\tau) = M_+(\tau) + M_-(\tau)$$

and applying Theorems 4.5.1, 4.5.4, 4.5.5 and their slight extensions, we have the following.

Proposition 4.6.4. *The system* (4.6.4) *with* $a = 1$ *has no real-analytic first integral depending analytically on* ε *in neighborhoods of* $S_\pm^k$ *for* $k \in R$*, of* $\tilde{S}^k$ *for* $k \in \tilde{R}$*, and of* $S_\pm^{\mathrm{h}}$ *near* $\varepsilon = 0$ *if* $\delta \neq 0$.

Proposition 4.6.5. *Let* $\hat{U}_\pm$ *(resp.* $\tilde{U}$*) be regions (resp. a region) in* $\mathbb{R}^2$ *such that* $\partial\hat{U}_\pm \supset \Gamma_\pm$ *(resp.* $\partial\tilde{U} \supset \Gamma_+ \cup \Gamma_-$*) and* $\hat{U}_\pm \supset \{x_\pm^k(t) \mid t \in [0, \hat{T}^\alpha)\}$ *(resp.* $\tilde{U} \supset \{\tilde{x}^k(t) \mid t \in [0, \tilde{T}^k)\}$ *for some* $k \in (0,1)$ *(resp.* $k \in (1/\sqrt{2}, 1)$*). For* $|\varepsilon| \neq 0$ *sufficiently small the system* (4.6.4) *with* $a = 1$ *is not real-analytically integrable in the regions* $\hat{U}_\pm \times \mathbb{S}^1$ *(resp.* $\tilde{U} \times \mathbb{S}^1$*) in the meaning of Theorem 4.2.4 if* $\beta \neq 0$.

If $\beta \neq 0$ and

$$\frac{\delta}{\beta} < \frac{3}{4}\sqrt{2}\pi\nu \operatorname{csch}\left(\frac{\pi\nu}{2}\right), \tag{4.6.7}$$

then $M_\pm(\tau)$ has a simple zero, so that for $|\varepsilon| > 0$ sufficiently small there exist transverse homoclinic orbits to a periodic orbit near the origin and chaotic dynamics may occur in (4.6.4) with $a = 1$, as stated in Section 4.5. From Proposition 4.6.5 we see that the system (4.6.4) is nonintegrable in the meaning of Theorem 4.2.4 even if condition (4.6.7) does not hold, i.e., there may exist no transverse homoclinic orbit to the periodic orbit, as stated in Remark 4.5.6(i). On the other hand, when the system (4.6.3) is Hamiltonian, i.e., $\delta = 0$, condition (4.6.7) always holds and such inconsistency does not occur. See also Remark 4.5.6(ii).

We turn to the case of $a = -1$. When $\varepsilon = 0$, in the phase plane there exists a one-parameter family of periodic orbits

$$\begin{aligned}\hat{x}^k(t) =& \left(\frac{\sqrt{2}k}{\sqrt{1-2k^2}} \operatorname{cn}\left(\frac{t}{\sqrt{1-2k^2}}\right),\right.\\ &\left. -\frac{\sqrt{2}k}{1-2k^2} \operatorname{sn}\left(\frac{t}{\sqrt{1-2k^2}}\right) \operatorname{dn}\left(\frac{t}{\sqrt{1-2k^2}}\right)\right), \quad k \in \big(0, 1/\sqrt{2}\big),\end{aligned}$$

as shown in Fig. 4.2(b), and their period is given by $\hat{T}^k = 4K(k)\sqrt{1-2k^2}$. See [64, 65]. Assume that the resonance conditions

$$l\hat{T}^k = \frac{2\pi n}{\nu}, \quad \text{i.e.,} \quad \nu = \frac{\pi n}{2lK(k)\sqrt{1-2k^2}} \tag{4.6.8}$$

holds for $l, n > 0$ relatively prime integers. We compute the subharmonic Melnikov function (4.5.5) for $\hat{x}^k(t)$ as

$$\hat{M}^{n/l}(\tau) = -\delta\hat{J}_1(k,l) \pm \beta\hat{J}_2(k,n,l)\sin\tau,$$

where

$$\begin{aligned}\hat{J}_1(k,l) &= \frac{8l[(2k^2-1)E(k) + k'^2K(k)]}{3(1-2k^2)^{3/2}},\\ \hat{J}_2(k,n,l) &= \begin{cases}\dfrac{\sqrt{2}\pi^2 n}{K(k)\sqrt{1-2k^2}} \operatorname{sech}\left(\dfrac{\pi n K(k')}{2K(k)}\right) & \text{(for } l = 1 \text{ and } n \text{ odd);}\\ 0 & \text{(for } l \neq 1 \text{ or } n \text{ even).}\end{cases}\end{aligned}$$

