Scalable Estimation on Linear and Nonlinear Regression Models via Decentralized Processing: Adaptive LMS Filter and Gaussian Process Regression

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Abstract

Large amounts of data collected from a variety of Internet of Things (IoT) devices or various services on the Internet give us rich information but in general require high computational and/or communication costs and complicated processing. Decentralization of the centralized data processing is a solution that enables scalable data treatments. The idea of the decentralization is also beneficial for in-network processing in terms of scalability in network size. However, performance of decentralized processing can be degraded compared with that of the centralized processing when the decentralized processing is derived by approximately dividing the centralized processing. The purpose of the thesis is to enhance performance of such decentralized processing.

The thesis covers two estimation problems, in-network adaptive least-mean-square (LMS) filter and Gaussian process regression (GPR), as the specific themes of the decentralization. The former aims at tracking an unknown deterministic vector by using measurements at nodes obtained via linear regression models in the network such as sensor networks. Full decentralization of the algorithm is achieved by approximately dividing a cost function and by performing estimation in cooperation with neighboring nodes. In the latter, desired outputs are predicted by using training data under the assumption that the relation follows nonlinear Gaussian process regression models. The algorithm yields flexible prediction but requires a high computational cost. Partial decentralization of GPR is realized by dividing the data and then aggregating locally processed estimations.

We tackle improvements of the decentralized algorithms. For the fully decentralized adaptive LMS filter, we adopt a more efficient cooperation method known as message propagation to reduce performance degradation caused by the approximate divisions of the problem. We propose two types of algorithms with different use of the message propagation method and extend the algorithm to be a sparsity-aware one. For the partially decentralized GPR, we provide a reinterpretation of a conventional decentralization by introducing the idea of sketching and propose a novel algorithm that allows to balance computational complexity and performance. The efficiency of the proposed algorithms is evaluated via computer simulations.

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Related Journal Papers

1. **A. Nakai-Kasai** and T. Tanaka, "Nested aggregation of experts using inducing points for approximated Gaussian process regression," Machine Learning, Springer. (accepted for publication)[1]

2. **A. Nakai-Kasai** and K. Hayashi, "An acceleration method of sparse diffusion LMS based on message propagation,"' IEICE Trans. Commun., vol. E104-B, no.2, pp. 141–148, Feb. 2021. Copyright© 2021 IEICE[2]

3. **A. Nakai-Kasai** and K. Hayashi, "Diffusion LMS based on message passing algorithm," IEEE Access, vol. 7, pp. 47022–47033, Apr. 2019. Copyright© 2019 IEEE

Related International Conference Proceedings

4. **A. Nakai-Kasai** and K. Hayashi, "Optimal combination weight for sparse diffusion least-mean-square based on consensus propagation," Asia-Pacific Signal and Information Processing Association Annual Summit and Conference (APSIPA ASC 2020), pp. 228–235, virtual conference, Dec. 2020. Copyright© 2020 IEEE

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[2] Part I in this dissertation is based on "An acceleration method of sparse diffusion LMS based on message propagation" [2], by the same author, which appeared in IEICE Transactions on Communications, Copyright©2021 IEICE. The material in this dissertation was presented in part at IEICE Transactions on Communications [2], and a part of the figures of this dissertation is reused from [2] under the permission of the IEICE.

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Other Journal Paper

8. R. Hayakawa, **A. Nakai-Kasai**, and K. Hayashi, "Discreteness and group sparsity aware detection for uplink overloaded MU-MIMO systems," APSIPA Transactions on Signal and Information Processing, vol. 9, no. E21, pp. 1–12, Oct. 2020.

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Contents

1: Introduction

1.1 Demand for Scalability

Today large amounts of data (or big data) are collected from a lot of sensors placed in equipment or spaces and from various services on the Internet. Improvement of hardware performance has been promoting centralized analysis of such data, which is providing new generation mobile communication systems or practical realization of machine learning technologies (Kambatla et al., 2014; Liu et al., 2020b). However, the more we collect such data, generally the more difficulties we are faced with in processing them. Large amounts of data require high computational and/or communication cost, and complicate the centralized processing operations. For scalability to the large amount of data, the idea of parallel or distributed computing can be incorporated into algorithm designs (Dean and Ghemawat, 2004; Low et al., 2010). The data themselves or (a part of) centralized processes are exactly or approximately divided into sub-data or sub-tasks, respectively, which are assigned to multiple processors or computers. This enables us to decentralize the processing and handle large amounts of data while keeping the computational cost low.

On the other hand, decentralized processing is also gathering much attention in a perspective of scalability to large networks, especially in wireless communication field. One of the important technologies for new standards, 5G or 6G, is communication including or among Internet of Things (IoT) devices having abilities of wireless communication, computation, and sensing (Gupta and Jha, 2015; Zhang et al., 2019). The number of sensors equipped with the IoT devices is said to become a million per square kilometer (Bi et al., 2015). This causes an enormous increase in data traffic in a general centralized processing, which may result in communication delays or failures of a central data processing unit. Moreover, the diversity of data collected from different IoT devices complicates the processing. Even in such a case, exact or approximate division of a part or all of the data processing can be one of the solutions of the problems for preventing the concentration of load at the central unit. The divided processes are assigned to local devices, which enables decentralized processing in networks. For instance, in edge computing (Li et al., 2018), data are partially processed at edge servers before being sent to the cloud; in federated learning (Konečnỳ et al., 2016), data are learned at local IoT devices and then sent to the central server; or in application to sensor networks or industrial IoT (Dimakis et al., 2008; Savazzi et al., 2020), devices process data by themselves and

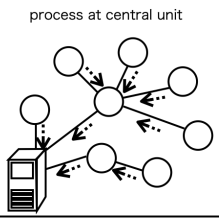
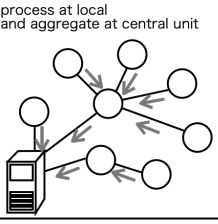
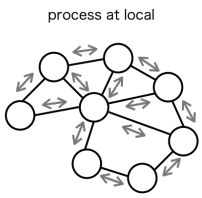
	Centralized	Decentralized	
		Partially distributed	Fully distributed
processing			
characteristics	• all at once • high speed/cost • highly-biased load	• sequentially or simultaneously • privacy consideration • low speed/cost	
		• little-biased load	• unbiased load
examples	• general communication • batch processing	• edge computing • federated learning • parallel processing	• sensor network • industrial IoT • machine-to-machine

Figure 1.1: Centralized and decentralized processing.

cooperate with the neighboring devices. Such decentralized processing may have an additional advantage in applications that deal with personal data with privacy, for example, as it does not exchange the raw data that may enable identification of an individual.

In the thesis, we focus on decentralized processing composed of divided processes of centralized processing for scalability in network size and the amounts of data. The decentralized processing includes partially distributed processing that supposes aggregation at the central processing unit and fully distributed processing that does not suppose the existence of the central unit. Fig. 1.1 summarizes conceptual diagrams, characteristics, and examples for the centralized, partially distributed, and fully distributed processing. If a problem that should be originally solved by centralized processing can be exactly divided into sub-problems, the solution by the decentralized processing according to the division becomes identical with that by the centralized one. On the other hand, if it cannot be done so and if an approximated division is employed, performance of the decentralized processing will be degraded. Therefore, it is required to develop a decentralized algorithm that can reduce the degradation as much as possible.

1.2 Thesis Overview and Contributions

1.2.1 Objective and Overview of Thesis

The objective of the thesis is to enhance performance of decentralized processing brought about by approximate divisions of the corresponding centralized

processing. In the thesis, we tackle two estimation problems, namely, in-network adaptive least-mean-square (LMS) filter and Gaussian Process Regression (GPR), as the specific themes of the decentralized processing. Decentralization of the former problem is in demand to enhance scalability in network size, which is motivated by IoT environments where the number of devices is increasing recently. On the other hand, the latter problem is desired to be scalable in the amount of data because it requires high computational complexity. Some approximation methods of GPR reduce the complexity by decentralization of a part of the processes. Decentralization of both of the problems includes in common some degradation coming from approximate divisions, so that there is some room for improving the performance of the algorithms. In the following, the decentralization of each problem and the contributions of the thesis are reviewed.

1.2.2 Part I: Fully Distributed LMS based on Message Propagation

Adaptive filters (Farhang-Boroujeny, 2013) aim to adaptively estimate filter coefficients $\boldsymbol{w}^{\mathrm{o}}$ so that a system can yield desirable outputs, according to measurements $d^{(i)}$ and inputs $\boldsymbol{u}^{(i)}$ that can be observed at each time i. One may notice that the settings are the same as online learning (Shalev-Shwartz, 2012) but the main interest in online learning is often to estimate outputs corresponding to test inputs instead of the filter coefficients. LMS is the simplest adaptive filter derived on the basis of a mean-square error of a linear measurement model $d^{(i)} = \boldsymbol{u}^{(i)\mathrm{H}}\boldsymbol{w}^{\mathrm{o}} + \boldsymbol{v}^{(i)}$, where $\boldsymbol{v}^{(i)}$ is additive noise, and has relatively low complexity because it does not include calculations of matrix inversion or expectation.

LMS itself does not necessarily require high computational cost but the decentralized estimation is in demand from the application side. In communication networks such as sensor networks and multi-agent systems (Olfati-Saber et al., 2007), LMS can be used for model parameter tracking or target localization (Sayed, 2014) on the basis of the measurements at the sensors or agents which are referred to as nodes. For the centralized LMS, a central processor (or fusion node) collects all the measurements and processes them. However, when the number of nodes in the network is large, most of the load concentrates around the fusion node. The heavily biased load may cause the failure of the fusion node and stop all the processing. Under such a situation, there have been proposed fully distributed LMS algorithms, where each node in the network iterates updates of its own estimate by adaptation and by communications with its neighbors, to make the process scalable and robust. The algorithms include approximated divisions of the centralized LMS in order to run only with locally available information at each node, so that the performance is generally degraded. Thus, it is important to reduce the degradation caused by the approximated divisions and improve the performance of the fully

distributed LMS.

In the thesis, to reduce the degradation, we focus on ways for exchanging information in a fully distributed LMS, which are significantly related to approximate divisions of the problem considered in the centralized LMS. Specifically, we introduce a message propagation algorithm as the method for information exchange to achieve faster convergence or lower error than the conventional distributed LMS. The detailed derivations are described and the performance is evaluated via computer simulation shown in Chapter 3. We extend the proposed algorithm to the case where the parameter of interest is sparse. The derivation and evaluation are shown in Chapter 4.

1.2.3 Part II: New Representation of Scalable GPR

GPR (Rasmussen and Williams, 2006) mainly aims to estimate output z corresponding to input $\boldsymbol{x}$ by using more flexible models than the linear measurement model used in LMS, whose use is not limited to time-dependent data. It supposes a nonlinear measurement model $z = f(\boldsymbol{x}) + \epsilon$, where $f(\cdot)$ follows a Gaussian process and where ϵ is additive noise. The prediction can be obtained as the posterior distribution of Bayesian estimation. GPR enables flexible estimation but is known to require high computational cost when the number N of the input data is large because of the inversion of an $N \times N$ matrix. Note that the linear measurement model in LMS is actually included in GPR as a special case.

There exist a lot of approximation methods of GPR. One approach called aggregation method divides all data into relatively small sub-datasets, runs GPR for each local sub-dataset, and then aggregates the local predictions. This can be interpreted as partially distributed GPR. Also in the aggregation method, the performance is generally degraded compared with the centralized GPR because a part of covariance information of the data is lost due to the division of the processing. Thus, it is desirable to develop an algorithm that balances high availability of the covariance information and low complexity.

In the thesis, we enhance performance of the conventional aggregation method by adopting richer covariance information to reduce the degradation. We show that the conventional aggregation method can be generalized via the concept of dimensionality reduction called sketching (Liberty, 2013; Woodruff, 2014) and propose a novel distributed GPR. The proposed algorithm can use richer information at the cost of a slightly higher computational complexity than the conventional method or can significantly reduce the complexity by using less information. The detailed derivation is described and the performance is evaluated on synthetic and real datasets shown in Chapter 5.

1.3 Notations

In the rest of the thesis, we use the following notations. Boldface indicates vector or matrix. Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ be the sets of real numbers, complex numbers, and nonnegative integers, respectively. Superscripts $(\cdot)^{\mathrm{T}}$ and $(\cdot)^{\mathrm{H}}$ denote the transpose and the Hermitian transpose, respectively. $\mathrm{E}[\cdot]$, $\det[\cdot]$, $\mathrm{Tr}[\cdot]$, and $\mathrm{Re}[\cdot]$ stand for the expectation, the determinant, the trace operators, and the real part of a complex number, respectively. $\|\boldsymbol{w}\|_p$ for a vector $\boldsymbol{w} = [w_1, \ldots, w_M]^{\mathrm{T}}$ and $p \geq 1$ is ℓ_p-norm defined by $\left(\sum_{m=1}^{M} |w_m|^p\right)^{1/p}$. The abbreviated form $\|\cdot\|$ represents ℓ_2-norm. $\|\boldsymbol{w}\|_0$ for a vector $\boldsymbol{w}$ is the number of nonzero elements of $\boldsymbol{w}$ and is called ℓ_0-norm. $\mathrm{vec}(\cdot)$ and $\mathrm{vec}^{-1}(\cdot)$ denote the vectorization and the inverse vectorization operators. $\otimes$ denotes Kronecker product. $\mathrm{diag}[\cdot]$ is the diagonal matrix composed of the elements in the square brackets. The largest eigenvalue of matrix $\boldsymbol{A}$ is represented as $\lambda_{\max}(\boldsymbol{A})$. $\mathbf{0}$, $\mathbf{1}$, $\boldsymbol{O}$, and $\boldsymbol{I}$ stand for the vectors where all elements are composed of 0 or 1, zero matrix, and identity matrix, respectively. $\mathbf{1}_M$ and $\boldsymbol{I}_M$ are also the vector with size M where all elements are composed of 1 and identity matrix with size $M \times M$, respectively. $\mathcal{N}(\boldsymbol{m}, \boldsymbol{\Sigma})$ is Gaussian distribution with mean $\boldsymbol{m}$ and covariance $\boldsymbol{\Sigma}$. $\mathrm{Cov}[\boldsymbol{x}, \boldsymbol{y}]$ means the covariance matrix of random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. $\ker \boldsymbol{A} := \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\}$ is the kernel (null space) of the matrix $\boldsymbol{A}$. $[\cdot]_{ab}$, $[\cdot]_a$, and $[\cdot]_{[a][b]}$ denote the (a, b)th element of the matrix, the ath row of the matrix or ath element of the vector, and the (a, b)th block of the block matrix, respectively. $\mathrm{sign}(\boldsymbol{w})$ for $\boldsymbol{w} = [w_1, \ldots, w_M]^{\mathrm{T}}$ is a vector of size M whose m-th element $[\mathrm{sign}(\boldsymbol{w})]_m$ is defined as

$$[\mathrm{sign}(\boldsymbol{w})]_m = \begin{cases} w_m/|w_m|, & \text{if } w_m \neq 0 \\ 0, & \text{if } w_m = 0. \end{cases} \quad (m = 1, \ldots, M)$$

1.4 Organization

This thesis is organized as follows. In Part I, we discuss fully distributed LMS based on message propagation method.

- Chapter 2: Network model and mathematical characterizations are presented. Diffusion LMS (D-LMS), which is one of the decentralized LMS algorithms, and consensus propagation (CP), which is employed in the thesis to improve the performance of D-LMS, are introduced.

- Chapter 3: We propose two algorithms: 1. A novel distributed LMS based on the exact version of CP, which runs by extracting a spanning tree from the original network. This work is based on Nakai and Hayashi (2017); Nakai-Kasai and Hayashi (2019). 2. D-LMS based on the suboptimal version of CP, which can run even in a network with cycles and which results

in novel combination weights for D-LMS. This work is based on Nakai and Hayashi (2018); Nakai-Kasai and Hayashi (2019). The convergence analysis and computational complexity of the two proposed algorithms are discussed. The performance is evaluated via computer simulation.

- Chapter 4: We propose an algorithm that extends the second algorithm in Chapter 3 to a sparsity-aware variant of D-LMS. The convergence analysis and computational complexity of the proposed algorithm are discussed. The performance is evaluated via computer simulation. This work is based on Nakai-Kasai and Hayashi (2020, 2021).

In Part II, we address the new representation of scalable GPR.

- Chapter 5: The centralized processing of GPR and its approximation methods are introduced. We show a generalization of one of the approximation methods through a sketching technique which lowers a dimension of a large-size vector by multiplying it by a matrix. The generalization brings about a novel scalable approximation that uses some auxiliary points. There are five options for the choice of the points, which affect the predictive performance and the computational complexity. The proposed method has a theoretical guarantee known as consistency. The performance of the proposed algorithm with the five options is evaluated by using synthetic and real data. This work is based on Nakai-Kasai and Tanaka (2021).

In Part III Chapter 6, the summary of the thesis and future directions are discussed.

Part I

Fully Distributed LMS based on Message Propagation

2: Introduction of Distributed LMS

2.1 Introduction

2.1.1 Background and Related Work

On large-scale communication networks composed of a number of small nodes, such as sensor networks and M2M (machine-to-machine), the subject of distributed signal processing (Hara et al., 2006) is gathering much attention recently. For the centralized method, all measurements at the nodes are processed at once in a fusion node, but it suffers from problems that nodes around the fusion node tend to consume much energy for the relaying and also that the network is vulnerable to the failure of the fusion node. On the other hand, for the distributed processing, each node processes by itself its own measurements and information obtained from neighboring nodes and estimates a desired parameter. This method is scalable in network size and robust because there is less imbalance of computation and communication loads among the nodes.

For tracking unknown parameters of interest in networks, several interpretations of adaptive filters (Farhang-Boroujeny, 2013) as distributed signal processing have been proposed (Sayed and Lopes, 2007; Cattivelli et al., 2008; Wang and Dekorsy, 2019; Lopes and Sayed, 2008; Schizas et al., 2009; Cattivelli and Sayed, 2010; Sayed et al., 2013). The most famous and simplest adaptive filter, least-mean-square (LMS) algorithm (Mandic et al., 2015), has also been decentralized in various ways (Lopes and Sayed, 2007, 2008; Schizas et al., 2009; Cattivelli and Sayed, 2010; Sayed et al., 2013). In these algorithms, each node iteratively updates its estimate by the LMS algorithm using local measurements and by averaging the estimates of its neighboring nodes obtained via wireless communications, and finally all nodes in the network achieve a common estimate. The incremental LMS (Lopes and Sayed, 2007) is applicable to any connected networks but it suffers from slow convergence because it assumes that each node communicates only with a single specific node among its neighboring nodes. In order to achieve faster convergence, the Combine-then-Adapt (CTA) diffusion LMS (D-LMS) (Lopes and Sayed, 2008) is derived by modifying the updating rules of the incremental LMS so that it allows each node to utilize the estimates of all neighboring nodes. Moreover, the consensus-based LMS (Schizas et al., 2009) achieves faster convergence than the CTA D-LMS (Lopes and Sayed, 2008) while it requires uneven processing of the nodes. Furthermore, a modified D-LMS named Adapt-then-Combine (ATC) D-LMS, which is obtained by just interchanging the order of the updating rules of

CTA diffusion (Lopes and Sayed, 2008), has been proposed in Cattivelli and Sayed (2010). The ATC D-LMS can outperform not only the original version (Lopes and Sayed, 2008) but also the consensus-based LMS (Schizas et al., 2009) in terms of convergence rate. The estimate of D-LMS converges to that of the centralized solution if sufficiently small step-size parameters are employed (Zhao and Sayed, 2012; Sayed et al., 2013; Sayed, 2014). More recently, some improvements have been proposed such as a doubly-compressed D-LMS (Harrane et al., 2019) for communication reduction and an Adapt-Multi-Combine (AMC) D-LMS (Kong et al., 2017) for better convergence performance at the cost of frequent communications. D-LMS for more specific settings has also attracted interests in the context of multitask learning where each node in the network estimates its own target vector and the neighboring nodes have related targets (Nassif et al., 2017).

In addition to this, in many applications, unknown parameters could be sparse in some representation domain, meaning that most elements of the parameters are zero or very small. It is known that compressed sensing (Donoho, 2006) is effective for the sparse signal estimation, and methods based on the compressed sensing have been proposed for the distributed signal processing as well (Duarte et al., 2005; Hayakawa et al., 2018; Wee and Yamada, 2013; Hu and Zhan, 2016). Among such methods, an extension of D-LMS for sparse estimation has been proposed and called sparse diffusion LMS (SD-LMS) (Lorenzo and Sayed, 2013). The theoretical analysis in Lorenzo and Sayed (2013) guarantees convergence in the mean sense and describes mean-square behaviors of SD-LMS. It has been also revealed that SD-LMS shows better convergence performance than that of D-LMS in the case of sparse estimation under realistic conditions.