See also [64, 65] for the computations of the Melnikov function. When $\delta \neq 0$, the subharmonic Melnikov function $\hat{M}^{n/l}(\tau)$ is not identically zero for any relatively prime integers $n, l > 0$ since $J_1(k, l)$ is not zero.

Let

$$\hat{R} = \left\{k \in \left(0, 1/\sqrt{2}\right) \mid k \text{ satisfies (4.6.8) for } n, l \in \mathbb{N}\right\},$$

and let

$$\hat{S}^k = \{(\hat{x}^k(t), \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \mid t \in [0, \hat{T}^k), \theta \in \mathbb{S}^1\}.$$

Applying Theorem 4.5.1, we obtain the following.

Proposition 4.6.6. *The system* (4.6.4) *with* $a = -1$ *has no real-analytic first integral depending analytically on* ε *in a neighborhood of* $\hat{S}^k$ *for* $k \in \hat{R}$ *near* $\varepsilon = 0$ *if* $\delta \neq 0$.

Remark 4.6.7.

(i) Since the subharmonic Melnikov function $\hat{M}^{n/l}(\tau)$ *is constant for* $l \neq 1$*, Theorem 4.5.2 is not applicable to* (4.6.4) *with* $a = -1$*. So we cannot exclude the possibility that the system* (4.6.4) *with* $a = -1$ *is* $(3, 0)$*-integrable in the meaning of Theorem 4.2.4 when* $\beta, \delta \neq 0$.

(ii) It was shown in [73] that the system (4.6.4) *with* $a = -1$ *is meromorphically non-integrable in a meaning similar to that of Theorem 4.2.4 when the independent and state variables are extended to complex ones.*

4.A Proof of Proposition 4.3.3

In this Appendix, we prove Proposition 4.3.3. We begin with the following lemma.

Lemma 4.A.1. *Let* Ω *be an open subset of* $\mathbb{R}^k$ *and let* $\chi_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, m$, *be analytic, where* $k, m \in \mathbb{N}$. *Let* $\chi(x) = (\chi_1(x), \ldots, \chi_m(x))$. *If* $\operatorname{rank} d\chi$ *is constant on* Ω *and less than* m*, then for any* $x \in \Omega$ *there exists a neighborhood* V *of* x *on which* $\chi_1, \ldots, \chi_m$ *are* analytically dependent, *i.e., there exist an open set* $\Omega' \subset \mathbb{R}^m$ *and a non-constant analytic map* $\zeta : \Omega' \to \mathbb{R}$ *such that* $\chi(V) \subset \Omega'$ *and* $\zeta(\chi(y)) = 0$ *for any* $y \in V$.

Proof. Using Theorem 1.3.14 of [50] and an argument in the proof of Theorem 1.4.15 of [50], we can immediately obtain the desired result as follows. The theorem says that there exist a neighborhood V (resp. V') of x (resp. of $\chi(x)$), a cube Q (resp. Q') in $\mathbb{R}^k$ (resp. in $\mathbb{R}^m$) and analytic isomorphisms $u : Q \to V$ and $u' : V' \to Q'$ such that the composite map $u' \circ \chi \circ u$ has the form $(x_1, \ldots, x_k) \to (x_1, \ldots, x_{m'}, 0, \ldots, 0)$, where x_j is the jth element of x for $j = 1, \ldots, k$ and $m' = \operatorname{rank} d\chi < m$. Here a *cube* in $\mathbb{R}^k$ is an open set of the form

$$\{x \mid |x_j - a_j| < r_j, j = 1, \ldots, k\}$$

for some $a_j \in \mathbb{R}$ and $r_j > 0$, $j = 1, \ldots, k$. Letting $u' = (u'_1, \ldots, u'_m)$ and $\zeta = u'_m$, we have $\zeta(\chi(y)) = 0$ for every $y \in V$. $\square$