Both D-LMS and SD-LMS are derived by approximately dividing each corresponding centralized problem into sub-problems that can be solved in a fully distributed manner. The performance degradation by the approximate divisions is compensated by cooperation with neighboring nodes. In D-LMS and SD-LMS, conventional average consensus protocol (Olfati-Saber et al., 2007) has been employed for the cooperation, where all nodes in the network aim to obtain the average of all nodes' initial state values by exchanging the current estimates of the average with their neighbors. This protocol is known to require a lot of iterations to achieve consensus, especially in large networks. It is also known that, even in small-scale networks, the choice of combination weights used in the updates has a great impact on the convergence performance of D-LMS (Takahashi et al., 2010). Thus, several combination rules have been proposed in the literature, such as uniform rule (Blondel et al., 2005), maximum degree rule (Scherber and Papadopoulos, 2004), Metropolis rule (Xiao and Boyd, 2004), and relative degree rule (Cattivelli and Sayed, 2010). More sophisticated static rules are considered

in Cattivelli and Sayed (2010); Takahashi et al. (2010); Zhao et al. (2012), which are derived by solving some optimization problems. In particular, a closed-form solution that minimizes the steady-state error has been derived in Tu and Sayed (2011) and Zhao et al. (2012), which is referred to as relative-variance rule. Since the relative-variance rule requires network statistics such as noise variance at all nodes, which are not locally available at each node in general, adaptive estimation methods of the parameters have been also proposed in Tu and Sayed (2011) and Zhao et al. (2012). On the other hand, there exists another kind of average consensus algorithms named consensus propagation (CP) (Moallemi and Roy, 2006). CP is based on the efficient exchange of information called belief propagation (Pearl, 1988). The algorithm can be interpreted in two ways. The one is exact CP that achieves exact average consensus with the minimum number of iterations required for message propagation through the network, i.e., the diameter of the tree, while it restricts the use to the specific structure of the network such as a tree. Another is loopy CP that can be applied to networks with some cycles but becomes suboptimal. Convergence of the loopy CP is controlled by some constants whose optimal values have not been known.

2.1.2 Contributions

In this thesis, we enhance the convergence performance of D-LMS (Cattivelli and Sayed, 2010) and SD-LMS (Lorenzo and Sayed, 2013) by improving ways for cooperation among nodes that compensate for performance degradation caused by approximated divisions of the centralized LMS. Specifically, we first propose a novel distributed LMS algorithm by applying the exact CP. Since networks for in-network signal processing do not necessarily have a tree structure in general, the proposed algorithm is applied to a spanning tree extracted from the original network using some spanning tree protocols (Herzen et al., 2011; Bui et al., 2004). Moreover, we apply the loopy CP to D-LMS and derive another novel distributed LMS algorithm which is directly applicable to the original networks having cycles. This thesis also extends it to an analytically tractable algorithm, which results in a novel combination rule of D-LMS. The constants involved in the loopy CP control the convergence property but they are known to be difficult to optimize in general. We thus select the constants by minimizing an upper bound of steady-state mean-squared-deviation (MSD) of D-LMS. We further extend the proposed combination rule to an adaptive version, which can be implemented in a fully distributed manner. Finally, we apply the above method to SD-LMS and optimize the constants in terms of MSD at the steady-state under practical assumptions. We derive two combination rules for SD-LMS according to different assumptions. Simulation results show the efficiency of the proposed methods. The proposed algorithms based

on the exact CP and the loopy CP can achieve faster convergence than the conventional ATC D-LMS for large-scale network. The proposed combination rule inspired by the loopy CP achieves better convergence performance than the conventional combination rules. SD-LMS using the proposed method also achieves faster convergence than that using conventional combination rules.

2.2 Preliminaries

2.2.1 Network Model

Consider a network $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is a set of nodes and $\mathcal{E}$ is a set of edges, and where $|\mathcal{V}| = N$. The network $\mathcal{G}$ is assumed to be fixed and connected, i.e., there is at least one path for any pair of nodes in the network. The nodes that are directly connected by edges can communicate and share information with each other. Each node $k \in \mathcal{V}$ ($k = 1, \ldots, N$) can perform only single-hop communications with its neighbors. $\mathcal{N}_k$ is the set of the neighbors of node k including node k itself, and $\mathcal{N}_k \setminus \{k\}$ is the set of the neighbors except for node k. Fig. 2.1 shows an example of a network with $N = 10$. Weighted adjacency matrix $\mathrm{adj}(\mathcal{G}) \in \mathbb{R}^{N \times N}$ of the network $\mathcal{G}$ is given by

$$[\mathrm{adj}(\mathcal{G})]_{lk} = \begin{cases} w_{lk}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ 0, & \text{if } l \notin \mathcal{N}_k \setminus \{k\}, \end{cases} \quad \text{for } k, l \in \mathcal{V}. \tag{2.2.1}$$

where $w_{lk} > 0$ is a weight of the edge between node l and k.

Figure 2.1: An example of a network with 10 nodes. $\mathcal{N}_k$ is the set of neighbors of node k (including node k itself).

2.2.2 Diffusion LMS

D-LMS Algorithm

On a network $\mathcal{G}$, consider a problem of tracking an unknown deterministic vector of interest $\boldsymbol{w}^{\mathrm{o}} \in \mathbb{C}^M$ by using measurements on the unknown vector and measurement vectors obtained at each node and each time that are related via linear regression model. For the centralized processing, the LMS filter (Mandic et al., 2015) is employed to track the unknown vector in a low computational cost. In this section, the fully distributed method, D-LMS (Lopes and Sayed, 2008; Cattivelli and Sayed, 2010; Sayed et al., 2013; Sayed, 2014), is introduced, where all nodes in the network track the unknown vector by themselves in a computationally efficient manner by exploiting LMS-like updates and by cooperating with the neighboring nodes. For simplicity, we assume that all communications between neighboring nodes are perfect, i.e., we do not consider any communication error, while the following discussion is applicable to networks with noisy links as considered in Zhao et al. (2012). Note that we also assume that nodes do not exchange the raw measurements and measurement vectors but only estimates processed in some way, which is convenient for private data for example, but the following discussion can be extended to the more general situation where each node exchanges the raw measurements and measurement vectors.

Consider random measurement process $\mathcal{D}_k^{(i)} \in \mathbb{C}$ at each node $k \in \mathcal{V}$ and time $i \in \mathbb{N}$ given by the following linear model:

$$\mathcal{D}_k^{(i)} = \boldsymbol{\mathcal{U}}_k^{(i)\mathrm{H}} \boldsymbol{w}^{\mathrm{o}} + \mathcal{V}_k^{(i)}. \tag{2.2.2}$$

$\boldsymbol{\mathcal{U}}_k^{(i)} \in \mathbb{C}^M$ is a random measurement vector process. $\mathcal{V}_k^{(i)} \in \mathbb{C}$ is a zero-mean additive complex noise process with variance of σ_k^2, which is pairwise independent of $\mathcal{V}_l^{(j)}$ for $l \in \mathcal{V} \setminus \{k\}$ and $j \in \mathbb{N} \setminus \{i\}$, and is pairwise independent of $\boldsymbol{\mathcal{U}}_l^{(j)}$ for all $k, l \in \mathcal{V}, i, j \in \mathbb{N}$. The variance σ_k^2 is assumed to be unknown to each node. The measurement and measurement vector processes $\{\mathcal{D}_k^{(i)}, \boldsymbol{\mathcal{U}}_k^{(i)}\}$ at each node k are assumed to be jointly wide-sense stationary and have zero-mean.

First, a global problem that should be considered at all nodes is shown and then it is approximated to derive a fully distributed algorithm. All nodes in the network aim to obtain a common estimate of $\boldsymbol{w}^{\mathrm{o}}$ basically by minimizing the following global cost function (Cattivelli and Sayed, 2010):

$$\mathcal{J}^{\mathrm{glob}}(\boldsymbol{w}) = \sum_{k=1}^{N} \mathrm{E}\left[\left|\mathcal{D}_k^{(i)} - \boldsymbol{\mathcal{U}}_k^{(i)\mathrm{H}} \boldsymbol{w}\right|^2\right]. \tag{2.2.3}$$

The function also does not depend on time index i because of the joint wide-sense stationarity. This includes all nodes' information of random processes, but, in the

fully distributed situation, each node can directly obtain only a part of the information, i.e., its own and neighbors' information. In D-LMS, the following approximated local cost function has been considered at each node k,

$$J_k^{\mathrm{loc}}(\boldsymbol{w}) = \mathrm{E}\left[\left|\mathcal{D}_k^{(i)} - \boldsymbol{\mathcal{U}}_k^{(i)\mathrm{H}}\boldsymbol{w}\right|^2\right] + \sum_{l\in\mathcal{N}_k\setminus\{k\}} b_{lk}\|\boldsymbol{w} - \boldsymbol{\phi}_l^{(i)}\|^2, \tag{2.2.4}$$

where $\boldsymbol{\phi}_l^{(i)} \in \mathbb{C}^M$ is the current estimate of $\boldsymbol{w}^{\mathrm{o}}$ at node l, and where $b_{lk} \geq 0$ is the weight to be determined later. The second term on the right-hand side of (2.2.4) means the penalty for the difference between node k's and the neighbors' estimates. The local function $J_k^{\mathrm{loc}}(\boldsymbol{w})$ is an approximate division of the global one $J^{\mathrm{glob}}(\boldsymbol{w})$ and includes compensation by the second term. If each node k took the first term on the right-hand side of (2.2.4) into account and if it knew moments related to $\mathcal{D}_k^{(i)}$ and $\boldsymbol{\mathcal{U}}_k^{(i)}$, it could obtain the optimal value $\hat{\boldsymbol{w}}$ from the derivative of the first term as

$$\hat{\boldsymbol{w}} = \boldsymbol{R}_{u_k}^{-1}\boldsymbol{r}_{d_k u_k},$$

where $\boldsymbol{r}_{d_k u_k} = \mathrm{E}[\mathcal{D}_k^{(i)}\boldsymbol{\mathcal{U}}_k^{(i)}]$ and $\boldsymbol{R}_{u_k} = \mathrm{E}[\boldsymbol{\mathcal{U}}_k^{(i)}\boldsymbol{\mathcal{U}}_k^{(i)\mathrm{H}}]$, which are independent of time index i because of the joint wide-sense stationarity of $\{\mathcal{D}_k^{(i)}, \boldsymbol{\mathcal{U}}_k^{(i)}\}$. It coincides with the unknown vector because the same formula can be obtained by multiplying $\boldsymbol{\mathcal{U}}_k^{(i)}$ and taking the expectation of both sides in (2.2.2). In other words, the unknown vector could be calculated at each node k if the moments $\{\boldsymbol{r}_{d_k u_k}, \boldsymbol{R}_{u_k}\}$ were known. However, in applications, each node cannot obtain the moments but realizations of the processes. The second term of the function (2.2.4) is motivated to improve convergence performance of the algorithm by cooperating with the neighboring nodes.

If we employ the steepest descent method (Farhang-Boroujeny, 2013) to minimize the function J_k^{loc}, node k's estimate $\boldsymbol{\phi}_k^{(i)}$ is updated by using the gradient $\nabla J_k^{\mathrm{loc}}(\boldsymbol{w})$ of (2.2.4) as

$$\begin{aligned}\boldsymbol{\phi}_k^{(i)} &= \boldsymbol{\phi}_k^{(i-1)} - \frac{\mu_k}{2}\nabla J_k^{\mathrm{loc}}(\boldsymbol{\phi}_k^{(i-1)}) \\ &= \boldsymbol{\phi}_k^{(i-1)} + \mu_k(\boldsymbol{r}_{d_k u_k} - \boldsymbol{R}_{u_k}\boldsymbol{\phi}_k^{(i-1)}) + \mu_k \sum_{l\in\mathcal{N}_k\setminus\{k\}} b_{lk}(\boldsymbol{\phi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)}),\end{aligned} \tag{2.2.5}$$

where $\mu_k > 0$ is a step-size parameter, which controls a balance between the convergence rate and the steady-state error. Larger μ_k accelerates convergence to the steady-state but error at the steady-state becomes larger. Smaller μ_k promotes smaller steady-state error but the convergence becomes slower. One of the famous adaptive filters, LMS algorithm (Farhang-Boroujeny, 2013; Mandic et al., 2015), replaces the moments in (2.2.5) with instantaneous values. Let a realization of $\{\mathcal{D}_k^{(i)}, \boldsymbol{\mathcal{U}}_k^{(i)}\}$ be $\{d_k^{(i)}, \boldsymbol{u}_k^{(i)}\}$. The moments are replaced as $\boldsymbol{r}_{d_k u_k} \simeq d_k^{(i)}\boldsymbol{u}_k^{(i)}$ and

$\boldsymbol{R}_{u_k} \simeq \boldsymbol{u}_k^{(i)} \boldsymbol{u}_k^{(i)\mathrm{H}}$, respectively. The LMS-type update is given by

$$\boldsymbol{\phi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)} (d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)}) + \mu_k \sum_{l \in \mathcal{N}_k \setminus \{k\}} b_{lk} (\boldsymbol{\phi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)}). \tag{2.2.6}$$

The equation (2.2.6) can be divided into two steps. Each node k first updates an immediate estimate $\boldsymbol{\psi}_k^{(i)} \in \mathbb{C}^M$ by using its own measurement, and then updates the estimate $\boldsymbol{\phi}_k^{(i)}$ by using $\boldsymbol{\psi}_k^{(i)}$ and $\boldsymbol{\psi}_l^{(i)}$ received from its neighbors $l \in \mathcal{N}_k \setminus \{k\}$.

$$\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)} (d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)}), \tag{2.2.7}$$

$$\boldsymbol{\phi}_k^{(i)} = \boldsymbol{\psi}_k^{(i)} + \mu_k \sum_{l \in \mathcal{N}_k \setminus \{k\}} b_{lk} (\boldsymbol{\psi}_l^{(i)} - \boldsymbol{\psi}_k^{(i)}), \tag{2.2.8}$$

where $\boldsymbol{\phi}_l^{(i)}$ and $\boldsymbol{\phi}_k^{(i-1)}$ in (2.2.6) are replaced with the immediate estimates $\boldsymbol{\psi}_l^{(i)}$ and $\boldsymbol{\psi}_k^{(i)}$, respectively. Here, let $\boldsymbol{A} = \{a_{lk}\} \in \mathbb{R}^{N \times N}$ be

$$a_{lk} = \begin{cases} \mu_k b_{lk}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ 1 - \mu_k \sum_{l \in \mathcal{N}_k \setminus \{k\}} b_{lk}, & \text{if } l = k, \\ 0, & \text{if } l \notin \mathcal{N}_k, \end{cases} \tag{2.2.9}$$

and then (2.2.8) can be rewritten as

$$\boldsymbol{\phi}_k^{(i)} = \sum_{l \in \mathcal{N}_k} a_{lk} \boldsymbol{\psi}_l^{(i)}, \tag{2.2.10}$$

where a_{lk} is the nonnegative combination weight defined in (2.2.9) and the matrix $\boldsymbol{A}$ satisfies

$$\mathbf{1}^{\mathrm{T}} \boldsymbol{A} = \mathbf{1}^{\mathrm{T}}. \tag{2.2.11}$$

The update (2.2.7) of the immediate estimate $\boldsymbol{\psi}_k^{(i)}$ is called adaptation step and that of $\boldsymbol{\phi}_k^{(i)}$ (2.2.10) is called combination step. From (2.2.4), (2.2.9), and (2.2.10), the second term in (2.2.4) corresponds to the combination step of D-LMS.

The combination weight a_{lk} does not have to be necessarily determined by the form in (2.2.9) but can be chosen freely as long as the constraint (2.2.11) is satisfied. A specific choice of the combination weights is called combination rule and has a great impact on the convergence performance of the algorithm (Takahashi et al., 2010). There are two types of the combination rules; a static rule that is independent of time index i and an adaptive rule that is dependent on i. Possible choices for the static one will be uniform rule (Blondel et al., 2005)

$$a_{lk}^{\mathrm{ave}} = \begin{cases} 1/|\mathcal{N}_k|, & \text{if } l \in \mathcal{N}_k, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2.12}$$

Metropolis rule (Xiao and Boyd, 2004)

$$a_{lk}^{\mathrm{met}} = \begin{cases} \frac{1}{\max\{|\mathcal{N}_k|, |\mathcal{N}_l|\}}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ 1 - \sum_{l \in \mathcal{N}_k \setminus \{k\}} a_{lk}^{\mathrm{met}}, & \text{if } l = k, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2.13}$$

Algorithm 1 D-LMS (ATC version) (Cattivelli and Sayed, 2010)

1: Initialize $\boldsymbol{\phi}_k^{(-1)} = 0,\ \forall k \in \mathcal{V}$
2: **for** each time $i \in \mathbb{N}$ and each node $k \in \mathcal{V}$ **do**
3: $\quad \boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)}(d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)})$ (adaptation step)
4: $\quad \boldsymbol{\phi}_k^{(i)} = \sum_{l \in \mathcal{N}_k} a_{lk} \boldsymbol{\psi}_l^{(i)}$ (combination step)
5: **end for**

relative degree rule (Cattivelli and Sayed, 2010)

$$a_{lk}^{\mathrm{rd}} = \begin{cases} \frac{|\mathcal{N}_l|}{\sum_{m \in \mathcal{N}_k} |\mathcal{N}_m|}, & \text{if } l \in \mathcal{N}_k, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2.14}$$

relative-variance rule (Tu and Sayed, 2011)

$$a_{lk}^{\mathrm{rv}} = \begin{cases} \frac{[\gamma_l^2]^{-1}}{\sum_{m \in \mathcal{N}_k} [\gamma_m^2]^{-1}}, & \text{if } l \in \mathcal{N}_k, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2.15}$$

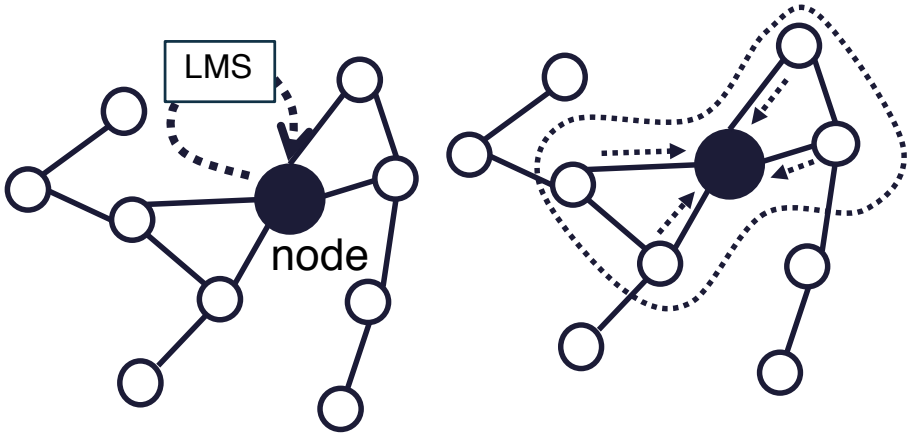
where $\gamma_l^2 = \mu_l^2 \sigma_l^2 \mathrm{Tr}[\boldsymbol{R}_{u_l}]$, and so on (Sayed, 2014). An example of the adaptive rule is adaptive relative-variance rule (Zhao et al., 2012)

$$a_{lk}^{\mathrm{rv}(i)} = \begin{cases} \frac{[(\gamma_{lk}^2)^{(i)}]^{-1}}{\sum_{m \in \mathcal{N}_k} [(\gamma_{mk}^2)^{(i)}]^{-1}}, & \text{if } l \in \mathcal{N}_k, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2.16}$$

where $(\gamma_{lk}^2)^{(i)} = (1 - \nu_k)(\gamma_{lk}^2)^{(i-1)} + \nu_k \left\| \boldsymbol{\psi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)} \right\|^2$ and where $0 < \nu_k < 1$.

Summarizing the adaptation step (2.2.7) and the combination step (2.2.10), the updating rules of D-LMS are shown in *Algorithm* 1 and outlined in Fig. 2.2. Note that this formulations are called ATC version of D-LMS (Cattivelli and Sayed, 2010) and an alternative algorithm obtained by interchanging the order of the updates (2.2.7) and (2.2.10) is referred to as CTA version (Lopes and Sayed, 2008). In this part, we focus on ATC because it generally outperforms CTA (Sayed, 2014, Table 6), (Zhao and Sayed, 2012), though the following discussion can hold in both versions.

AMC diffusion LMS algorithm (Kong et al., 2017) has been also proposed and is constructed on the basis of D-LMS. The algorithm iterates the combination steps to collect more estimates and increase available information at each node.



(a) Adaptation step (b) Combination step

Figure 2.2: Two steps of D-LMS.

The update rules are described as

$$\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)} (d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)}), \tag{2.2.17}$$

$$\boldsymbol{\phi}_k^{(i)[1]} = \sum_{l \in \mathcal{N}_k} a_{lk}^{[1]} \boldsymbol{\psi}_l^{(i)}, \tag{2.2.18}$$

$$\boldsymbol{\phi}_k^{(i)[2]} = \sum_{l \in \mathcal{N}_k} a_{lk}^{[2]} \boldsymbol{\phi}_l^{(i)[1]}, \tag{2.2.19}$$

$$\vdots$$

$$\boldsymbol{\phi}_k^{(i)} = \boldsymbol{\phi}_k^{(i)[J']} = \sum_{l \in \mathcal{N}_k} a_{lk}^{[J']} \boldsymbol{\phi}_l^{(i)[J'-1]}, \tag{2.2.20}$$

where J' denotes the number of iterations of the combination steps, and where $\boldsymbol{\psi}_k^{(i)[j']} \in \mathbb{C}^M$ and $a_{lk}^{[j']} > 0$ are the immediate estimate and a combination weight at j'th iteration ($j' = 1, \ldots, J'$), respectively. The equations (2.2.18)–(2.2.20) are summarized as follows:

$$\boldsymbol{\phi}_k^{(i)} = \sum_{l_1 \in \mathcal{N}_k} a_{l_1 k}^{[J']} \sum_{l_2 \in \mathcal{N}_{l_1}} a_{l_2 l_1}^{[J'-1]} \cdots \sum_{l_{J'} \in \mathcal{N}_{l_{J'-1}}} a_{l_{J'} l_{J'-1}}^{[1]} \boldsymbol{\psi}_{l_{J'}}^{(i)}. \tag{2.2.21}$$

The adaptation step of AMC (2.2.17) is the same as that of D-LMS (2.2.7), but the

AMC diffusion LMS communicates J' times to collect immediate estimates of the nodes within J' hops.

Connection with Discrete-time Average Consensus Protocol

The combination step (2.2.10) of D-LMS is closely related to conventional average consensus protocol using a weighted average of neighbors' values in multi-agent networked systems (Olfati-Saber et al., 2007). In the protocol, all nodes in the network aim to obtain the average of all nodes' initial state values by exchanging the current estimates of the average with their neighbors. This problem is called average consensus problem. An iterative update equation of the conventional discrete-time protocol is given by

$$\hat{x}_k^{(i'+1)} = \bar{x}_k^{(i')} + \mu' \sum_{l \in \mathcal{N}_k \setminus \{k\}} w_{lk} (\hat{x}_l^{(i')} - \hat{x}_k^{(i')}), \tag{2.2.22}$$

where $\hat{x}_k^{(i')}$ is the estimate at node k and time i' and can be vector or scalar, and where μ' is a small positive number. In order to relate the combination step equation (2.2.10) with (2.2.22), we substitute $\boldsymbol{\phi}_k^{(i)}$ to $\hat{x}_k^{(i'+1)}$ and $\boldsymbol{\psi}_k^{(i)}$ to $\hat{x}_k^{(i')}$. Then, we have

$$\begin{aligned}
\boldsymbol{\phi}_k^{(i)} &= \boldsymbol{\psi}_k^{(i)} + \mu' \sum_{l \in \mathcal{N}_k \setminus \{k\}} w_{lk} (\boldsymbol{\psi}_l^{(i)} - \boldsymbol{\psi}_k^{(i)}) \\
&= (1 - \sum_{l \in \mathcal{N}_k \setminus \{k\}} a_{lk}) \boldsymbol{\psi}_k^{(i)} + \sum_{l \in \mathcal{N}_k \setminus \{k\}} a_{lk} \boldsymbol{\psi}_l^{(i)} \\
&= \sum_{l \in \mathcal{N}_k} a_{lk} \boldsymbol{\psi}_l^{(i)},
\end{aligned}$$

where we set $a_{lk} = \mu' w_{lk}$. Therefore, the combination step (2.2.10) can be regarded as an example of the conventional average consensus protocol.

Convergence Analysis

In this section, the convergence of D-LMS is discussed. We define convergence in the mean sense as in the following definition.

Definition 1. *Let $\{\boldsymbol{\phi}\}_{i \in \mathbb{N}}$ be a sequence of estimates of $\boldsymbol{w}^{\mathrm{o}}$ generated by an algorithm. We say that the algorithm converges in the mean sense when the estimation error $\tilde{\boldsymbol{w}}^{(i)} = \boldsymbol{w}^{\mathrm{o}} - \boldsymbol{\phi}^{(i)}$ satisfies $\lim_{i \to \infty} \mathrm{E}[\tilde{\boldsymbol{w}}^{(i)}] = 0$.*

The estimation error of D-LMS at node k and time i is given by

$$\tilde{\boldsymbol{w}}_k^{(i)} = \boldsymbol{w}^{\mathrm{o}} - \boldsymbol{\phi}_k^{(i)}. \tag{2.2.23}$$

Moreover, for the mean-square error analysis, we evaluate behaviors of $\mathrm{E}[\|\tilde{\boldsymbol{w}}_k^{(i)}\|^2]$ at time i or at the steady-state. We define instantaneous network MSD, MSD at each node k at time i, network MSD at time i, the steady-state MSD at each node k, and the steady-state network MSD as

$$\mathrm{MSD}^{\mathrm{nw,in}} = \frac{1}{N}\sum_{k=1}^{N}\|\tilde{\boldsymbol{w}}_k^{(i)}\|^2, \tag{2.2.24}$$

$$\mathrm{MSD}_{k,i} = \mathrm{E}[\|\tilde{\boldsymbol{w}}_k^{(i)}\|^2], \tag{2.2.25}$$

$$\mathrm{MSD}_i^{\mathrm{nw}} = \frac{1}{N}\sum_{k=1}^{N}\mathrm{MSD}_{k,i} = \mathrm{E}\left[\mathrm{MSD}^{\mathrm{nw,in}}\right], \tag{2.2.26}$$

$$\mathrm{MSD}_k = \lim_{i\to\infty}\mathrm{MSD}_{k,i}, \tag{2.2.27}$$

$$\mathrm{MSD}^{\mathrm{nw}} = \frac{1}{N}\sum_{k=1}^{N}\mathrm{MSD}_k = \lim_{i\to\infty}\mathrm{MSD}_i^{\mathrm{nw}} = \lim_{i\to\infty}\mathrm{E}\left[\mathrm{MSD}^{\mathrm{nw,in}}\right], \tag{2.2.28}$$

respectively.

We introduce the following two reasonable assumptions.

Assumption 1 (Cattivelli and Sayed (2010); Sayed (2014))**.** *The measurement vector process* $\{\boldsymbol{\mathcal{U}}_k^{(i)}\}$ *is spatially independent and temporally white, i.e.,* $\mathrm{E}\left[\boldsymbol{\mathcal{U}}_k^{(i)}\boldsymbol{\mathcal{U}}_l^{(j)\mathrm{H}}\right] = \boldsymbol{R}_{u_k}\delta_{kl}\delta_{ij}$.

Assumption 2 (Lorenzo and Sayed (2013))**.** *The step-sizes* $\{\mu_k\}$ *are sufficiently small so that higher order terms of the step-sizes can be ignored.*

Assumption 1 does not match actual measurement situations. However, not only it simplifies the analysis of adaptive filters but also in general the analyzed performance under Assumption 2 reproduces the actual situations reasonably well (Sayed, 2011).

The convergence of D-LMS in the mean sense is guaranteed under Assumption 1.

Theorem 1 (Cattivelli and Sayed (2010, Theorem 1), Sayed (2014, Theorem 6.1))**.** *Consider the measurement model given by* (2.2.2) *and the combination weight satisfying* (2.2.11)*. The convergence of D-LMS in the mean sense is guaranteed under Assumption 1 if the step-sizes satisfy*

$$0 < \mu_k < \frac{2}{\lambda_{\max}(\boldsymbol{R}_{u_k})} \quad (k \in \mathcal{V}).$$

Proof sketch: The error recursion can be expressed as

$$\mathrm{E}\left[\tilde{\boldsymbol{w}}^{(i)}\right] = \mathcal{A}^{\mathrm{T}}(\boldsymbol{I}_{NM} - \mathcal{M}\mathcal{D})\mathrm{E}\left[\tilde{\boldsymbol{w}}^{(i-1)}\right], \tag{2.2.29}$$

where $\tilde{\boldsymbol{w}}^{(i)} = \left[(\tilde{\boldsymbol{w}}_1^{(i)})^{\mathrm{T}}, \ldots, (\tilde{\boldsymbol{w}}_N^{(i)})^{\mathrm{T}}\right]^{\mathrm{T}}$, $\mathcal{A} = \boldsymbol{A} \otimes \boldsymbol{I}_M$, $\mathcal{D} = \mathrm{diag}\,[\boldsymbol{R}_{u_1}, \ldots, \boldsymbol{R}_{u_N}]$, and $\mathcal{M} = (\mathrm{diag}\,[\mu_1, \ldots, \mu_N]) \otimes \boldsymbol{I}_M$. The recursion converges to zero if all the eigenvalue of $\boldsymbol{I}_{MN} - \mathcal{M}\mathcal{D}$ lie in a unit disc because $\mathcal{A}$ is composed of stochastic matrices, the magnitude of whose eigenvalues is less than or equal to 1. Thus, the theorem is derived by constraining the maximum eigenvalue of $\boldsymbol{I}_{MN} - \mathcal{M}\mathcal{D}$ to be less than 1. □

Moreover, the transient behavior of the mean-square error is represented in Cattivelli and Sayed (2010, Sect. 4) and Sayed (2014, Sect. 6) under Assumptions 1 and 2. Now let

$$\boldsymbol{F}' = (\boldsymbol{I}_{(MN)^2} - \boldsymbol{I}_{MN} \otimes (\mathcal{D}\mathcal{M}) - (\mathcal{D}^{\mathrm{T}}\mathcal{M}) \otimes \boldsymbol{I}_{MN})\,(\mathcal{A} \otimes \mathcal{A}), \tag{2.2.30}$$

and then the theoretical network MSD learning curve of D-LMS is given by

$$\mathrm{MSD}_i^{\mathrm{nw}} = \mathrm{MSD}_{i-1}^{\mathrm{nw}} + \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}}\boldsymbol{F}'^{i}\boldsymbol{q}' - \frac{1}{N}\boldsymbol{w}^{\mathrm{T}}\left[\mathrm{vec}^{-1}\left(\boldsymbol{F}'^{i}\left[\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}'\right]\boldsymbol{q}'\right)\right]\boldsymbol{w}, \tag{2.2.31}$$

where $\boldsymbol{r}' = \mathrm{vec}\left(\mathcal{A}^{\mathrm{T}}\mathcal{M}\mathcal{G}\mathcal{M}\mathcal{A}\right)$, $\mathcal{G} = \mathrm{diag}\left[\sigma_1^2\boldsymbol{R}_{u_1}, \ldots, \sigma_N^2\boldsymbol{R}_{u_N}\right]$, $\boldsymbol{w} = \boldsymbol{1}_N \otimes \boldsymbol{w}^{\mathrm{o}}$, and $\boldsymbol{q}' = \mathrm{vec}(\boldsymbol{I}_{MN})$. The performance at the steady-state is given by

$$\mathrm{MSD}^{\mathrm{nw}} = \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{q}', \tag{2.2.32}$$

$$\mathrm{MSD}_k = \boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{p}', \tag{2.2.33}$$

where $\boldsymbol{p}' = \mathrm{vec}(C_k)$ and C_k is a block diagonal matrix with size $MN \times MN$ whose blocks are $M \times M$ matrices that are zero matrices for all blocks except for the identity matrix on the k-th diagonal block.

Proof sketch: Consider weighted ℓ_2-norm of the error recursion (2.2.29) with any weight Σ. If $\Sigma = \frac{1}{N}\boldsymbol{I}_{NM}$, the norm coincides with $\mathrm{MSD}_i^{\mathrm{nw}}$ and if $\Sigma = C_k$, it coincides with $\mathrm{MSD}_{k,i}$. By calculating $\mathrm{MSD}_i^{\mathrm{nw}}$ directly under Assumptions 1 and 2, $\boldsymbol{F}'$ is extracted as a coefficient matrix that does not depend on time indices. The learning curve of the network MSD can be derived by the recursion for $\Sigma = \frac{1}{N}\boldsymbol{I}_{NM}$. Taking the limit $i \to \infty$ yields the steady-state network MSD. □

SD-LMS Algorithm

SD-LMS is an extension of D-LMS which is applicable when the unknown vector $\boldsymbol{w}^{\mathrm{o}}$ is known to be sparse. All nodes in the network aim to obtain a common estimate of $\boldsymbol{w}^{\mathrm{o}}$ basically by minimizing the following global cost function (Lorenzo and Sayed, 2013):

$$\mathcal{J}^{\mathrm{glob}}(\boldsymbol{w}) = \sum_{k=1}^{N} \mathrm{E}\left[\left|\mathcal{D}_k^{(i)} - \mathcal{U}_k^{(i)\mathrm{H}}\boldsymbol{w}\right|^2\right] + \lambda' f(\boldsymbol{w}), \tag{2.2.34}$$

where $\lambda' > 0$ is a regularization parameter and $f(\boldsymbol{w})$ is a real-valued sparsity-promoting regularization function such as ℓ_1-norm of $\boldsymbol{w}$ motivated by least absolute shrinkage and selection operator (LASSO) (Tibshirani, 1996), which is further discussed later. In the same way as in Sect. 2.2.2, the alternative problem considered to perform at each node k is minimizing the following approximated local cost function

$$\mathcal{J}_k^{\mathrm{loc}}(\boldsymbol{w}) = \mathrm{E}\left[\left|\mathcal{D}_k^{(i)} - \boldsymbol{\mathcal{U}}_k^{(i)\mathrm{H}}\boldsymbol{w}\right|^2\right] + \lambda f(\boldsymbol{w}) + \sum_{l\in\mathcal{N}_k\setminus\{k\}} b'_{lk}\|\boldsymbol{w} - \boldsymbol{\phi}_l^{(i)}\|^2, \tag{2.2.35}$$

where $\lambda > 0$ is a regularization parameter and $b'_{lk} \geq 0$ is the weight to be determined later. The LMS-type update for the local cost function can be derived by using the gradient and replacing the moments with the instantaneous values, and also divided into two steps as before, as

$$\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)}(d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}}\boldsymbol{\phi}_k^{(i-1)}) - \mu_k\lambda\partial f(\boldsymbol{\phi}_k^{(i-1)}), \tag{2.2.36}$$

$$\boldsymbol{\phi}_k^{(i)} = \boldsymbol{\psi}_k^{(i)} + \mu_k \sum_{l\in\mathcal{N}_k\setminus\{k\}} b'_{lk}(\boldsymbol{\psi}_l^{(i)} - \boldsymbol{\psi}_k^{(i)}), \tag{2.2.37}$$

where $\partial f(\cdot)$ is the sub-gradient of $f(\cdot)$. Letting $b'_{lk} = b_{lk}$ and introducing $\boldsymbol{A} = \{a_{lk}\}$ in (2.2.9) lead to

$$\boldsymbol{\phi}_k^{(i)} = \sum_{l\in\mathcal{N}_k} a_{lk}\boldsymbol{\psi}_l^{(i)}. \tag{2.2.38}$$

Summarizing the adaptation step (2.2.36) and the combination step (2.2.38), the updating rules of SD-LMS (Lorenzo and Sayed, 2013) are shown in *Algorithm* 2.

Remark 1. *SD-LMS is an extended algorithm of D-LMS (Cattivelli and Sayed, 2010) by adding the regularization term to the adaptation step.*

As the regularization function $f(\boldsymbol{w})$, a widely used form is ℓ_1-norm which enables us to avoid solving ℓ_0-regularization problems, which are known to be NP-hard, but in this thesis, we employ the following regularization function

$$f(\boldsymbol{w}) = \|\boldsymbol{W}_\epsilon\boldsymbol{w}\|_1 = \sum_{m=1}^{M} \frac{1}{\epsilon + |w_m|}|w_m|, \tag{2.2.39}$$

where w_m is the m-th element of $\boldsymbol{w}$, $\epsilon > 0$ is a parameter, and $\boldsymbol{W}_\epsilon = \mathrm{diag}\left[\frac{1}{\epsilon+|w_1|}, \ldots, \frac{1}{\epsilon+|w_M|}\right]$. This is regarded as reweighting of ℓ_1-norm (Candes et al., 2008) and, according to Lorenzo and Sayed (2013), achieves better performance than ℓ_1-norm, as anticipated by the fact that it is a better approximation of $\|\boldsymbol{w}\|_0$ than $\|\boldsymbol{w}\|_1$ as long as ϵ is sufficiently small. As an alternative to sub-gradient of (2.2.39), the following formula is exploited:

$$\partial f(\boldsymbol{w}) = \boldsymbol{W}_\epsilon \mathrm{sign}(\boldsymbol{w}), \tag{2.2.40}$$

where $\boldsymbol{W}_\epsilon$ in (2.2.39) are regarded as weights independent of $\boldsymbol{w}$.

The convergence of SD-LMS in the mean sense is also guaranteed in Lorenzo and Sayed (2013) under the same assumption as D-LMS.

Theorem 2 (Lorenzo and Sayed (2013, Theorem 1))**.** *Consider the measurement model given by* (2.2.2) *and the combination weight satisfying* (2.2.11)*. Assume the regularization function $f(\cdot)$ so that $\partial f(\cdot)$ is bounded. The convergence of SD-LMS in the mean sense is guaranteed under Assumption 1 if the step-sizes satisfy*

$$0 < \mu_k < \frac{2}{\lambda_{\max}(\boldsymbol{R}_{u_k})} \quad (k \in \mathcal{V}).$$

Proof sketch: The error recursion can be expressed as

$$\begin{aligned}\mathrm{E}\left[\tilde{\boldsymbol{w}}^{(i)}\right] &= \mathcal{B}\mathrm{E}\left[\tilde{\boldsymbol{w}}^{(i-1)}\right] + \lambda\mathcal{A}^{\mathrm{T}}\mathcal{M}\mathrm{E}\left[\partial f(\boldsymbol{\phi}^{(i-1)})\right] \\ &= \mathcal{B}^i\mathrm{E}\left[\tilde{\boldsymbol{w}}^{(0)}\right] + \lambda\sum_{n=0}^{i-1}\mathcal{B}^n\mathcal{A}^{\mathrm{T}}\mathcal{M}\mathrm{E}\left[\partial f(\boldsymbol{\phi}^{(i-n-1)})\right], \end{aligned} \tag{2.2.41}$$

where $\mathcal{B} = \mathcal{A}^{\mathrm{T}}(\boldsymbol{I}_{NM} - \mathcal{M}\mathcal{D})$ and $\boldsymbol{\phi}^{(i-1)} = \left[(\boldsymbol{\phi}_1^{(i-1)})^{\mathrm{T}}, \ldots, (\boldsymbol{\phi}_N^{(i-1)})^{\mathrm{T}}\right]^{\mathrm{T}}$. The first term on the right-hand side can be handled in the same manner as in Theorem 1. For the convergence of the second term, comparison test is employed that bounds the term by another series where the absolute convergence can be shown. From the triangle inequality, each element of $\mathcal{B}^n\lambda\mathcal{A}^{\mathrm{T}}\mathcal{M}\mathrm{E}\left[\partial f(\boldsymbol{\phi}^{(i-n-1)})\right]$ can be bounded by the norms of $\mathcal{B}$ and $\partial f(\boldsymbol{\phi}^{(\cdot)})$. The norm of $\mathcal{B}$ is bounded under the discussion in Theorem 1 and that of $\partial f(\boldsymbol{\phi}^{(\cdot)})$ is also bounded from the assumption of the theorem. Considering a series composed of these bounds, it becomes absolutely convergent, so that the series of $\mathcal{B}^n\lambda\mathcal{A}^{\mathrm{T}}\mathcal{M}\mathrm{E}\left[\partial f(\boldsymbol{\phi}^{(i-n-1)})\right]$ in the second term of (2.2.41) is bounded by the absolutely convergent series. Therefore the second term is also absolutely convergent. □

The theoretical network MSD learning curve is expressed under Assumptions 1 and 2 as in D-LMS but has a more complicated form.