Let $f_\varepsilon : \mathscr{M} \to \mathbb{R}$ be an analytic function such that it depends on ε analytically. We expand it near $\varepsilon = 0$ as $f_\varepsilon(x) = \sum_{j=0}^\infty f^j(x)\varepsilon^j$, where $f^j(x)$, $j \in \mathbb{Z}_0 := \mathbb{N} \cup \{0\}$, are analytic functions on $\mathscr{M}$. Define the order function $\sigma(f_\varepsilon)$ by

$$\sigma(f_\varepsilon) := \min\{j \in \mathbb{Z}_0 \mid f^j(x) \not\equiv 0\}$$

if $f_\varepsilon \not\equiv 0$ and $\sigma(0) := +\infty$, as in [13].

Lemma 4.A.2. *Suppose that $f_\varepsilon(x)$ is a nonconstant analytic first integral of (4.1.4) depending analytically on ε near $\varepsilon = 0$. Then there exists an analytic first integral $\tilde{f}_\varepsilon(x) = \tilde{f}^0(x) + O(\varepsilon)$ depending analytically on ε near $\varepsilon = 0$ such that $\tilde{f}^0(x)$ is not constant.*

Proof. Since f_ε is not constant, $\sigma(df_\varepsilon)$ takes a finite value. Let $k = \sigma(df_\varepsilon)$ and $f_\varepsilon(x) = \sum_{j=0}^\infty \varepsilon^j f^j(x)$. Define

$$\tilde{f}_\varepsilon(x) := \frac{1}{\varepsilon^k}\left(f_\varepsilon(x) - \sum_{j=0}^{k-1} \varepsilon^j f^j(x)\right).$$

Then $\tilde{f}_\varepsilon(x) = f^k(x) + O(\varepsilon)$ and $\tilde{f}^0(x) = f^k(x)$ is not constant. Moreover, $\tilde{f}_\varepsilon(x)$ is a first integral of (4.1.4) since $X_\varepsilon(f_\varepsilon) = 0$ and $\sum_{j=0}^{k-1} \varepsilon^j f^j$ is constant. $\square$

Proof of Proposition 4.3.3. Modifying the proof of Ziglin's lemma [11, 22, 79] slightly, we prove this proposition. For $k = 1$ the statement of the proposition holds by Lemma 4.A.2. Let $k > 1$ and suppose that it is true up to $k - 1$. Let $G_1^\varepsilon(x), \dots, G_k^\varepsilon(x)$ be analytic first integrals of (4.1.4) in a neighborhood of $F^{-1}(c)$ near $\varepsilon = 0$ such that they are functionally independent for $\varepsilon \neq 0$ and depend analytically on ε. Without loss of generality, we assume that $G_1^0(x), \dots, G_{k-1}^0(x)$ are functionally independent near $F^{-1}(c)$. Letting $G^\varepsilon(x) = (G_1^\varepsilon(x), \dots, G_k^\varepsilon(x))$, we see that $G^0(\varphi(I, \theta))$ depends only on I, as in the proof of Lemma 4.3.1(i), where φ denotes the analytic diffeomorphism in Proposition 4.2.1(ii). Let $\tilde{G}_j(I) = G_j^0(\varphi(I, \theta))$ for $j = 1, \dots, k$. Note that, if $d\tilde{G}_1(I), \dots, d\tilde{G}_k(I)$ are linearly independent at $I = I_0 \in U$, then so are $dG_1(x), \dots, dG_k(x)$ on $\varphi(\{I_0\} \times \mathbb{T}^q) \subset \mathcal{U}$.