$$\begin{aligned}\mathrm{MSD}_i^{\mathrm{nw}} = \mathrm{MSD}_{i-1}^{\mathrm{nw}} &+ \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}}\boldsymbol{F}'^{i}\boldsymbol{q}' - \frac{1}{N}\boldsymbol{w}^{\mathrm{T}}\left[\mathrm{vec}^{-1}\left(\boldsymbol{F}'^{i}\left[\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}'\right]\boldsymbol{q}'\right)\right]\boldsymbol{w} \\ &+ g^{(i)}_{\boldsymbol{q}'/N} + \sum_{j=0}^{i-1}\left(g^{(j)}_{(\boldsymbol{F}')^{i-j}\boldsymbol{q}'/N} - g^{(j)}_{(\boldsymbol{F}')^{i-j-1}\boldsymbol{q}'/N}\right), \end{aligned} \tag{2.2.42}$$

where $g^{(j)}_{\boldsymbol{\sigma}} = \frac{\lambda}{N}\left(\lambda g^{(j)}_{1,\boldsymbol{\sigma}} - g^{(j)}_{2,\boldsymbol{\sigma}}\right)$, and where

$$g^{(j)}_{1,\boldsymbol{\sigma}} = \mathrm{E}\left[\partial f(\boldsymbol{\phi}^{(j-1)})^{\mathrm{H}}\mathcal{M}\mathcal{A}\left(\mathrm{vec}^{-1}(\boldsymbol{\sigma})\right)\mathcal{A}^{\mathrm{T}}\mathcal{M}\partial f(\boldsymbol{\phi}^{(j-1)})\right], \tag{2.2.43}$$

Algorithm 2 SD-LMS (Lorenzo and Sayed, 2013)

1: Initialize $\boldsymbol{\phi}_k^{(-1)} = 0,\ \forall k \in \mathcal{V}$
2: **for** each time $i \in \mathbb{N}$ and each node $k \in \mathcal{V}$ **do**
3: $\quad \boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)}(d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}}\boldsymbol{\phi}_k^{(i-1)}) - \mu_k \lambda \partial f(\boldsymbol{\phi}_k^{(i-1)})$ (adaptation step)
4: $\quad \boldsymbol{\phi}_k^{(i)} = \sum_{l \in \mathcal{N}_k} a_{lk} \boldsymbol{\psi}_l^{(i)}$ (combination step)
5: **end for**

$$g_{2,\boldsymbol{\sigma}}^{(j)} = -2\mathrm{E}\Big[\partial f(\boldsymbol{\phi}^{(j-1)})^{\mathrm{H}} \mathcal{M}\mathcal{A}\Big(\mathrm{vec}^{-1}(\boldsymbol{\sigma})\Big)\mathcal{B}\tilde{\boldsymbol{w}}^{(j-1)}\Big], \tag{2.2.44}$$

for any vector $\boldsymbol{\sigma}$. The steady-state network MSD and the steady-state MSD at each node k are shown in Lorenzo and Sayed (2013, Sect. 3) and given by

$$\mathrm{MSD}^{\mathrm{nw}} = \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{q}' + \frac{1}{N}\lambda g_{1,\Sigma,\infty}\left(\lambda - \frac{g_{2,\Sigma,\infty}}{g_{1,\Sigma,\infty}}\right), \tag{2.2.45}$$

$$\mathrm{MSD}_k = \boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{p}' + \lambda g_{1,\Sigma_k,\infty}\left(\lambda - \frac{g_{2,\Sigma_k,\infty}}{g_{1,\Sigma_k,\infty}}\right), \tag{2.2.46}$$

respectively, where

$$g_{1,\Sigma,\infty} = \lim_{i\to\infty} g_{1,(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{q}'/N}^{(i)}, \tag{2.2.47}$$

$$g_{1,\Sigma_k,\infty} = \lim_{i\to\infty} g_{1,(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}C_k}^{(i)}. \tag{2.2.48}$$

$$g_{2,\Sigma,\infty} = \lim_{i\to\infty} g_{2,(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{q}'/N}^{(i)}, \tag{2.2.49}$$

$$g_{2,\Sigma_k,\infty} = \lim_{i\to\infty} g_{2,(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}C_k}^{(i)}, \tag{2.2.50}$$

The derivation is done in the same manner as D-LMS. Compared with (2.2.32) and (2.2.33), the last terms in (2.2.45) and (2.2.46) result from the regularization term and complicate the analysis of the steady-state performance. If the last terms are negative, SD-LMS is superior to the original D-LMS. In Lorenzo and Sayed (2013, Sect. 3-C), this is likely to be true in sparse estimation under the condition that $g_{2,\Sigma_k,\infty} > 0$ under Assumption 2 and that the regularization parameter satisfies $0 < \lambda < \frac{g_{2,\Sigma_k,\infty}}{g_{1,\Sigma_k,\infty}}$.

2.2.3 Consensus Propagation

CP (Moallemi and Roy, 2006) is an iterative algorithm to achieve the average consensus by regarding an undirected network as a bidirectional network and by using the idea of message passing algorithm (Pearl, 1988). Assume that each node k has an initial state value $x_k \in \mathbb{C}$. The goal of CP is the same as that of average consensus protocol (Olfati-Saber et al., 2007), that each node obtains the average of the initial state values $\bar{x} = \frac{1}{N}\sum_{k=1}^{N} x_k$. The formulations of CP can be categorized

into two schemes: one is for a tree structured network and another is for the other structures that include some cycles. We name in the thesis the former exact CP and the latter loopy CP.

In the network with a tree structured network, exact CP can obtain the exact average. CP consists of two types of updates, namely, message update between neighboring nodes and state update at each node, to calculate the average using locally available information only. The updates of exact CP at the j-th iteration ($j \in \mathbb{N} \setminus \{0\}$) are given as follows:

$$K_{(u\to k)}^{[j]} = 1 + \sum_{m\in\mathcal{N}_u\setminus\{k,u\}} K_{(m\to u)}^{[j-1]}, \tag{2.2.51}$$

$$\theta_{(u\to k)}^{[j]} = \frac{x_u + \sum_{m\in\mathcal{N}_u\setminus\{k,u\}} K_{(m\to u)}^{[j-1]}\theta_{(m\to u)}^{[j-1]}}{1 + \sum_{m\in\mathcal{N}_u\setminus\{k,u\}} K_{(m\to u)}^{[j-1]}}, \tag{2.2.52}$$

where $K_{(u\to k)}^{[0]} = 0,\ \forall u, k \in \mathcal{V}$, and where $K_{(u\to k)}^{[j]}$ and $\theta_{(u\to k)}^{[j]}$ are the messages sent from node u to k. After iterating (2.2.51) and (2.2.52) between all neighboring nodes with the same number J as the diameter of the tree, the estimate of the average $\bar{x}$ at node k is given by

$$x_k^{[J]} = \frac{x_k + \sum_{u\in\mathcal{N}_k\setminus\{k\}} K_{(u\to k)}^{[J]}\theta_{(u\to k)}^{[J]}}{1 + \sum_{u\in\mathcal{N}_k\setminus\{k\}} K_{(u\to k)}^{[J]}}, \tag{2.2.53}$$

$x_k^{[J]}$ is exactly identical with $\bar{x}$ meaning that all nodes obtain $\bar{x}$. Figs. 2.3 and 2.4 show the updates at node k and its neighbors. Since the diameter is the minimum number of iterations required for the message propagation through the entire network, exact CP is a fast and efficient algorithm to achieve average consensus.

For networks involving some cycles, loopy CP is employed where the update of the message $K_{(u\to k)}^{[j]}$ is replaced with

$$K_{(u\to k)}^{[j]} = \frac{1 + \sum_{m\in\mathcal{N}_u\setminus\{k,u\}} K_{(m\to u)}^{[j-1]}}{1 + \frac{1}{\beta_k}(1 + \sum_{m\in\mathcal{N}_u\setminus\{k,u\}} K_{(m\to u)}^{[j-1]})}. \tag{2.2.54}$$

Here, β_k is a positive constant and it should be noted that β_k can be also made dependent on u as $\beta_{k,u}$ so that it depends not only on k but also on u. However, unlike the exact CP, what the sufficient number of iterations is has not been understood yet. By iterating (2.2.54) and (2.2.52) T times, the messages are shown to converge to unique constants as $T \to \infty$ and the resulting value of $x_k^{[T]}$ also becomes a constant, which is an estimate of the average. The convergence time for networks of any structure has not been fully understood.[1] It is known that β_k in (2.2.54)

[1]The convergence time for regular graphs has been analyzed in Moallemi and Roy (2006) but we now consider broader classes of networks.

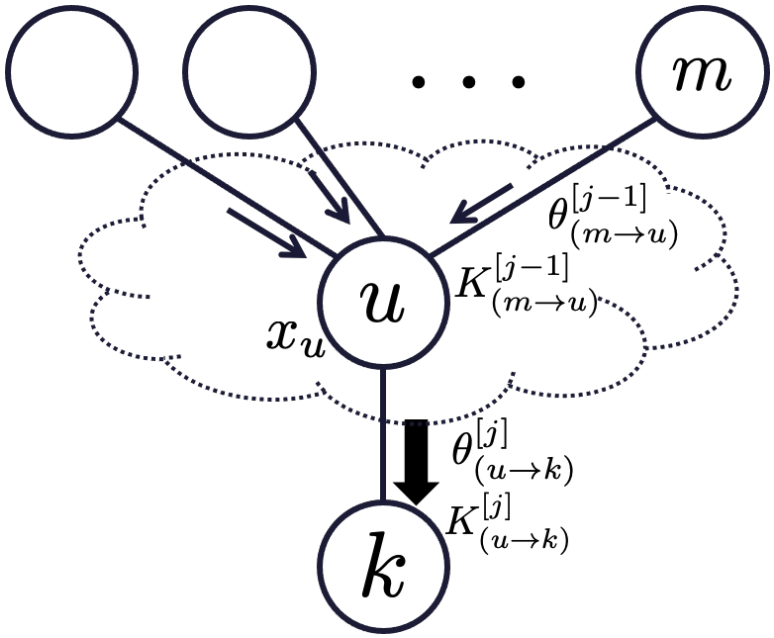


Figure 2.3: Update of message from node u to node k in the j-th iteration. The message is generated from the messages received in the $(j-1)$-th iteration from its neighbors except for node k, and its initial value x_u.

plays an important role for the convergence, but, to the best of our knowledge, the optimal value of β_k in terms of the convergence time has not been derived.

The message updates of CP can be represented by the non-backtracking operator (Coja-Oghlan et al., 2009; Decelle et al., 2011; Krzakala et al., 2013), which is a matrix defined in terms of connections between edges. We expect that the representation may enable the convergence analysis of loopy CP by using a spectrum of the operator (Bordenave et al., 2015).

2.3 Overview of Proposed Methods

In this part, we aim to reduce performance degradation of D-LMS and SD-LMS caused by approximate divisions of the cost functions and improve the part related to the compensation of the local cost functions in the algorithms, i.e., the combination steps which are closely related to the conventional average consensus protocol. Specifically, we propose three novel distributed LMS algorithms that employ CP as the alternative to the average consensus protocol to achieve better convergence performance.

The first algorithm is based on exact CP which converges to the exact average on the network with a tree structure. It requires extraction of a spanning tree

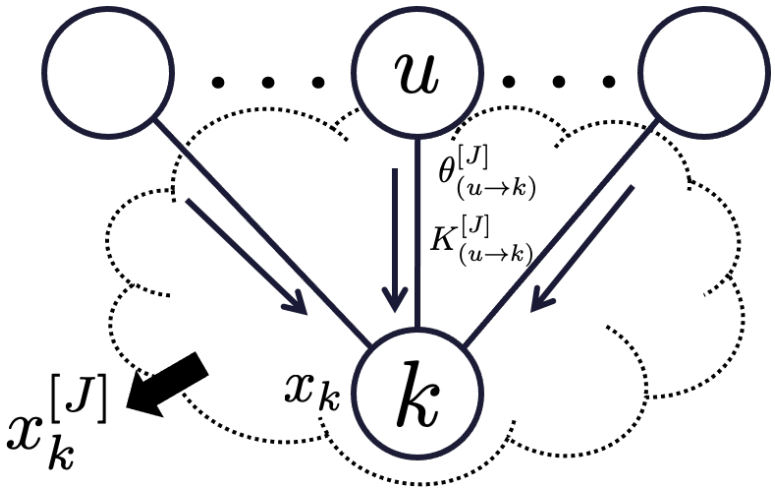


Figure 2.4: State update at node k. Node k calculates the estimate $x_k^{[J]}$ using the messages received in the J-th iteration from its all neighbors.

beforehand because the network of interest is not necessarily a tree. We then construct the algorithm by substituting the combination step of D-LMS with exact CP. Unlike the original D-LMS that iterates a pair of one adaptation step and one combination step, the proposed algorithm iterates a set of one adaptation step and J updates of exact CP (i.e., iterations of exact CP until achieving average consensus). Although in small networks the proposed algorithm may take more iterations than D-LMS, it expects faster convergence, especially in large networks, as long as the diameter of the tree is not so large. The algorithm is named CP-LMS.

The second algorithm is based on loopy CP. It does not require extraction of a spanning tree and can perform on networks of general structure. There can be two approaches to apply loopy CP to the distributed LMS. One approach is to replace the exact CP of CP-LMS with the loopy CP. However, constants involved in the approach and the required iteration number cannot be determined because appropriate values for them in loopy CP have not been known. Another approach considers a special case that employs only one iteration of loopy CP. This approach actually results in the proposal of a novel combination rule of D-LMS in this thesis and enables us to analyze convergence performance. The combination rule is named CP rule. Moreover, it enables optimization of the constant involved in loopy CP and extension to an adaptive version of the combination rule. The adaptive extension is named adaptive CP rule.

The last algorithm is an extension to SD-LMS. We apply the latter approach mentioned above to SD-LMS. In this case, there are two methods for the optimization of the constant involved in loopy CP. One method introduces a rough approximation, which results in the same combination rule as the above, adaptive CP rule.

Another method relaxes the approximation and yields a novel combination rule of SD-LMS. The rule is named adaptive CP with Optimization (CPO) rule.

3: D-LMS based on CP

3.1 D-LMS based on Exact CP

3.1.1 Proposed Algorithm

In the first approach CP-LMS, we firstly extract a spanning tree from the original network which possibly has some cycles using some centralized or distributed spanning tree protocol such as Herzen et al. (2011) and Bui et al. (2004). For example, we can use the distributed algorithm in Bui et al. (2004) to find a minimum diameter spanning tree of any graph $G = \{V, E\}$ with $O(|V|)$ time complexity and $O(|V||E|)$ message complexity. We then apply exact CP in the combination step of D-LMS on the extracted tree network. Every time each node obtains a linear measurement, the estimate $\boldsymbol{\psi}_k^{(i)}$ at node k is updated by the adaptation step (2.2.7) as in the conventional method. The subsequent estimate $\boldsymbol{\phi}_k^{(i)}$ is obtained as the consensus value achieved by exact CP on each element of the vector $\boldsymbol{\psi}_k^{(i)}$, i.e., the average of all nodes' estimates at time i. After each node performs the same number of exaxt CP updates as the diameter J of the spanning tree, the average is substituted to the new estimate $\boldsymbol{\phi}_k^{(i)}$, and then the algorithm proceeds to the next step.

The proposed updating rules are summarized in *Algorithm* 3. $K_{p,(k\to l)}^{(i)[j]}$ and $\theta_{p,(k\to l)}^{(i)[j]}$ are the p-th elements ($p = 1, \ldots, M$) of the messages transmitted from node k to node l at the j-th iteration of exact CP at time i. $(\boldsymbol{\phi}_k^{(i)})_p$ becomes the p-th element of the average of all nodes' estimates at time i.

Algorithm 3 CP-LMS

Extract a spanning tree
Initialize $\boldsymbol{\phi}_k^{(-1)} = 0,\ \forall k \in \mathcal{V}$
for each time $i \in \mathbb{N}$, each node $k \in \mathcal{V}$, and each element $p = 1, \ldots, M$ **do**
    $\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)} (d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)})$
    $\theta_{p,(k\to l)}^{(i)[0]} = 0,\ K_{p,(k\to l)}^{(i)[0]} = 0$
    for $j = 1$ to J **do**
        $K_{p,(k\to l)}^{(i)[j]} = 1 + \sum_{u\in\mathcal{N}_k\setminus\{l,k\}} K_{p,(u\to k)}^{(i)[j-1]}$
        $\theta_{p,(k\to l)}^{(i)[j]} = \frac{(\boldsymbol{\psi}_k^{(i)})_p + \sum_{u\in\mathcal{N}_k\setminus\{l,k\}} K_{p,(u\to k)}^{(i)[j-1]} \theta_{p,(u\to k)}^{(i)[j-1]}}{K_{p,(k\to l)}^{(i)[j]}}$
    end for
    $(\boldsymbol{\phi}_k^{(i)})_p = \frac{(\boldsymbol{\psi}_k^{(i)})_p + \sum_{u\in\mathcal{N}_k\setminus\{k\}} K_{p,(u\to k)}^{(i)[J]} \theta_{p,(u\to k)}^{(i)[J]}}{1 + \sum_{u\in\mathcal{N}_k\setminus\{k\}} K_{p,(u\to k)}^{(i)[J]}}$
end for

3.1.2 Convergence Analysis

In this section, we analyze the convergence in the mean sense and the mean-square behavior of the proposed CP-LMS. In order to make the analysis tractable, we appropriately use the assumptions in Sect. 2.2.2.

It should be noted here that the proposed CP-LMS realizes the centralized fusion-based solution known as block LMS (Cattivelli and Sayed, 2011; Zhao and Sayed, 2012), where a fusion node collects measurements and measurement vectors of all other nodes, in a fully distributed manner if we introduce further assumption:

Assumption 3. *All nodes have the same step-size parameter, i.e., $\mu_k = \mu$ for all k,*

because all nodes can collect all nodes' estimates at each iteration after J updates of exact CP. To be more specific, under Assumption 3, the update equations of CP-LMS can be summarized as

$$\begin{cases} \boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu \boldsymbol{u}_k^{(i)} (d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)}), \\ \boldsymbol{\phi}_k^{(i)} = \frac{1}{N} \sum_{l=1}^{N} \boldsymbol{\psi}_l^{(i)}. \end{cases}$$

Since $\boldsymbol{\phi}_k^{(i)}$ does not depend on k, we replace $\boldsymbol{\phi}_k^{(i)}$ with a new notation $\boldsymbol{\psi}^{(i)}$, and then the whole update is simplified as

$$\begin{aligned} \boldsymbol{\psi}^{(i)} &= \frac{1}{N} \sum_{l=1}^{N} [\boldsymbol{\psi}^{(i-1)} + \mu \boldsymbol{u}_l^{(i)} (d_l^{(i)} - \boldsymbol{u}_l^{(i)\mathrm{H}} \boldsymbol{\psi}^{(i-1)})] \\ &= \boldsymbol{\psi}^{(i-1)} + \frac{\mu}{N} \sum_{l=1}^{N} \boldsymbol{u}_l^{(i)} (d_l^{(i)} - \boldsymbol{u}_l^{(i)\mathrm{H}} \boldsymbol{\psi}^{(i-1)}), \end{aligned} \tag{3.1.1}$$

which corresponds to the centralized block LMS algorithm. This means that the conventional performance analysis for the block LMS (Cattivelli and Sayed, 2011, Sect. 4-B) is directly applicable to the proposed CP-LMS. For example, the following theorem, which was originally proved for the block LMS, is valid for the CP-LMS as well.

Theorem 3. *Consider the measurement model given by* (2.2.2)*. The convergence of CP-LMS in the mean sense is guaranteed under Assumptions 1 and 3 if the step-size satisfies*

$$0 < \mu < \frac{2}{\lambda_{\max}(\boldsymbol{R})},$$

where $\boldsymbol{R} = \sum_{k=1}^{N} \boldsymbol{R}_{u_k}$.

The transient behavior of CP-LMS can be also represented by the same expression as the conventional block LMS (Cattivelli and Sayed, 2011, Sect. 4-C) under Assumptions 1 and 3. Now let $\boldsymbol{r}_{u_k} = \mathrm{vec}(\boldsymbol{R}_{u_k})$ and

$$\boldsymbol{F} = \boldsymbol{I}_{M^2} - \mu'(\boldsymbol{I}_M \otimes \boldsymbol{R}) - \mu'(\boldsymbol{R}^{\mathrm{T}} \otimes \boldsymbol{I}_M) + \mu'^2(\boldsymbol{R}^{\mathrm{T}} \otimes \boldsymbol{R}) + \mu'^2 \sum_{k=1}^{N} \boldsymbol{r}_{u_k} \boldsymbol{r}_{u_k}^{\mathrm{H}},$$

where $\mu' = \mu/N$. The theoretical network MSD learning curve of the CP-LMS algorithm is given by

$$\mathrm{MSD}_i^{\mathrm{nw}} = \mathrm{MSD}_{i-1}^{\mathrm{nw}} + \mu'^2 \sum_{k=1}^{N} \sigma_k^2 \boldsymbol{r}_{u_k}^{\mathrm{H}} \boldsymbol{F}^i \boldsymbol{q} - \boldsymbol{w}^{\mathrm{oH}} \left[\mathrm{vec}^{-1}\left(\boldsymbol{F}^i \left[\boldsymbol{I}_{M^2} - \boldsymbol{F}\right] \boldsymbol{q}\right)\right] \boldsymbol{w}^{\mathrm{o}}, \tag{3.1.2}$$

where $\boldsymbol{q} = \mathrm{vec}(\boldsymbol{I}_M)$. It is clear that $\mathrm{MSD}_k = \mathrm{MSD}^{\mathrm{nw}}$ holds in the case of CP-LMS because all nodes obtain the same estimate $\boldsymbol{\psi}^{(i)}$ at each step i. Thus, the steady-state MSD is given by

$$\mathrm{MSD}_k = \mathrm{MSD}^{\mathrm{nw}} = \mu'^2 \sum_{l=1}^{N} \sigma_l^2 \boldsymbol{r}_{u_l}^{\mathrm{H}} (\boldsymbol{I}_{M^2} - \boldsymbol{F})^{-1} \boldsymbol{q}. \tag{3.1.3}$$

Note that it is one of the important merits of CP-LMS that the transient behavior can be described by the same expression as that of the centralized solution. Unlike the conventional D-LMS, the calculation of the theoretical transient behavior of the centralized solution requires much smaller computational complexity than that of the conventional D-LMS.

3.2 D-LMS based on Loopy CP

3.2.1 Proposed Algorithm – General Case

The extraction of a spanning tree in the algorithm in Sect. 3.1.1 can ensure the perfect consensus at each iteration of LMS. However, the use of a spanning tree may also degrade the convergence performance because some communication links available in the original network are not utilized at all in the algorithm working on the extracted spanning tree. In this section, we consider applying loopy CP to D-LMS without extracting any spanning tree.

Here, we describe a modified CP-LMS that can be directly applicable to networks with some cycles. Every time each node obtains a linear measurement, the estimate $\boldsymbol{\psi}_k^{(i)}$ at node k is updated by LMS (2.2.7) as in the conventional method. Since it is difficult to know the required number of updates for the case of loopy CP, we set a fixed number of updates. After each node performs the updates of loopy

Algorithm 4 LCP-LMS

Set $T > 0, \beta_k > 0, \forall k \in \mathcal{V}$
Initialize $\boldsymbol{\phi}_k^{(-1)} = 0,\ \beta_k > 0,\ \forall k \in \mathcal{V},\ T > 0$
for each time $i \in \mathbb{N}$, each node $k \in \mathcal{V}$, and each element $p = 1, \ldots, M$ **do**
  $\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)} (d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)})$
  $\theta_{p,(k\to l)}^{(i)[0]} = 0,\ K_{p,(k\to l)}^{(i)[0]} = 0$
  for $t = 1$ to T **do**
    $K_{p,(k\to l)}^{(i)[t]} = \frac{1+\sum_{u\in\mathcal{N}_k\setminus\{l,k\}} K_{p,(u\to k)}^{(i)[t-1]}}{1+\frac{1}{\beta_l}(1+\sum_{u\in\mathcal{N}_k\setminus\{l,k\}} K_{p,(u\to k)}^{(i)[t-1]})}$
    $\theta_{p,(k\to l)}^{(i)[t]} = \frac{(\boldsymbol{\psi}_k^{(i)})_p + \sum_{u\in\mathcal{N}_k\setminus\{l,k\}} K_{p,(u\to k)}^{(i)[t-1]} \theta_{p,(u\to k)}^{(i)[t-1]}}{1+\sum_{u\in\mathcal{N}_k\setminus\{l,k\}} K_{p,(u\to k)}^{(i)[t-1]}}$
  end for
  $(\boldsymbol{\phi}_k^{(i)})_p = \frac{(\boldsymbol{\psi}_k^{(i)})_p + \sum_{u\in\mathcal{N}_k\setminus\{k\}} K_{p,(u\to k)}^{(i)[T]} \theta_{p,(u\to k)}^{(i)[T]}}{1+\sum_{u\in\mathcal{N}_k\setminus\{k\}} K_{p,(u\to k)}^{(i)[T]}}$
end for

CP T times, the calculated estimate is assigned to the p-th element of the new estimate $\boldsymbol{\phi}_k^{(i)}$, and then the algorithm proceeds to the next step. Note that $(\boldsymbol{\phi}_k^{(i)})_p$ will not be exactly the same as the average of the p-th elements of $\boldsymbol{\psi}_k^{(i)}$ in general for any T.

The proposed updating rules are summarized in *Algorithm* 4. We call this algorithm *Loopy CP-LMS (LCP-LMS)*. As mentioned in Sect. 2.2.3, the sufficient number of iterations for convergence has not been understood and the optimal value of constants β_k that minimizing convergence time has not been derived, so that the behavior of LCP-LMS cannot be also analyzed. It is necessary to adjust values of T and β_k ($\forall k \in \mathcal{V}$) depending on a structure of a network or desired performance.

3.2.2 Proposed Algorithm – Special Case

Derivation

LCP-LMS has difficulties in analyzing the performance and determining the constants β_k because the behavior of loopy CP has not been fully understood. However, we can analytically evaluate the convergence performance if we consider a special case that the number T of iterations of loopy CP updates is limited to one. This idea also brings about an advantage to track fluctuations of an unknown vector more quickly than CP-LMS or the general case of LCP-LMS which repeat communications while keeping information of measurements fixed. We describe the algorithm for the special case that employs only one update of loopy CP and then the method turns out to be D-LMS using a novel combination rule. We show how to determine the constants β_k in terms of the performance of D-LMS. This idea that T is limited to one does not enjoy benefits of CP but we focus on the analytical

aspect and further extend the proposed combination rule to an adaptive one.

From the first updates of the messages in Algorithm 4, we have

$$K_{p,(u\to k)}^{(i)[1]} = \frac{\beta_k}{1+\beta_k}, \tag{3.2.1}$$

$$\theta_{p,(u\to k)}^{(i)[1]} = x_{p,u}^{(i)[0]} = (\boldsymbol{\psi}_u^{(i)})_p. \tag{3.2.2}$$

We omit the subscript p in $K_{p,(u\to k)}^{(i)[1]}$ as $K_{(u\to k)}^{(i)[1]}$ because $K_{p,(u\to k)}^{(i)[1]}$ does not depend on p, and thus the element-wise update rule in Algorithm 4 can be rewritten as a vector-wise update as

$$K_{(u\to k)}^{(i)[1]} = \frac{\beta_k}{1+\beta_k}, \tag{3.2.3}$$

$$\theta_{(u\to k)}^{(i)[1]} = \boldsymbol{\psi}_u^{(i)}. \tag{3.2.4}$$

Moreover, deriving $x_k^{(i)[1]}$, namely, $\boldsymbol{\phi}_k^{(i)}$ by using (3.2.3) and (3.2.4) gives

$$\boldsymbol{\phi}_k^{(i)} = \frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}\boldsymbol{\psi}_k^{(i)} + \frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}\sum_{u\in\mathcal{N}_k\setminus\{k\}}\boldsymbol{\psi}_u^{(i)} = \sum_{l\in\mathcal{N}_k} a_{lk}\boldsymbol{\psi}_l^{(i)}. \tag{3.2.5}$$

This can be regarded as a novel combination rule summarized as

$$a_{lk} = \begin{cases} \frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ \frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}, & \text{if } k = l, \\ 0, & \text{otherwise}, \end{cases} \tag{3.2.6}$$

which satisfies $\mathbf{1}^{\mathrm{T}}\boldsymbol{A} = \mathbf{1}^{\mathrm{T}}$.

Optimal Combination Weights

As mentioned in Sect. 2.2.3, how to select β_k has been an open issue (Moallemi and Roy, 2006). In this section, we choose β_k in terms of the steady-state network MSD of D-LMS (2.2.28). The steady-state network MSD itself includes an infinite sum and has a complicated form, but an upper bound of it can be expressed simply by a product of some parameters. We thus optimize β_k by minimizing the upper bound of the steady-state network MSD.

Under Assumptions 1 and 2 in Sect. 2.2.2, the steady-state network MSD can be rewritten by expanding $\left(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}'\right)^{-1}$ as

$$\mathrm{MSD}^{\mathrm{nw}} = \frac{1}{N}\sum_{j=0}^{\infty}\mathrm{Tr}\left[\mathcal{B}^j\mathcal{A}^{\mathrm{T}}\mathcal{M}\mathcal{G}\mathcal{M}\mathcal{A}\left(\mathcal{B}^{\mathrm{H}}\right)^j\right], \tag{3.2.7}$$

where $\mathcal{B} = \mathcal{A}^{\mathrm{T}}(\boldsymbol{I} - \mathcal{M}\mathcal{D})$. The right-hand side can be shown to be bounded (Sayed, 2014, Sect. 8.2) such that

$$\mathrm{MSD}^{\mathrm{nw}} \le c\mathrm{Tr}\left[\mathcal{A}^{\mathrm{T}}\mathcal{M}\mathcal{G}\mathcal{M}\mathcal{A}\right], \tag{3.2.8}$$

where c is a positive constant. Recalling the definitions $\mathcal{A} = \boldsymbol{A} \otimes \boldsymbol{I}_M$, $\mathcal{M} = (\mathrm{diag}[\mu_1, \ldots, \mu_N]) \otimes \boldsymbol{I}_M$, and $\mathcal{G} = \mathrm{diag}[\sigma_1^2 \boldsymbol{R}_{u_1}, \ldots, \sigma_N^2 \boldsymbol{R}_{u_N}]$, the trace can be calculated as follows (Tu and Sayed, 2011; Zhao et al., 2012; Sayed, 2014):

$$\mathrm{Tr}\left[\mathcal{A}^{\mathrm{T}} \mathcal{M} \mathcal{G} \mathcal{M} \mathcal{A}\right] = \sum_{k=1}^{N} \sum_{l=1}^{N} \gamma_l^2 a_{lk}^2, \tag{3.2.9}$$

where $\gamma_l^2 = \mu_l^2 \sigma_l^2 \mathrm{Tr}[\boldsymbol{R}_{u_l}]$. For the proposed combination rule (3.2.6), the optimization problem to determine $\{\beta_k\}$ is written as

$$\begin{aligned} \{\beta_k^{\mathrm{opt}}\}_{k=1}^N = \quad & \arg\min_{\{\beta_k\}_{k=1}^N} \sum_{k=1}^{N} \sum_{l=1}^{N} \gamma_l^2 a_{lk}^2, \\ & \text{s.t. (3.2.6)}, \ \beta_k > 0 \ (k = 1, \ldots, N), \end{aligned} \tag{3.2.10}$$

where the constraint $\beta_k > 0$ comes from the definition in loopy CP. By substituting (3.2.6) into the problem, this optimization problem can be separated into N independent subproblems as

$$\beta_k^{\mathrm{opt}} = \arg\min_{\beta_k} \ h(\beta_k), \tag{3.2.11}$$

where

$$h(\beta_k) = \gamma_k^2 \left(\frac{1 + \beta_k}{1 + |\mathcal{N}_k| \beta_k} \right)^2 + \sum_{l \in \mathcal{N}_k \setminus \{k\}} \gamma_l^2 \left(\frac{\beta_k}{1 + |\mathcal{N}_k| \beta_k} \right)^2.$$

The above function $h(\beta_k)$ is differentiable and we have

$$\frac{\partial h}{\partial \beta_k} = \frac{2}{(1 + |\mathcal{N}_k| \beta_k)^3} \left(\left(\sum_{l \in \mathcal{N}_k} \gamma_l^2 - |\mathcal{N}_k| \gamma_k^2 \right) \beta_k - (|\mathcal{N}_k| - 1) \gamma_k^2 \right). \tag{3.2.12}$$

In line with this equation, the shape of the cost function $h(\beta_k)$ largely depends on $\Gamma_k = \sum_{l \in \mathcal{N}_k} \gamma_l^2 - |\mathcal{N}_k| \gamma_k^2 \neq 0$ in the first term of the right-hand side of (3.2.12). When $\Gamma_k > 0$, the function $h(\beta_k)$ has a global minimum in $\beta_k > 0$ and the optimal value is obtained as

$$\beta_k^{\min} = \frac{(|\mathcal{N}_k| - 1) \gamma_k^2}{\Gamma_k}. \tag{3.2.13}$$

On the other hand, it becomes a monotonically decreasing function of $\beta_k > 0$ when $\Gamma_k < 0$. Thus, in summary, the optimum β_k is given by

$$\beta_k^{\mathrm{opt}} = \begin{cases} \beta_k^{\min}, & \text{if } \Gamma_k > 0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.2.14}$$

and the corresponding combination weights are given by

$$a_{lk}^{\mathrm{cp}} = \begin{cases} \frac{\beta_k^{\mathrm{opt}}}{1 + |\mathcal{N}_k| \beta_k^{\mathrm{opt}}}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ \frac{1 + \beta_k^{\mathrm{opt}}}{1 + |\mathcal{N}_k| \beta_k^{\mathrm{opt}}}, & \text{if } k = l, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2.15}$$

We name this combination rule as *CP rule*. Note that if $\Gamma_k < 0$, i.e., $\beta_k^{\mathrm{opt}} = +\infty$, the weight a_{lk}^{cp} coincides with the uniform rule (2.2.12).

In the existing works (Tu and Sayed, 2011; Zhao et al., 2012; Sayed, 2014), $\{a_{lk}\}$ in (2.2.10) is directly derived by minimizing the upper bound (3.2.9) of $\mathrm{MSD}^{\mathrm{nw}}$ with respect to $\{a_{lk}\}$ as

$$\begin{aligned} \{a_{lk}^{\mathrm{opt}}\}_{k=1}^N = \quad & \arg\min_{\{a_{lk}\}_{k=1}^N} \sum_{l=1}^N \gamma_l^2 a_{lk}^2, \\ \text{s.t.} \quad & \sum_{l=1}^N a_{lk} = 1,\ a_{lk} = 0 \text{ if } l \notin \mathcal{N}_k. \end{aligned} \tag{3.2.16}$$

The solution results in the relative-variance rule (2.2.15) (Tu and Sayed, 2011) given by

$$a_{lk}^{\mathrm{rv}} = \begin{cases} \frac{[\gamma_l^2]^{-1}}{\sum_{m\in\mathcal{N}_k}[\gamma_m^2]^{-1}}, & \text{if } l \in \mathcal{N}_k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.17}$$

Adaptive Combination Weights

The CP rule (3.2.15) and the relative-variance rule (3.2.17) require the knowledge of γ_l^2, which depends on locally unavailable network statistics such as the correlation matrices $\boldsymbol{R}_{u_l}$ of the measurement vectors and the measurement noise variance σ_l^2. Thus, we employ adaptive estimations of γ_l^2 for D-LMS proposed in Tu and Sayed (2011); Zhao et al. (2012); Sayed (2014). The estimate $\boldsymbol{\phi}_k^{(i)}$ should approach the unknown vector $\boldsymbol{w}^{\mathrm{o}}$ as the algorithm iterates (2.2.36) and (2.2.38), and reach the steady-state under Assumption 2. By using (2.2.36) and (2.2.2), the estimate $\boldsymbol{\psi}_l^{(i)}$ can be rewritten as

$$\boldsymbol{\psi}_l^{(i)} \approx \boldsymbol{w}^{\mathrm{o}} + \mu_l \boldsymbol{\mathcal{U}}_l^{(i)} \boldsymbol{\mathcal{V}}_l^{(i)}.$$

Taking the expectation leads to

$$\mathrm{E}\left[\left\|\boldsymbol{\psi}_l^{(i)} - \boldsymbol{w}^{\mathrm{o}}\right\|^2\right] \approx \mu_l^2 \sigma_l^2 \mathrm{Tr}(\boldsymbol{R}_{u_l}).$$

The right-hand side is γ_l^2 itself. To estimate it adaptively, the instantaneous values are substituted into the expectation. Let $(\gamma_{lk}^2)^{(i)}$ be an estimate of γ_l^2 at node k and time i. Such an estimate $(\gamma_{lk}^2)^{(i)}$ is obtained by employing the following update:

$$(\gamma_{lk}^2)^{(i)} = (1-\nu_k)(\gamma_{lk}^2)^{(i-1)} + \nu_k \left\|\boldsymbol{\psi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)}\right\|^2, \tag{3.2.18}$$

where ν_k ($0 < \nu_k < 1$) is the forgetting factor. By using this estimate, the adaptive version of (3.2.17) named the adaptive relative-variance rule (2.2.16) is proposed

Algorithm 5 D-LMS with adaptive relative-variance rule (Zhao et al., 2012)

1: Initialize $\boldsymbol{\phi}_k^{(-1)} = 0,\ \forall k \in \mathcal{V}$
2: **for** each time $i \in \mathbb{N}$, each node $k \in \mathcal{V}$, and each neighbor $l \in \mathcal{N}_k$ **do**
3: $\quad \boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)}(d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}}\boldsymbol{\phi}_k^{(i-1)})$
4: $\quad (\gamma_{lk}^2)^{(i)} = (1-\nu_k)(\gamma_{lk}^2)^{(i-1)} + \nu_k\|\boldsymbol{\psi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)}\|^2$
5: $\quad a_{lk}^{(i)} = \frac{[(\gamma_{lk}^2)^{(i)}]^{-1}}{\sum_{m\in\mathcal{N}_k}[(\gamma_{mk}^2)^{(i)}]^{-1}}$
6: $\quad \boldsymbol{\phi}_k^{(i)} = \sum_{l\in\mathcal{N}_k} a_{lk}^{(i)}\boldsymbol{\psi}_l^{(i)}$
7: **end for**

in Zhao et al. (2012) as

$$a_{lk}^{\mathrm{rv}(i)} = \begin{cases} \frac{[(\gamma_{lk}^2)^{(i)}]^{-1}}{\sum_{m\in\mathcal{N}_k}[(\gamma_{mk}^2)^{(i)}]^{-1}}, & \text{if } l \in \mathcal{N}_k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.19}$$

In the same manner, the adaptive version of the CP rule is given by

$$\beta_k^{(i)} = \begin{cases} \frac{(|\mathcal{N}_k|-1)(\gamma_{kk}^2)^{(i)}}{\Gamma_k^{(i)}}, & \text{if } \Gamma_k^{(i)} > 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.2.20}$$

$$a_{lk}^{\mathrm{cp}(i)} = \begin{cases} \frac{\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ \frac{1+\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}, & \text{if } k = l, \\ 0, & \text{otherwise,} \end{cases} \tag{3.2.21}$$

where $\Gamma_k^{(i)} = \sum_{l\in\mathcal{N}_k}(\gamma_{lk}^2)^{(i)} - |\mathcal{N}_k|(\gamma_{kk}^2)^{(i)}$.

The algorithms of D-LMS using the conventional adaptive combination rule in (3.2.19) and the proposed rule in (3.2.21) are summarized in *Algorithm* 5 and *Algorithm* 6, respectively. The computational complexity of Algorithm 5 and Algorithm 6 are almost the same because the complexity of the proposed rule (3.2.21) becomes comparable to that of the conventional rule (3.2.19) by substituting (3.2.20) into (3.2.21).

Convergence Analysis

In this section, we discuss the convergence of D-LMS using static (i.e., non-adaptive) CP rule (3.2.15) in the mean sense and the mean-square behavior. Note that the transient behavior for the adaptive combination rules (not only adaptive CP rule but also adaptive relative-variance rule) remains an open issue. So, the theoretical network MSD learning curve below will not be valid for the adaptive CP rule, while theoretical results at the steady-state are applicable to the adaptive CP rule as well. The conventional performance analysis framework for D-LMS in

Algorithm 6 D-LMS with proposed adaptive CP rule

Initialize $\boldsymbol{\phi}_k^{(-1)} = 0,\ \forall k \in \mathcal{V}$
for each time $i \in \mathbb{N}$, each node $k \in \mathcal{V}$, and each neighbor $l \in \mathcal{N}_k$ **do**
  $\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)} (d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}} \boldsymbol{\phi}_k^{(i-1)})$
  $(\gamma_{lk}^2)^{(i)} = (1-\nu_k)(\gamma_{lk}^2)^{(i-1)} + \nu_k \|\boldsymbol{\psi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)}\|^2$
  if $\sum_{l\in\mathcal{N}_k} (\gamma_{lk}^2)^{(i)} - (\gamma_{kk}^2)^{(i)} |\mathcal{N}_k| > 0$ **then**
    $\beta_k^{(i)} = \frac{(|\mathcal{N}_k|-1)(\gamma_{kk}^2)^{(i)}}{\sum_{l\in\mathcal{N}_k}(\gamma_{lk}^2)^{(i)} - (\gamma_{kk}^2)^{(i)}|\mathcal{N}_k|}$
    $a_{lk}^{(i)} = \frac{\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}\ (l \in \mathcal{N}_k \setminus \{k\}),\quad \frac{1+\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}\ (l = k)$
  else
    $a_{lk}^{(i)} = \frac{1}{|\mathcal{N}_k|}$
  end if
  $\boldsymbol{\phi}_k^{(i)} = \sum_{l\in\mathcal{N}_k} a_{lk}^{(i)} \boldsymbol{\psi}_l^{(i)}$
end for

Sect. 2.2.2 is applicable to the proposed D-LMS using CP rule because CP rule can be regarded as one of the combination rules.