Assume that $\tilde{G}_1(I), \dots, \tilde{G}_{k-1}(I), \tilde{G}_k(I)$ are functionally dependent in an open set $U' \subset U$. So $\Omega := \{p \in U' \mid \operatorname{rank} d_p\tilde{G} = k - 1\}$ contains a dense open set in U' since $d\tilde{G}_1(I), \dots, d\tilde{G}_{k-1}(I)$ are functionally independent on U. By Lemma 4.A.1, there exist an open set $\Omega' \subset \mathbb{R}^k$ and a nonzero analytic function $\zeta : \Omega' \to \mathbb{R}$ such that $\tilde{G}(V) = (\tilde{G}_1(V), \dots, \tilde{G}_k(V)) \subset \Omega'$ and

$$\zeta(\tilde{G}_1(I), \dots, \tilde{G}_k(I)) = 0$$

in a neighborhood V of $p \in \Omega$. Moreover, there is a positive integer s such that $(\partial^s\zeta/\partial y_k^s)(\tilde{G}(I)) \neq 0$, since if not, then $\zeta(\tilde{G}_1(I), \dots, \tilde{G}_{k-1}(I), y_k)$ depends on y_k near $\tilde{G}(V)$ and consequently $\tilde{G}_1(I), \dots, \tilde{G}_{k-1}(I)$ are functionally dependent. Let s be the smallest one of such integers and let $\tilde{\zeta}(y) = (\partial^{s-1}\zeta/\partial y_k^{s-1})(y)$. Then $\tilde{\zeta}$ satisfies

$$\tilde{\zeta}(\tilde{G}_1(I), \dots, \tilde{G}_k(I)) = 0$$

and $(\partial\tilde{\zeta}/\partial y_k)(\tilde{G}(I)) \neq 0$ on V. Hence,

$$\tilde{\zeta}(G^0(x)) = 0 \tag{4.7.1}$$

and $(\partial\tilde{\zeta}/\partial y_k)(G^0(x)) \neq 0$ on $\varphi(V \times \mathbb{T}^q)$.

Let $\hat{G}_k^\varepsilon(x) = \tilde{\zeta}(G^\varepsilon(x))/\varepsilon$. By (4.7.1) $\hat{G}_k^\varepsilon$ is an analytic first integral depending analytically on ε. We have

$$d\hat{G}_k^\varepsilon = \varepsilon^{-1} d(\tilde{\zeta}(G^\varepsilon(x))) = \varepsilon^{-1} \sum_{j=1}^{k} \frac{\partial\tilde{\zeta}}{\partial y_j}(G^\varepsilon(x))dG_j^\varepsilon$$

and

$$N(\hat{G}^\varepsilon) := dG_1^\varepsilon \wedge \ldots \wedge dG_{k-1}^\varepsilon \wedge d\hat{G}_k^\varepsilon = \varepsilon^{-1}\frac{\partial\tilde{\zeta}}{\partial y_k}(G^\varepsilon(x))dG_1^\varepsilon \wedge \ldots \wedge dG_k^\varepsilon.$$

Since $(\partial\tilde{\zeta}/\partial y_k)(G^0(x)) \neq 0$ on $\varphi(V \times \mathbb{T}^q)$, we have $\sigma((\partial\tilde{\zeta}/\partial y_k)(G^0(x))) = 0$, so that

$$\sigma(N(\hat{G}^\varepsilon)) = \sigma(\varepsilon^{-1}N(G^\varepsilon)) + \sigma\left(\frac{\partial\tilde{\zeta}}{\partial y_k}(G^\varepsilon(x))\right) = \sigma(N(G^\varepsilon)) - 1.$$

Repeating this procedure till $\sigma(N(\hat{G}^\varepsilon)) = 0$, we obtain

$$dG_1^0 \wedge \ldots \wedge dG_{k-1}^0 \wedge d\hat{G}_k^0 \neq 0,$$

which means the desired result. □

Chapter 5

Conclusions

In this thesis, we considered a class of ordinary differential equations including non-Hamiltonian systems and nonlinear oscillators with parametric or external forcing, and developed theories for their nonintegrability and related dynamics. In particular, we gave several necessary conditions for persistence of periodic and homoclinic orbits, first integrals and commutative vector fields in perturbed systems, and for real-analytic integrability of nearly integrable systems in the sense of Bogoyavlenskij [15] such that the first integrals and commutative vector fields also depend analytically on the perturbation parameter.