Corollary 1. *Consider the measurement model given by* (2.2.2) *and the combination weight given by CP rule* (3.2.15)*. The convergence of D-LMS using the proposed CP rule in the mean sense is guaranteed under Assumption 1 if the step-sizes satisfy*

$$0 < \mu_k < \frac{2}{\lambda_{\max}(\boldsymbol{R}_{u_k})} \quad (k \in \mathcal{V}).$$

The theoretical network MSD learning curve of D-LMS using CP rule (3.2.15) can be represented by the same expression as (2.2.31) given by

$$\mathrm{MSD}_i^{\mathrm{nw}} = \mathrm{MSD}_{i-1}^{\mathrm{nw}} + \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}} \boldsymbol{F}'^{i} \boldsymbol{q}' - \frac{1}{N}\boldsymbol{w}^{\mathrm{T}} \left[\mathrm{vec}^{-1}\left(\boldsymbol{F}'^{i}\left[\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}'\right]\boldsymbol{q}'\right)\right]\boldsymbol{w}. \tag{3.2.22}$$

The steady-state network MSD and the steady-state MSD at each node k are also represented by the same expressions as (2.2.32) and (2.2.33) given by

$$\mathrm{MSD}^{\mathrm{nw}} = \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{q}', \tag{3.2.23}$$

$$\mathrm{MSD}_k = \boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{p}', \tag{3.2.24}$$

respectively.

3.3 Computational Complexity and Communication Cost

We compare the computational complexity and the communication cost of the proposed schemes with those of the conventional methods in this section. Table 3.1

shows the complexity and the cost of the proposed algorithms and other algorithms in each combination step at a node. The computational complexity is defined as the number of scalar multiplication (division) or addition (subtraction) par one-way synchronous communications to node k from its neighboring nodes (for example for CP-LMS, not added up J times). The communication cost is defined as the number of scalar values that node k has to receive par one synchronous communication (for CP-LMS, not added up J times). Note that, since the adaptation step is common for all algorithms including the proposed schemes, the complexities only related to the combination step are evaluated. We can find that the proposed CP-LMS and LCP-LMS require higher computational complexity and communication cost than other algorithms because they contain exchanges of two types of messages between nodes. However, for all the algorithms compared here the communication cost is of the same order. As the communication time typically dominates the execution time in applications such as wireless sensor networks, one may conclude that the required time is comparable.

Here, we also mention communication cost of centralized block LMS. In this case, a fusion node collects measurements and measurement vectors of all other nodes, so that it receives $M + 1$ scalar values from $N - 1$ nodes, namely, the communication cost is $(M + 1)(N - 1)$. This is higher than those of the distributed methods discussed here in terms of cost par a processing node and the difference will be remarkable, especially in large-scale networks.

Table 3.1: Computational complexity and communication cost for the proposed algorithms and other conventional algorithms par one synchronous communication at node k.

Algorithm	$\times,\div$	$+,-$	Communication cost
D-LMS w/ static rules	$M\lvert\mathcal{N}_k\rvert$	$M(\lvert\mathcal{N}_k\rvert - 1)$	$M(\lvert\mathcal{N}_k\rvert - 1)$
AMC diffusion	$M\lvert\mathcal{N}_k\rvert$	$M(\lvert\mathcal{N}_k\rvert - 1)$	$M(\lvert\mathcal{N}_k\rvert - 1)$
CP-LMS (proposed)	$M((\lvert\mathcal{N}_k\rvert - 1)^2$ $+\lvert\mathcal{N}_k\rvert/J)$	$2M(\lvert\mathcal{N}_k\rvert - 1)$ $\cdot(\lvert\mathcal{N}_k\rvert - 2 + 1/J)$	$2M(\lvert\mathcal{N}_k\rvert - 1)$
LCP-LMS (proposed)	$M(\lvert\mathcal{N}_k\rvert^2 - 1$ $+\lvert\mathcal{N}_k\rvert/T)$	$M(\lvert\mathcal{N}_k\rvert - 1)$ $\cdot(3\lvert\mathcal{N}_k\rvert - 5 + 2/T)$	$2M(\lvert\mathcal{N}_k\rvert - 1)$
D-LMS w/ adaptive relative-variance rule	$(2M + 5)\lvert\mathcal{N}_k\rvert$	$(3M+2)\lvert\mathcal{N}_k\rvert - M - 1$	$M(\lvert\mathcal{N}_k\rvert - 1)$
D-LMS w/ adaptive CP rule (proposed)	$(2M + 4)\lvert\mathcal{N}_k\rvert + 3$	$(3M+3)\lvert\mathcal{N}_k\rvert - M + 2$	$M(\lvert\mathcal{N}_k\rvert - 1)$

3.4 Simulation Results

3.4.1 Settings

In this section, we compare the learning curves of instantaneous network MSD, $\mathrm{MSD}^{\mathrm{nw,in}}$, of the proposed schemes with the theoretical results and those of the conventional schemes via computer simulations. All the simulation results were obtained by using MATLAB, and we implemented all algorithms by ourselves without any toolbox. We assumed that the measurement vectors $\{\boldsymbol{u}_k^{(i)}\}$ were zero-mean real Gaussian random vectors with size $M = 5$ and had time-correlated shift structures (Lopes and Sayed, 2007). The specific structure was given by

$$\boldsymbol{u}_k^{(i)} = [u_k(i),\ u_k(i-1),\ \cdots,\ u_k(i-M+1)]^{\mathrm{T}}, \tag{3.4.1}$$

where

$$u_k(i) = \alpha_k u_k(i-1) + \sqrt{\sigma_{u_k}^2(1-\alpha_k^2)}z_k(i). \quad (i \in \mathbb{N}) \tag{3.4.2}$$

$\alpha_k \in [0,1)$ and $\sigma_{u_k}^2 \in (0,1]$ were chosen from uniform distribution. $z_k(i)$ was a white Gaussian process with zero mean and unit variance, and independent of $z_l(j)$ for $l \in \mathcal{V}\backslash\{k\}$ and $j \neq i$. The initial value $u_k(0)$ was also chosen from i.i.d. Gaussian with zero mean and unit variance. In this case, the trace of the correlation matrix is $\mathrm{Tr}[\boldsymbol{R}_{u_k}] = M\sigma_{u_k}^2$. We used the common step-size parameters $\mu_k = \mu$, the common forgetting factors $\nu_k = \nu$, and the common initial values $(\gamma_{lk}^2)^{(0)} = (\gamma^2)^{(0)}$, for all k, l. The unknown vector was set to be $\boldsymbol{w}^{\mathrm{o}} = \frac{1}{\sqrt{M}}\mathbf{1}_M$. All simulation results were obtained by averaging over 100 independent trials. All networks considered in the following sections were connected and the topologies were fixed throughout the simulations.

3.4.2 Comparison with Theoretical Learning Curve

In Figs. 3.1 and 3.2, we compare the learning curves of the CP-LMS and D-LMS using the static CP rule (3.2.15) obtained by simulations with the theoretical curves using (3.1.2) and (3.2.22), respectively. We used only a small network with $N = 20$ in Fig. 3.3 due to the computational difficulty in obtaining the theoretical values. The simulation of CP-LMS and its theoretical calculation were performed on the spanning tree shown in Fig. 3.3 (b), and that of D-LMS using CP rule and its theoretical calculation were performed on the original network shown in Fig. 3.3 (a). The step-size parameters were $\mu = 0.08$ ($\mu' = 0.004$) in CP-LMS and $\mu = 0.01$ in D-LMS with CP rule, respectively. The measurement noise power was set to $\sigma_k^2 = 10^{-3}$ ($k = 1, \ldots, N$). Fig. 3.1 shows that the simulation result agrees well with the theoretical curve. On the other hand, in Fig. 3.2, the curve before the steady-state matches the theoretical one but we found a slight disagreement between the simulation and the theoretical results at the steady-state. This may be

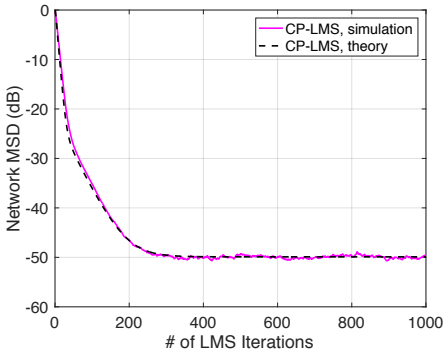


Figure 3.1: Network MSD learning curves of the proposed CP-LMS.

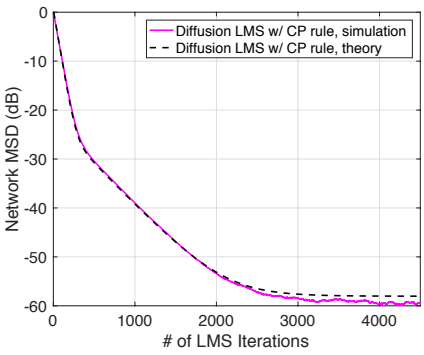
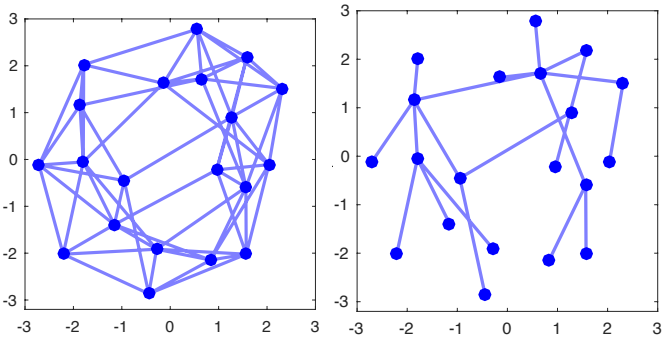


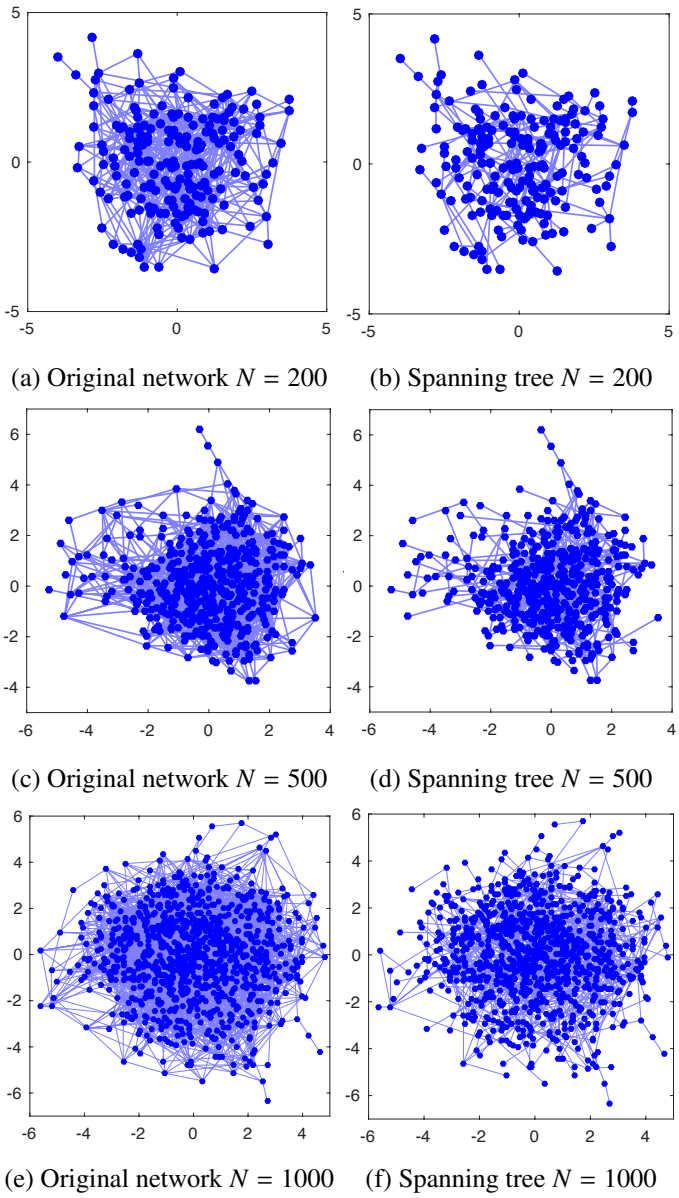
Figure 3.2: Network MSD learning curves of D-LMS with proposed CP rule.



(a) Original network (b) Spanning tree

Figure 3.3: Network topologies with $N = 20$.

due to the gap between the model of measurement vectors (3.4.1) and Assumption 1 that imposes the temporal whiteness. Although this gap also affects Fig. 3.1, the disagreement becomes more pronounced in Fig. 3.2 because, even before the assumption is introduced, the theoretical result of D-LMS is more complicated than that of CP-LMS so that more information is lost in the former in approximation by introducing the assumption.



(a) Original network $N = 200$ (b) Spanning tree $N = 200$

(c) Original network $N = 500$ (d) Spanning tree $N = 500$

(e) Original network $N = 1000$ (f) Spanning tree $N = 1000$

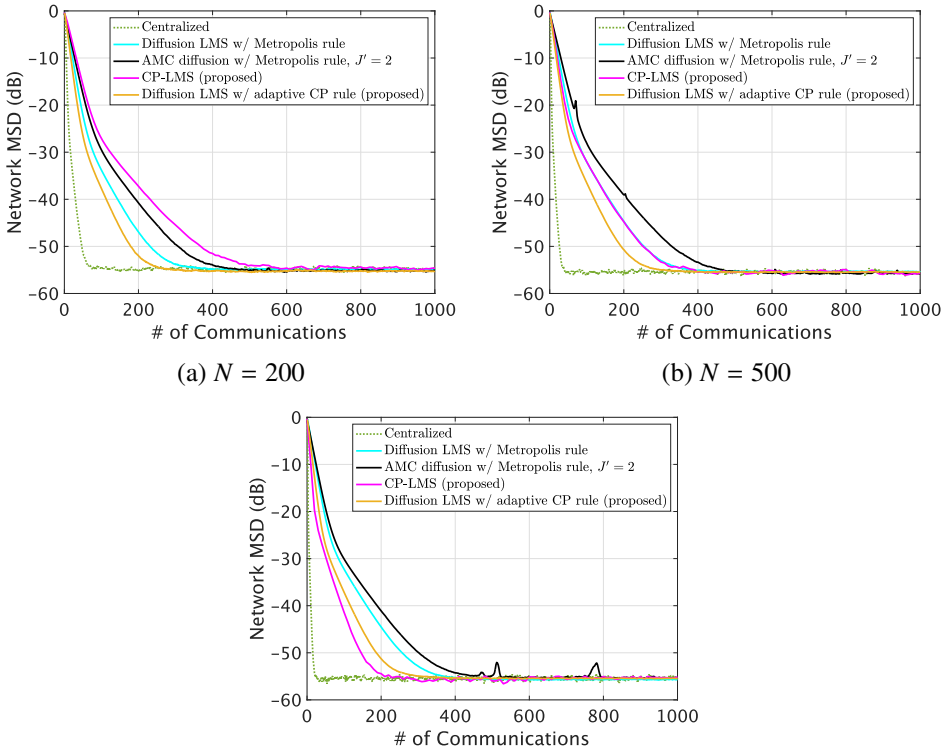
Figure 3.4: Network topologies.

3.4.3 Performance of Proposed CP-LMS and D-LMS w/ Adaptive CP rule

Here, we compare the performance of the proposed CP-LMS and D-LMS using the proposed adaptive CP rule (3.2.21) with that of D-LMS and the AMC diffusion

LMS using the conventional Metropolis rule (2.2.13). In order to compare the performance in networks of different sizes, we have generated Erdős-Rényi random networks with $N = 200,\ 500$, and 1000 as shown in Figs. 3.4 (a), (c), and (e), whose average degree was set to 6. Figs. 3.4 (b), (d), and (f) show the spanning trees extracted from the original networks so that the diameter of the tree becomes the smallest among all possible choices, i.e., the minimum diameter spanning trees. The diameters of the trees with $N = 200,\ 500$, and 1000 were $J = 8, 10$, and 10, respectively. Here, we adopted the minimum diameter spanning trees to CP-LMS and the original connected networks to D-LMS algorithms. The reason of the latter is that the original networks have more edges than the trees, which increases available information at the combination step in D-LMS. The number of iterations of the AMC diffusion LMS and the measurement noise power were set to $J' = 2$ and $\sigma_k^2 = 10^{-3}$ $(k = 1, \ldots, N)$, respectively. The performance of centralized block LMS in a star topology where measurements of all nodes were collected in a single synchronous communication was also evaluated.

Fig. 3.5 shows the network MSD learning curves of the centralized block LMS at a star topology, the proposed methods, and the conventional methods for the networks with different sizes versus the number of communications. The number of communications is defined as the number of the combination steps, where the exchanges of measurements, messages, or immediate estimates are assumed to be synchronous among all nodes. We have controlled the step-size parameters of the algorithms so that the steady-state performance becomes comparable for all algorithms. In the figures, D-LMS using the proposed adaptive CP rule outperformed the conventional diffusion methods for all cases. As for CP-LMS, though the conventional methods achieved faster convergence in Fig. 3.5 (a), CP-LMS converged faster as the number of nodes increases and it outperformed the conventional methods and D-LMS using the proposed adaptive CP rule in Fig. 3.5 (c). These results are consistent with the expectation that the proposed CP-LMS would be useful, especially for large-scale networks. This is because the consensus protocol employed in the conventional LMS generally achieves slower convergence when the network size is large, while the convergence rate of CP depends only on the diameter of the graph. Note that the reason why the AMC diffusion LMS converged slower than D-LMS was that this chapter evaluated the network MSD with respect to the number of communications instead of the number of the adaptation steps. The centralized block LMS achieved the fastest convergence if it worked on a star topology but the convergence became slower in a case of relaying measurements in the networks shown in Fig. 3.4, which was understood from the reason that the situation can be regarded as CP-LMS.



(a) $N = 200$

(b) $N = 500$

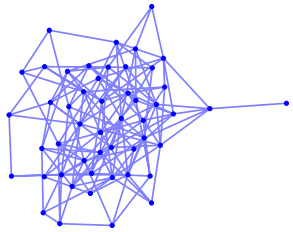
(c) $N = 1000$

Figure 3.5: Network MSD learning curves of proposed methods and conventional methods versus number of communications.

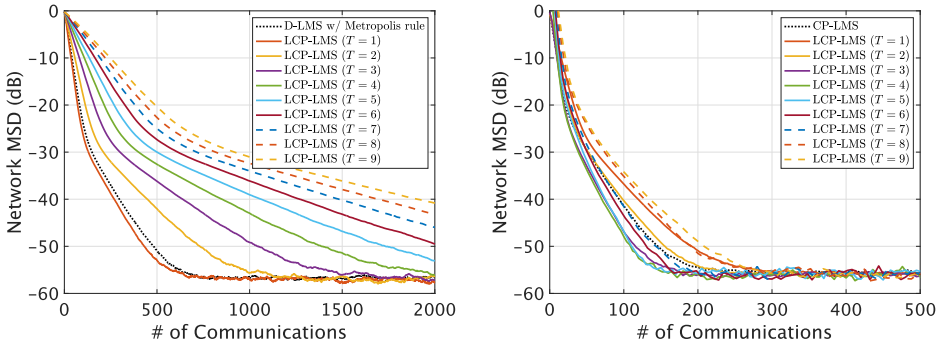
3.4.4 Performance of Proposed LCP-LMS

Next, we evaluated the performance of LCP-LMS where the iteration number T was varied from 1 to 9. For performance comparison in a small network, we used Erdős-Rényi random networks with $N = 50$ shown in Fig. 3.6 (a) and, for that in a large network, we used the network with $N = 1000$ in Fig. 3.4 (e). The constant and the measurement noise power were set to $\beta_k = 10^4$ and $\sigma_k^2 = 10^{-3}$ ($k = 1, \ldots, N$), respectively.

Figs. 3.6 (b) and (c) show the network MSD learning curves for the proposed LCP-LMS for the networks with different sizes versus the number of communications. We compare the performance of LCP-LMS in the small network with the conventional D-LMS in Fig. 3.6 (b). In this case, convergence of LCP-LMS with $T > 1$ was slower than D-LMS, so that the method could not work well in the small network. In Fig. 3.6 (c), we compare the performance of LCP-LMS in the large network with the proposed CP-LMS that achieved the best performance in Fig. 3.5 (c). The performance of LCP-LMS with $T = 4$ and $T = 5$ was the best among the



(a) Network $N = 50$



(b) Learning curves ($N = 50$) (c) Learning curves ($N = 1000$)

Figure 3.6: Network MSD learning curves of LCP-LMS versus number of communications.

methods in this case. The figure suggests that LCP-LMS achieves faster convergence than CP-LMS and conventional diffusion schemes in large scale networks if we can set an appropriate value of T.

3.4.5 Proposed Rules vs Conventional Rules

Finally, we compare the performance of the proposed D-LMS using the static CP rule (3.2.15) and the adaptive CP rule (3.2.21) with that of conventional D-LMS using the static Metropolis rule (2.2.13), the static relative-variance rule (3.2.17), and the adaptive relative-variance rule (3.2.19) in a small network. We used the network with $N = 20$ shown in Fig. 3.3 (a). The measurement noise power σ_k^2 was varied among nodes in order to confirm the estimation ability of the proposed adaptive CP rule. We set σ_k^2 in proportion to the square of the distance of node k from the origin $(0, 0)$ of Fig. 3.3 (a). This setting can be alternatively understood as assuming the situation that the observation target is located at the origin, and the power of the target signal decays with the square of the distance while keeping the measurement noise power to be uniform for all observation nodes. The initial value and the forgetting factor of the adaptation were $(\gamma^2)^{(0)} = 4.5 \times 10^{-2}$ and $\nu = 0.07$ in

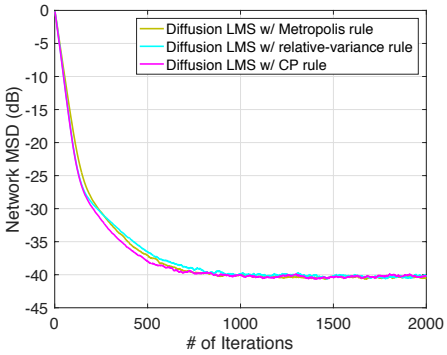


Figure 3.7: Network MSD learning curves of proposed CP rule and conventional static combination rules with $N = 20$.

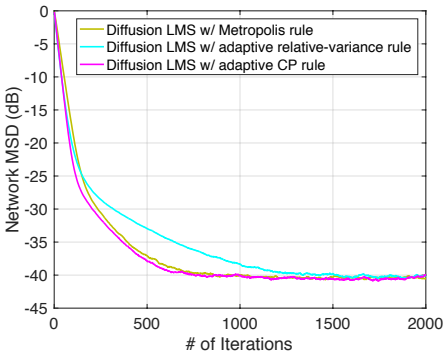


Figure 3.8: Network MSD learning curves of proposed adaptive CP rule and conventional combination rules with $N = 20$.

both Algorithms 5 and 6. We have controlled the step-size parameters of the algorithms so that the steady-state performance becomes comparable for all methods. Fig. 3.7 shows the learning curves of D-LMS using the static CP rule, Metropolis rule, and the static relative-variance rule in terms of the network MSD assuming that true γ_l^2s were known to each node. In the figure, we see that D-LMS using the CP rule converged faster than that using other rules. Fig. 3.8 shows the learning curves of D-LMS using the proposed adaptive CP rule, Metropolis rule, and the adaptive relative-variance rule in order to verify the influence of weight adaptation. The adaptive CP rule achieved comparable performance as in Fig. 3.7, while the performance of the adaptive relative-variance rule was significantly degraded in the settings here.

3.5 Conclusion

In this chapter, we have proposed novel D-LMS algorithms for in-network signal processing on the basis of the idea of the message passing algorithm of CP. By using exact CP on the spanning tree of the original network, CP-LMS can achieve the same solution as the centralized LMS in a fully distributed manner. LCP-LMS and its special case of CP rule are based on loopy CP, and we have optimized the constants involved in CP rule in terms of the steady-state MSD of D-LMS. Moreover, we have shown that their theoretical learning curves and steady-state MSDs can be obtained using existing frameworks. Also, we have extended the CP rule to an adaptive implementation named adaptive CP rule. Simulation results have shown that the proposed CP-LMS and LCP-LMS with an appropriate iteration number achieved better performance than the conventional D-LMS, especially

in large-scale networks, and that D-LMS with the static and the adaptive CP rules achieved better performance than that with the conventional combination rules.

Future work includes extension of the proposed CP rule to more flexible weight control methods using asymmetric updates in loopy CP, i.e., using different constant at each node depending on the direction of the messages.

4: SD-LMS based on CP

4.1 SD-LMS based on Loopy CP

4.1.1 Derivation

We discuss the case that the unknown vector $\boldsymbol{w}^{\mathrm{o}}$ is known to be sparse. SD-LMS (Lorenzo and Sayed, 2013) is an extension of D-LMS for estimating sufficiently sparse unknown vectors. To improve the performance of SD-LMS, we apply loopy CP, instead of the conventional average consensus protocol, to SD-LMS. Especially in this chapter, to focus on analytical tractability of the algorithm, we deal with a special case of loopy CP, where only the first iteration ($T = 1$) is employed, in the same way as Sect. 3.2.2. This section shows that combining the special case of loopy CP with SD-LMS results in an SD-LMS that employs the same combination weight as that derived in Sect. 3.2.2, which includes the constant β_k. Optimization of β_k is discussed in the next section.

We apply the special case of loopy CP to the combination step of SD-LMS (2.2.38) and describe the update by using the messages of loopy CP, $K^{[1]}_{(u\to k)}$ and $\theta^{[1]}_{(u\to k)}$. The former is directly calculated as

$$K^{[1]}_{(u\to k)} = \frac{\beta_k}{1+\beta_k}. \tag{4.1.1}$$

Substituting the current estimate $\boldsymbol{\psi}_k^{(i)}$ in (2.2.36) for the initial value x_k of loopy CP in (2.2.52) ($k = 1, 2, \ldots, N$), we have

$$\theta^{[1]}_{(u\to k)} = x_u = \boldsymbol{\psi}_u^{(i)}. \tag{4.1.2}$$

Moreover, the estimate $\boldsymbol{\phi}_k^{(i)}$ is obtained by the derivation of $x_k^{[1]}$ in (2.2.53) instead of (2.2.38) and can be calculated as

$$\begin{aligned}
\boldsymbol{\phi}_k^{(i)} &= \frac{x_k + \sum_{u\in\mathcal{N}_k\setminus\{k\}} K^{[1]}_{(u\to k)}\theta^{[1]}_{(u\to k)}}{1 + \sum_{u\in\mathcal{N}_k\setminus\{k\}} K^{[1]}_{(u\to k)}} \\
&= \frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}\boldsymbol{\psi}_k^{(i)} + \frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}\sum_{u\in\mathcal{N}_k\setminus\{k\}}\boldsymbol{\psi}_u^{(i)} \\
&= \sum_{l\in\mathcal{N}_k} a^{\mathrm{cp}}_{lk}\boldsymbol{\psi}_l^{(i)},
\end{aligned}$$

where

$$a_{lk} = \begin{cases} \frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ \frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}, & \text{if } l = k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.3}$$

This coincides with (3.2.6) for D-LMS. Therefore, the proposed method results in SD-LMS algorithm that uses the parameter β_k for the combination weights as in Sect. 3.2.2.

4.1.2 Optimization and Adaptive Combination Weights

Optimization Problem

In the following sections, we optimize β_k in (4.1.3) in terms of an upper bound of the steady-state network MSD of SD-LMS in the same way as Sect. 3.2.2. First, we introduce a specific formula of the steady-state network MSD of SD-LMS, $\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}}$, that has been concretely calculated in Lorenzo and Sayed (2013) as

$$\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}} = \mathrm{MSD}_{\mathrm{dif}}^{\mathrm{nw}} + \frac{1}{N}\lambda g_{\lambda,\epsilon}(\boldsymbol{A}), \tag{4.1.4}$$

where $\mathrm{MSD}_{\mathrm{dif}}^{\mathrm{nw}}$ is the steady-state network MSD of D-LMS (2.2.32), where $g_{\lambda,\epsilon}(\boldsymbol{A})$ is a function depending on the parameters λ and ϵ, and where $\boldsymbol{A}$ is the matrix composed of the combination weights $\{a_{lk}\}$. The specific formula of $g_{\lambda,\epsilon}(\boldsymbol{A})$ has been shown in Lorenzo and Sayed (2013) as

$$g_{\lambda,\epsilon}(\mathbf{A}) = \lambda g_{1,\boldsymbol{\Sigma},\infty} - g_{2,\boldsymbol{\Sigma},\infty}, \tag{4.1.5}$$

where

$$g_{1,\boldsymbol{\Sigma},\infty} = \lim_{i\to\infty} \mathrm{E}\left[\partial f(\boldsymbol{\phi}^{(i-1)})^{\mathrm{H}} \mathcal{M}\mathcal{A}\boldsymbol{\Sigma}\mathcal{A}^{\mathrm{T}}\mathcal{M}\partial f(\boldsymbol{\phi}^{(i-1)})\right], \tag{4.1.6}$$

$$g_{2,\boldsymbol{\Sigma},\infty} = -2\lim_{i\to\infty} \mathrm{E}\Big[\partial f(\boldsymbol{\phi}^{(i-1)})^{\mathrm{H}} \mathcal{M}\mathcal{A}\boldsymbol{\Sigma}\mathcal{A}^{\mathrm{T}}(\boldsymbol{I}_{NM} - \mathcal{M}\mathcal{D})\tilde{\boldsymbol{w}}^{(i-1)}\Big], \tag{4.1.7}$$

and where $\boldsymbol{\Sigma} = \mathrm{vec}^{-1}\left((\boldsymbol{I}_{(MN)^2} - \boldsymbol{F}')^{-1}\boldsymbol{q}'/N\right)$. Equations (4.1.6) and (4.1.7) are the same as (2.2.47) and (2.2.49), respectively, but they are reproduced here for convenience. Remark 1 in Sect. 2.2.2 demonstrates that the second term of (4.1.4) is due to the regularization term of SD-LMS.

Adaptive CP rule

The function $g_{\lambda,\epsilon}(\boldsymbol{A})$ has the complicated form so that it is difficult to use $\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}}$ directly for the optimization of β_k. We thus derive an upper bound of $\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}}$ by introducing some approximations.

In this section, we consider approximating $g_{\lambda,\epsilon}(\boldsymbol{A}) \simeq g_{\lambda,\epsilon}(\boldsymbol{I}_N)$, that is, the second term in (4.1.4) is assumed not to depend on $\boldsymbol{A}$, i.e., β_k. This is motivated by the fact that only the adaptation step includes the regularization term and that any information is not exchanged between nodes in the step. From the approximation, we only need to find an upper bound of $\mathrm{MSD}_{\mathrm{dif}}^{\mathrm{nw}}$ because the second term of (4.1.4)

is independent of β_k. The optimization problem is identical with (3.2.10), so that we can derive the optimal parameter β_k^{opt} in the same manner, namely,

$$\beta_k^{\text{opt}} = \begin{cases} \frac{(|\mathcal{N}_k|-1)\gamma_k^2}{\Gamma_k}, & \text{if } \Gamma_k > 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.1.8}$$

The resulting combination weight using (4.1.8) has been named CP rule. As discussed in Sect. 3.2.2, in the case of $\beta_k^{\text{opt}} = \infty$, the resulting combination weight (4.1.3) coincides with the uniform rule (2.2.12).

Furthermore, we can also derive an adaptive solution to estimate $\gamma_l^2 = \mu_l^2 \sigma_l^2 \text{Tr}(\boldsymbol{R}_{u_l})$ and then update β_k in a similar way to that of D-LMS (shown in Sect. 3.2.2) to avoid the direct calculation of γ_l^2 that depends on locally unavailable network statistics such as noise variance σ_l^2 and the correlation matrix $\boldsymbol{R}_{u_l}$. The estimate $\boldsymbol{\phi}_k^{(i)}$ approaches the unknown vector $\boldsymbol{w}^{\text{o}}$ as the algorithm iterates (2.2.36) and (2.2.38), and reaches steady-state under the assumption that the step-sizes are sufficiently small. By using (2.2.36) and (2.2.2), we can rewrite

$$\boldsymbol{\psi}_l^{(i)} \approx \boldsymbol{w}^{\text{o}} + \mu_l \boldsymbol{\mathcal{U}}_l^{(i)} \boldsymbol{\mathcal{V}}_l^{(i)} - \mu_l \lambda \partial f(\boldsymbol{w}^{\text{o}}).$$

Taking the expectation leads to

$$\text{E}\left[\left\| \boldsymbol{\psi}_l^{(i)} - \boldsymbol{w}^{\text{o}} + \mu_l \lambda \partial f(\boldsymbol{w}^{\text{o}}) \right\|^2 \right] \approx \mu_l^2 \sigma_l^2 \text{Tr}(\boldsymbol{R}_{u_l}).$$

We substitute the instantaneous values into the expectation and use it to estimate γ_l^2. Let $(\gamma_{lk}^2)^{(i)}$ be the estimate of γ_l^2 at node k and time i. We adaptively obtain the estimate by employing the following update:

$$(\gamma_{lk}^2)^{(i)} = (1 - \nu_k)(\gamma_{lk}^2)^{(i-1)} + \nu_k \left\| \boldsymbol{\psi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)} + \mu_l \lambda \partial f(\boldsymbol{\phi}_k^{(i-1)}) \right\|^2, \tag{4.1.9}$$

where ν_k $(0 < \nu_k < 1)$ is the forgetting factor.

The subsequent estimate $\beta_k^{(i)}$ of β_k^{opt} and the combination weight $a_{lk}^{\text{cp},(i)}$ at time i are described as

$$\beta_k^{(i)} = \begin{cases} \frac{(|\mathcal{N}_k|-1)(\gamma_{kk}^2)^{(i)}}{\Gamma_k^{(i)}}, & \text{if } \Gamma_k^{(i)} > 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.1.10}$$

$$a_{lk}^{\text{cp}(i)} = \begin{cases} \frac{\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ \frac{1+\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}, & \text{if } k = l, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.11}$$

This combination weight has been named adaptive CP rule. SD-LMS using the adaptive CP rule is summarized in *Algorithm* 7.

Algorithm 7 SD-LMS with adaptive CP rule

Initialize $\boldsymbol{\phi}_k^{(-1)} = 0$, $(\gamma_{lk}^2)^{(-1)}$, $\forall k \in \mathcal{V}, l \in \mathcal{N}_k$
for each time $i \in \mathbb{N}$, each node $k \in \mathcal{V}$, and each neighbor $l \in \mathcal{N}_k$ **do**
  $\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)}(d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}}\boldsymbol{\phi}_k^{(i-1)}) - \mu_k \lambda \partial f(\boldsymbol{\phi}_k^{(i-1)})$
  $(\gamma_{lk}^2)^{(i)} = (1-\nu_k)(\gamma_{lk}^2)^{(i-1)} + \nu_k \left\| \boldsymbol{\psi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)} + \mu_l \lambda \partial f(\boldsymbol{\phi}_k^{(i-1)}) \right\|^2$
  if $\sum_{l\in\mathcal{N}_k}(\gamma_{lk}^2)^{(i)} - (\gamma_{kk}^2)^{(i)}|\mathcal{N}_k| > 0$ **then**
    $\beta_k^{(i)} = \frac{(|\mathcal{N}_k|-1)(\gamma_{kk}^2)^{(i)}}{\sum_{l\in\mathcal{N}_k}(\gamma_{lk}^2)^{(i)} - (\gamma_{kk}^2)^{(i)}|\mathcal{N}_k|}$
  else
    $\beta_k^{(i)} = +\infty$ (large positive constant)
  end if
  $a_{lk}^{(i)} = \frac{\beta_k^{(i)}}{1+\beta_k^{(i)}|\mathcal{N}_k|}$ $(l \in \mathcal{N}_k \setminus \{k\})$, $\frac{1+\beta_k^{(i)}}{1+\beta_k^{(i)}|\mathcal{N}_k|}$ $(l = k)$
  $\boldsymbol{\phi}_k^{(i)} = \sum_{l\in\mathcal{N}_k} a_{lk}^{(i)} \boldsymbol{\psi}_l^{(i)}$
end for

Adaptive CPO rule

The optimization in Sect. 4.1.2 has some room for improvement because it has been derived by ignoring the influence of the sparsity-promoting regularization term in $\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}}$ (4.1.4) by approximating $g_{\lambda,\epsilon}(\boldsymbol{A}) \simeq g_{\lambda,\epsilon}(\boldsymbol{I}_N)$. In this section, we introduce another approximation to capture the contribution of $g_{\lambda,\epsilon}(\boldsymbol{A})$ on $\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}}$ and optimize the constant β_k.

For the first term of (4.1.4), we utilize the upper bound as in the discussion above. The following upper bound has been already derived in Cattivelli and Sayed (2010) as

$$\mathrm{MSD}_{\mathrm{dif}}^{\mathrm{nw}} \leq c \sum_{k=1}^{N} \sum_{l\in\mathcal{N}_k} \gamma_l^2 a_{lk}^2. \tag{4.1.12}$$

By substituting (4.1.3), the equation (4.1.12) can be rewritten as

$$\mathrm{MSD}_{\mathrm{dif}}^{\mathrm{nw}} \leq \sum_{k=1}^{N} \left(c\gamma_k^2 \left(\frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k} \right)^2 + \sum_{l\in\mathcal{N}_k\setminus\{k\}} c\gamma_l^2 \left(\frac{\beta_k}{1+|\mathcal{N}_k|\beta_k} \right)^2 \right). \tag{4.1.13}$$

The second term of (4.1.4) is difficult to directly express as a function of β_k because it takes the limit $i \to \infty$ and the expectation. Thus, we consider simplifying (4.1.6) and (4.1.7) by introducing three approximations. First, since the step-sizes are sufficiently small under Assumption 2, we ignore the term in (4.1.7) which is quadratic in $\mathcal{M}$ and approximate it as

$$g_{2,\boldsymbol{\Sigma},\infty} \simeq -2 \lim_{i\to\infty} \mathrm{E}\Big[\partial f(\boldsymbol{\phi}^{(i-1)})^{\mathrm{T}} \mathcal{M}\mathcal{A}\boldsymbol{\Sigma}\mathcal{A}^{\mathrm{T}} \tilde{\boldsymbol{w}}^{(i-1)} \Big]. \tag{4.1.14}$$

Second, we remove the limit and the expectation in (4.1.6) and (4.1.7), and use the instantaneous values. Third, we approximate $\boldsymbol{\Sigma} = \boldsymbol{I}$. This approximation of $\boldsymbol{\Sigma}$ is reasonable because the instantaneous approximation of $g_{\lambda,\epsilon}(\boldsymbol{A})$ actually corresponds to the fourth term of (2.2.42) when the coefficient is ignored and because $\mathrm{vec}^{-1}(\boldsymbol{q}') = \mathrm{vec}^{-1}(\mathrm{vec}(\boldsymbol{I})) = \boldsymbol{I}$. Specifically, the approximated values of $g_{1,\boldsymbol{\Sigma},\infty}$ and $g_{2,\boldsymbol{\Sigma},\infty}$ are given by

$$g_{1,i} \simeq \partial f(\boldsymbol{\phi}^{(i-1)})^{\mathrm{H}} \mathcal{M}\mathcal{A}\mathcal{A}^{\mathrm{T}}\mathcal{M}\partial f(\boldsymbol{\phi}^{(i-1)}), \tag{4.1.15}$$

$$g_{2,i} \simeq -2\partial f(\boldsymbol{\phi}^{(i-1)})^{\mathrm{H}} \mathcal{M}\mathcal{A}\mathcal{A}^{\mathrm{T}}\tilde{\boldsymbol{w}}^{(i-1)}, \tag{4.1.16}$$

respectively. By using these terms, we have the approximated $g_{\lambda,\epsilon}(\boldsymbol{A})$ as

$$\begin{aligned} g_{\lambda,\epsilon}(\boldsymbol{A}) &\approx \lambda g_{1,i} - g_{2,i} \\ &= \sum_{k=1}^{N} \sum_{j\in\mathcal{N}_k} \sum_{l\in\mathcal{N}_k} \Big[\lambda\mu_j\mu_l a_{jk}a_{lk} \partial f(\boldsymbol{\phi}_j^{(i-1)})^{\mathrm{H}} \partial f(\boldsymbol{\phi}_l^{(i-1)}) \\ &\quad + 2\mu_j a_{jk}a_{lk} \partial f(\boldsymbol{\phi}_j^{(i-1)})^{\mathrm{H}} \left(\boldsymbol{w}^{\mathrm{o}} - \boldsymbol{\phi}_l^{(i-1)}\right) \Big]. \end{aligned} \tag{4.1.17}$$

It seems hard to optimize N^2 combination weights $\{a_{lk}\}$ directly from (4.1.17) but it can be captured by the optimization of N coefficients $\{\beta_k\}$ in our framework assuming (4.1.3). Substituting (4.1.3) into (4.1.17) leads to

$$g_{\lambda,\epsilon}(\boldsymbol{A}) \approx \sum_{k=1}^{N} \left[\tilde{\varrho}_k^{(i)} \left(\frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k} \right) \left(\frac{\beta_k}{1+|\mathcal{N}_k|\beta_k} \right) + \tilde{\varepsilon}_k^{(i)} \left(\frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k} \right)^2 + \tilde{\varsigma}_k^{(i)} \left(\frac{\beta_k}{1+|\mathcal{N}_k|\beta_k} \right)^2 \right], \tag{4.1.18}$$

where

$$\begin{aligned} \tilde{\varrho}_k^{(i)} &= \sum_{l\in\mathcal{N}_k\setminus\{k\}} \mathrm{Re}\left[\mu_k \partial f(\boldsymbol{\phi}_k^{(i-1)})^{\mathrm{H}} \left\{ \lambda\mu_l \partial f(\boldsymbol{\phi}_l^{(i-1)}) + 2\left(\boldsymbol{w}^{\mathrm{o}} - \boldsymbol{\phi}_l^{(i-1)}\right) \right\} \right] \\ &\quad + \sum_{l\in\mathcal{N}_k\setminus\{k\}} \mathrm{Re}\left[\mu_l \partial f(\boldsymbol{\phi}_l^{(i-1)})^{\mathrm{H}} \left\{ \lambda\mu_k \partial f(\boldsymbol{\phi}_k^{(i-1)}) + 2\left(\boldsymbol{w}^{\mathrm{o}} - \boldsymbol{\phi}_k^{(i-1)}\right) \right\} \right], \\ \tilde{\varepsilon}_k^{(i)} &= \mathrm{Re}\left[\mu_k \partial f(\boldsymbol{\phi}_k^{(i-1)})^{\mathrm{H}} \left\{ \lambda\mu_k \partial f(\boldsymbol{\phi}_k^{(i-1)}) + 2\left(\boldsymbol{w}^{\mathrm{o}} - \boldsymbol{\phi}_k^{(i-1)}\right) \right\} \right], \\ \tilde{\varsigma}_k^{(i)} &= \sum_{l\in\mathcal{N}_k\setminus\{k\}} \sum_{j\in\mathcal{N}_k\setminus\{k\}} \mathrm{Re}\left[\mu_j \partial f(\boldsymbol{\phi}_j^{(i-1)})^{\mathrm{H}} \left\{ \lambda\mu_l \partial f(\boldsymbol{\phi}_l^{(i-1)}) + 2\left(\boldsymbol{w}^{\mathrm{o}} - \boldsymbol{\phi}_l^{(i-1)}\right) \right\} \right]. \end{aligned}$$

In the above equations, we extract real parts of terms to guarantee that $\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}}$ and resulting weights are real numbers.