In Chapter 2, we discussed nonintegrability of parametrically forced nonlinear oscillators which are represented by second-order homogeneous differential equations with trigonometric coefficients and contain the Duffing and van der Pol oscillators as special cases. Specifically, we gave sufficient conditions for their rational nonintegrability in the meaning of Bogoyavlenskij, using Kovacic's algorithm [34] as well as an extension of the Morales-Ramis theory [43,47] due to Ayoul and Zung [12]. In application of the extended Morales-Ramis theory, for the associated variational equations, the identity components of their differential Galois groups were shown to be not commutative even if the differential Galois groups are triangularizable, i.e., they can be solved by quadratures. The obtained results were very general and reveal their rational nonintegarbility for the wide class of parametrically forced nonlinear oscillators. We also gave two examples for the van der Pol and Duffing oscillators to demonstrate our results.

In Chapter 3, we studied persistence of periodic and homoclinic orbits, first integrals and commutative vector fields in dynamical systems depending on a small parameter and gave several necessary conditions for their persistence. Here we treated homoclinic orbits not only to equilibria but also to periodic orbits. We also discussed some relationships of these results with the standard subharmonic and homoclinic Melnikov methods [26,42,62,65] for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems, and with another version of the homoclinic Melnikov method [61] for autonomous perturbations of multi-degree-of-freedom Hamiltonian systems. In particular, we showed that a first integral which converges to the Hamiltonian or another first integral as the perturbation tends to zero does not exist near the unperturbed periodic or homoclinic orbits in the perturbed systems if the subharmonic or homoclinic Melnikov functions are not identically zero on connected open sets. We illustrated our theory for four examples: The periodically forced Duffing oscillator, two identical pendula coupled with a harmonic oscillator, a periodically forced rigid body and a three-mode truncation of a buckled beam.

In Chapter 4, we studied the existence of first integrals and integrability for perturbations of integrable systems in the sense of Bogoyavlenskij including non-Hamiltonian ones. We especially assumed that there exists a family of periodic orbits on a regular level set of the first integrals which has a connected and compact component and gave sufficient conditions for nonexistence of the same number of first integrals in the perturbed systems as the unperturbed ones and for their nonintegrability near the level set such that the first integrals and commutative vector fields depend analytically on the small parameter. We compared our results with classical results of Poincaré [51, 52] and Kozlov [35, 36] for systems written in action and angle coordinates and discussed their relationships with the subharmonic and homoclinic Melnikov methods for periodic perturbations of single-degree-of-freeedom Hamiltonian systems. We illustrated our theory for three examples containing the periodically forced Duffing oscillator.

In closing this thesis, we give some comments on future work. In Chapter 2, using the extension due to Ayoul and Zung of the Morales-Ramis theory, we showed that parametrically forced nonlinear oscillators of the form (2.1.1) are rationally nonintegrable under weak conditions even if the parametric forcing is not small, as stated in Theorem 2.2.2. So one may expect a similar result for externally forced nonlinear oscillators such as Eq. (4.6.3) although the nonintegrability of (4.6.3) in the meaning of Theorem 4.2.4 or in a similar one, i.e., such that the first integrals and commutative vector fields also depend analytically or meromorphically on the parameter ε, was proved for $a = 1$ in Chapter 4 and for $a = -1$ in [73]. However, it is difficult to apply the extended Morales-Ramis theory to them since their particular solutions are not easily found. To prove their nonintegrability when ε is fixed or not small, we need to develop a new technique.

In Chapter 3, we discussed the persistence of periodic orbits, homoclinic orbits to equilibria or periodic orbits, and first integrals and commutative vector fields near these orbits. So it is interesting to extend the results of Chapter 3 to more general invariant sets such as invariant tori and to homoclinic orbits to such invariant sets. Moreover, one may be interested in relationships of the results to be obtained with several versions of the Melnikov method developed in [24, 25, 61] for such homoclinic orbits and with the KAM theory [10].

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Chapter 2 consists of

> S. Motonaga and K. Yagasaki, Nonintegrability of parametrically forced nonlinear oscillators, *Regular and Chaotic Dynamics*, **23** (2018), 291–303.

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> S. Motonaga and K. Yagasaki, Persistence of periodic and homoclinic orbits, first integrals and commutative vector fields in dynamical systems, *Nonlinearity*, **34** (2021), 7574–7608.

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