By incorporating (4.1.13) and (4.1.18), we can obtain an approximated upper

bound of $\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}}$ as

$$\mathrm{MSD}_{\mathrm{spa}}^{\mathrm{nw}} \lessapprox c\sum_{k=1}^{N}\sum_{l\in\mathcal{N}_k}\gamma_l^2 a_{lk}^2 + \frac{\lambda}{N}g_{\lambda,\epsilon}(\boldsymbol{A}) \tag{4.1.19}$$

$$\begin{aligned}
&= \sum_{k=1}^{N}\Big[\frac{\lambda}{N}\tilde{\varrho}_k^{(i)}\left(\frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}\right)\left(\frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}\right)\\
&\quad + \left(c\gamma_k^2 + \frac{\lambda}{N}\tilde{\varepsilon}_k^{(i)}\right)\left(\frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}\right)^2 + \left(\sum_{l\in\mathcal{N}_k\setminus\{k\}} c\gamma_l^2 + \frac{\lambda}{N}\tilde{\varsigma}_k^{(i)}\right)\left(\frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}\right)^2\Big]\\
&= \sum_{k=1}^{N}\left[\varrho_k^{(i)}\left(\frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}\right)\left(\frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}\right) + \varepsilon_k^{(i)}\left(\frac{1+\beta_k}{1+|\mathcal{N}_k|\beta_k}\right)^2 + \varsigma_k^{(i)}\left(\frac{\beta_k}{1+|\mathcal{N}_k|\beta_k}\right)^2\right]\\
&:= \sum_{k=1}^{N} F_k(\beta_k),
\end{aligned} \tag{4.1.20}$$

where $\varrho_k^{(i)} = \frac{\lambda}{N}\tilde{\varrho}_k^{(i)}$, $\varepsilon_k^{(i)} = c\gamma_k^2 + \frac{\lambda}{N}\tilde{\varepsilon}_k^{(i)}$, and $\varsigma_k^{(i)} = \sum_{l\in\mathcal{N}_k\setminus\{k\}} c\gamma_l^2 + \frac{\lambda}{N}\tilde{\varsigma}_k^{(i)}$. Hence, we optimize $\{\beta_k\}_{k=1}^N$ by minimizing the approximated upper bound (4.1.20), i.e.,

$$\min_{\{\beta_k\}_{k=1}^N}\sum_{k=1}^{N} F_k(\beta_k). \tag{4.1.21}$$

We can divide the problem into the following N problems,

$$\beta_k^{\mathrm{opt}} = \arg\min_{\beta_k} F_k(\beta_k) \quad k = 1,\ldots,N. \tag{4.1.22}$$

The derivative of F_k with respect to β_k can be calculated as

$$\begin{aligned}
\frac{\partial F_k}{\partial \beta_k} = \Big[&\left\{2\varsigma_k^{(i)} - 2(|\mathcal{N}_k|-1)\varepsilon_k^{(i)} - (|\mathcal{N}_k|-2)\varrho_k^{(i)}\right\}\beta_k\\
&- \left\{2(|\mathcal{N}_k|-1)\varepsilon_k^{(i)} - \varrho_k^{(i)}\right\}\Big]\cdot\frac{1}{(1+|\mathcal{N}_k|\beta_k)^3}.
\end{aligned} \tag{4.1.23}$$

Since the denominator is positive, $\frac{\partial F_k}{\partial \beta_k} = 0$ when

$$\beta_k = \frac{2(|\mathcal{N}_k|-1)\varepsilon_k^{(i)} - \varrho_k^{(i)}}{2\varsigma_k^{(i)} - 2(|\mathcal{N}_k|-1)\varepsilon_k^{(i)} - (|\mathcal{N}_k|-2)\varrho_k^{(i)}}. \tag{4.1.24}$$

We put $\Gamma_k' = 2\varsigma_k^{(i)} - 2(|\mathcal{N}_k|-1)\varepsilon_k^{(i)} - (|\mathcal{N}_k|-2)\varrho_k^{(i)}$ and $\Lambda_k' = 2(|\mathcal{N}_k|-1)\varepsilon_k^{(i)} - \varrho_k^{(i)}$. Considering $\beta_k > 0$, we can derive the following optimal parameter:

$$\begin{cases}
\beta_k^{\mathrm{opt}} = \frac{\Lambda_k'}{\Gamma_k'}, & \text{if } \Gamma_k' > 0 \text{ and } \Lambda_k' > 0,\\
\beta_k^{\mathrm{opt}} \to +0, & \text{if } \Gamma_k' > 0 \text{ and } \Lambda_k' \le 0,\\
\beta_k^{\mathrm{opt}} \to +\infty, & \text{if } \Gamma_k' \le 0.
\end{cases} \tag{4.1.25}$$

Note that sufficiently small β_k implies that the node does not use the neighbors' estimates at the combination step but only uses its own information. When β_k is very large, the resulting combination weight (4.1.3) coincides with the conventional uniform rule (2.2.12).

The optimal parameter (4.1.25) includes unavailable information such as the unknown vector $\boldsymbol{w}^{\mathrm{o}}$, noise variance $\{\sigma_k^2\}$, correlation matrices $\{\boldsymbol{R}_{u_k}\}$, and the number N of all nodes in the network. Therefore, we consider approximating the unknown vector and plugging in adaptive estimates to the other factors, and derive an adaptive algorithm. First, as an adaptive update of the estimate of γ_l^2, we adopt the same equation as (4.1.9). Second, we approximate the unknown vector $\boldsymbol{w}^{\mathrm{o}}$ and the number N of all nodes with the instantaneous estimate $\boldsymbol{\psi}_k^{(i)}$ and the number $|\mathcal{N}_k|$ of neighbors, respectively. The coefficients $\varepsilon_k^{(i)}$, $\varrho_k^{(i)}$, and $\varsigma_k^{(i)}$ are redefined as

$$\hat{\varepsilon}_k^{(i)} = c(\gamma_{kk}^2)^{(i)} + \frac{\lambda}{|\mathcal{N}_k|}\mu_k \mathrm{Re}\left[\partial f(\boldsymbol{\phi}_k^{(i-1)})^{\mathrm{H}}\left\{\lambda\mu_k \partial f(\boldsymbol{\phi}_k^{(i-1)}) + 2\left(\boldsymbol{\psi}_k^{(i)} - \boldsymbol{\phi}_k^{(i-1)}\right)\right\}\right], \tag{4.1.26}$$

$$\hat{\varrho}_k^{(i)} = \frac{\lambda}{|\mathcal{N}_k|}\mathrm{Re}\left[\boldsymbol{\zeta}_k^{(i)\mathrm{H}}\left\{\lambda\mu_k \partial f(\boldsymbol{\phi}_k^{(i-1)}) + 2\left(\boldsymbol{\psi}_k^{(i)} - \boldsymbol{\phi}_k^{(i-1)}\right)\right\} + \mu_k \partial f(\boldsymbol{\phi}_k^{(i-1)})^{\mathrm{H}}\boldsymbol{\iota}_k^{(i)}\right], \tag{4.1.27}$$

$$\hat{\varsigma}_k^{(i)} = c\sum_{l\in\mathcal{N}_k\setminus\{k\}}(\gamma_{lk}^2)^{(i)} + \frac{\lambda}{|\mathcal{N}_k|}\mathrm{Re}\left[\boldsymbol{\zeta}_k^{(i)\mathrm{T}}\boldsymbol{\iota}_k^{(i)}\right], \tag{4.1.28}$$

where $\boldsymbol{\zeta}_k^{(i)} = \sum_{j\in\mathcal{N}_k\setminus\{k\}}\mu_j \partial f(\boldsymbol{\phi}_j^{(i-1)})$ and $\boldsymbol{\iota}_k^{(i)} = \lambda\boldsymbol{\zeta}_k^{(i)} + 2(|\mathcal{N}_k|-1)\boldsymbol{\psi}_k^{(i)} - 2\sum_{j\in\mathcal{N}_k\setminus\{k\}}\boldsymbol{\phi}_j^{(i-1)}$. We further redefine Γ_k' and Λ_k' by using (4.1.26)–(4.1.28) as

$$\Gamma_k^{'(i)} = 2\hat{\varsigma}_k^{(i)} - 2(|\mathcal{N}_k|-1)\hat{\varepsilon}_k^{(i)} - (|\mathcal{N}_k|-2)\hat{\varrho}_k^{(i)}, \tag{4.1.29}$$

$$\Lambda_k^{'(i)} = 2(|\mathcal{N}_k|-1)\hat{\varepsilon}_k^{(i)} - \hat{\varrho}_k^{(i)}, \tag{4.1.30}$$

respectively.

Summarizing these plug-ins, we can derive an adaptive form of the parameter β_k as follows:

$$\begin{cases} \beta_k^{\mathrm{o},(i)} = \frac{\Lambda_k^{'(i)}}{\Gamma_k^{'(i)}}, & \text{if } \Gamma_k^{'(i)} > 0 \text{ and } \Lambda_k^{'(i)} > 0, \\ \beta_k^{\mathrm{o},(i)} \to +0, & \text{if } \Gamma_k^{'(i)} > 0 \text{ and } \Lambda_k^{'(i)} \le 0, \\ \beta_k^{\mathrm{o},(i)} \to +\infty, & \text{if } \Gamma_k^{'(i)} \le 0. \end{cases} \tag{4.1.31}$$

The corresponding combination weight $a_{lk}^{\mathrm{cpo}(i)}$ is described as

$$a_{lk}^{\mathrm{cpo}(i)} = \begin{cases} \frac{\beta_k^{\mathrm{o},(i)}}{1+|\mathcal{N}_k|\beta_k^{\mathrm{o},(i)}}, & \text{if } l \in \mathcal{N}_k \setminus \{k\}, \\ \frac{1+\beta_k^{\mathrm{o},(i)}}{1+|\mathcal{N}_k|\beta_k^{\mathrm{o},(i)}}, & \text{if } l = k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.32}$$

Algorithm 8 SD-LMS with adaptive CPO rule

Initialize $\boldsymbol{\phi}_k^{(-1)} = 0$, $\{(\gamma_{lk}^2)^{(-1)}\}$, $\forall k \in \mathcal{V}, l \in \mathcal{N}_k$
for each time $i \in \mathbb{N}$, each node $k \in \mathcal{V}$, and each neighbor $l \in \mathcal{N}_k$ **do**
  $\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\phi}_k^{(i-1)} + \mu_k \boldsymbol{u}_k^{(i)}(d_k^{(i)} - \boldsymbol{u}_k^{(i)\mathrm{H}}\boldsymbol{\phi}_k^{(i-1)}) - \mu_k \lambda \partial f(\boldsymbol{\phi}_k^{(i-1)})$
  $(\gamma_{lk}^2)^{(i)} = (1-\nu_k)(\gamma_{lk}^2)^{(i-1)} + \nu_k \left\| \boldsymbol{\psi}_l^{(i)} - \boldsymbol{\phi}_k^{(i-1)} + \mu_l \lambda \partial f(\boldsymbol{\phi}_k^{(i-1)}) \right\|^2$
  Calculate $\hat{\varepsilon}_k^{(i)}, \hat{\varrho}_k^{(i)}, \hat{\varsigma}_k^{(i)}$ as in (4.1.26)–(4.1.28)
  Calculate $\Gamma_k^{'(i)}$ and $\Lambda_k^{'(i)}$ as in (4.1.29) and (4.1.30)
  if $\Gamma_k^{'(i)} > 0$ **then**
    if $\Lambda_k^{'(i)} > 0$ **then**
      $\beta_k^{(i)} = \frac{\Lambda_l^{(i)}}{\Gamma_k^{'(i)}}$
    else
      $\beta_k^{(i)} = +0$ (small positive constant)
    end if
  else
    $\beta_k^{(i)} = +\infty$ (large positive constant)
  end if
  $a_{lk}^{(i)} = \frac{\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}$ $(l \in \mathcal{N}_k \setminus \{k\})$, $\frac{1+\beta_k^{(i)}}{1+|\mathcal{N}_k|\beta_k^{(i)}}$ $(l = k)$
  $\boldsymbol{\phi}_k^{(i)} = \sum_{l \in \mathcal{N}_k} a_{lk}^{(i)} \boldsymbol{\psi}_l^{(i)}$
end for

SD-LMS using the proposed adaptive optimization is summarized in *Algorithm* 8. We name this weight *adaptive CPO rule*.

4.1.3 Convergence Analysis

In this section, we discuss the convergence of SD-LMS using static CP rule (3.2.15), adaptive CP rule (4.1.11), and adaptive CPO rule (4.1.32) in the mean sense and the mean-square behaviors. The conventional performance analysis framework for SD-LMS in Sect. 2.2.2 is applicable to the proposed SD-LMS with CP and CPO rules because the rules can be regarded as ones of the combination rules.

Corollary 2. *Consider the measurement model given by* (2.2.2) *and the combination weight given by CP rule* (3.2.15)*, adaptive CP rule* (4.1.11)*, and adaptive CPO rule* (4.1.32)*. The convergence of SD-LMS using the proposed rules in the mean sense is guaranteed under Assumption 1 if the step-sizes satisfy*

$$0 < \mu_k < \frac{2}{\lambda_{\max}(\boldsymbol{R}_{u_k})} \quad (k \in \mathcal{V}).$$

If we use the static CP rule (3.2.15), the mean-square behaviors can be expressed as in Sect 2.2.2. The theoretical network MSD learning curve of SD-LMS

Table 4.1: Computational complexity and communication cost for the proposed and conventional rules par one synchronous communication at node k.

Algorithm	$\times,\div$	$+,-$	Communication cost
SD-LMS w/ static rules	$M\lvert\mathcal{N}_k\rvert$	$M(\lvert\mathcal{N}_k\rvert-1)$	$M(\lvert\mathcal{N}_k\rvert-1)$
SD-LMS w/ adaptive relative-variance rule	$(3M+6)\lvert\mathcal{N}_k\rvert$	$(4M+2)\lvert\mathcal{N}_k\rvert-M-1$	$M(\lvert\mathcal{N}_k\rvert-1)$
SD-LMS w/ adaptive CP rule (proposed)	$(3M+5)\lvert\mathcal{N}_k\rvert+3$	$(4M+3)\lvert\mathcal{N}_k\rvert-M+2$	$M(\lvert\mathcal{N}_k\rvert-1)$
SD-LMS w/ adaptive CPO rule (proposed)	$(4M+5)\lvert\mathcal{N}_k\rvert+10M+20$	$(6M+3)\lvert\mathcal{N}_k\rvert+6M+4$	$2M(\lvert\mathcal{N}_k\rvert-1)$

for the static CP rule can be represented by the same expression as (2.2.42) given by

$$\begin{aligned}\mathrm{MSD}_i^{\mathrm{nw}} = \mathrm{MSD}_{i-1}^{\mathrm{nw}} + \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}}\boldsymbol{F}'^{i}\boldsymbol{q}' - \frac{1}{N}\boldsymbol{w}^{\mathrm{T}}\left[\mathrm{vec}^{-1}\left(\boldsymbol{F}'^{i}\left[\boldsymbol{I}_{(MN)^2}-\boldsymbol{F}'\right]\boldsymbol{q}'\right)\right]\boldsymbol{w} \\ + g_{\boldsymbol{q}'}^{(i)} + \sum_{j=0}^{i-1}\left(g_{(\boldsymbol{F}')^{i-j}\boldsymbol{q}'}^{(j)} - g_{(\boldsymbol{F}')^{i-j-1}\boldsymbol{q}'}^{(j)}\right). \end{aligned} \tag{4.1.33}$$

The steady-state network MSD and the steady-state MSD at each node k under Assumptions 1 and 2 are also the same as (2.2.45) and (2.2.46) given by

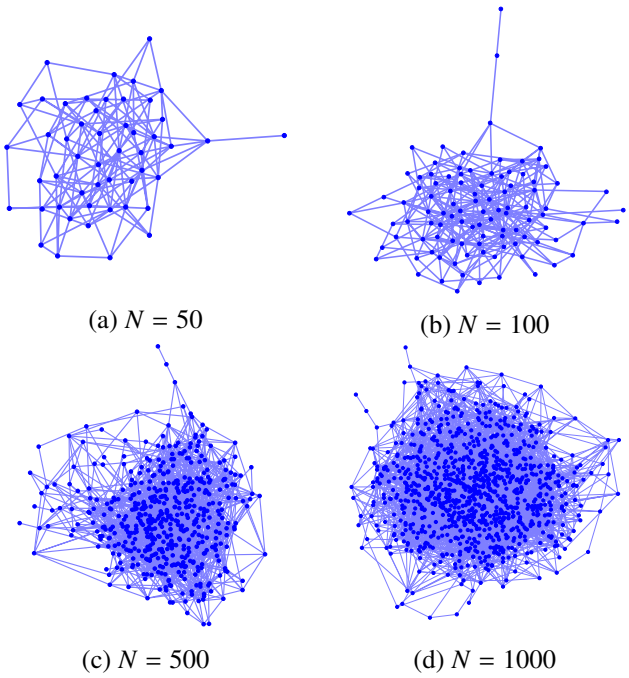
$$\mathrm{MSD}^{\mathrm{nw}} = \frac{1}{N}\boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2}-\boldsymbol{F}')^{-1}\boldsymbol{q}' + \frac{1}{N}\lambda\beta_{\Sigma,\infty}\left(\lambda-\frac{\alpha_{\Sigma,\infty}}{\beta_{\Sigma,\infty}}\right), \tag{4.1.34}$$

$$\mathrm{MSD}_k = \boldsymbol{r}'^{\mathrm{T}}(\boldsymbol{I}_{(MN)^2}-\boldsymbol{F}')^{-1}\boldsymbol{p}' + \lambda\beta_{\Sigma_k,\infty}\left(\lambda-\frac{\alpha_{\Sigma_k,\infty}}{\beta_{\Sigma_k,\infty}}\right), \tag{4.1.35}$$

respectively.

4.1.4 Computational Complexity

We discuss the computational complexity and the communication cost of SD-LMS using the proposed and conventional rules. Table 4.1 shows the complexity and the cost in each combination step at a node, whose definitions are the same as in Sect. 3.3. Note that, since the common adaptation step and calculation of $\partial f(\cdot)$ are required for all algorithms, the complexities only related to the combination step are evaluated. Adaptive CPO rule (Algorithm 8) requires higher computational complexity than the conventional rules and adaptive CP rule (Algorithm 7) because it computes additional coefficients given by (4.1.26)–(4.1.28). Moreover, the



(a) $N = 50$ (b) $N = 100$

(c) $N = 500$ (d) $N = 1000$

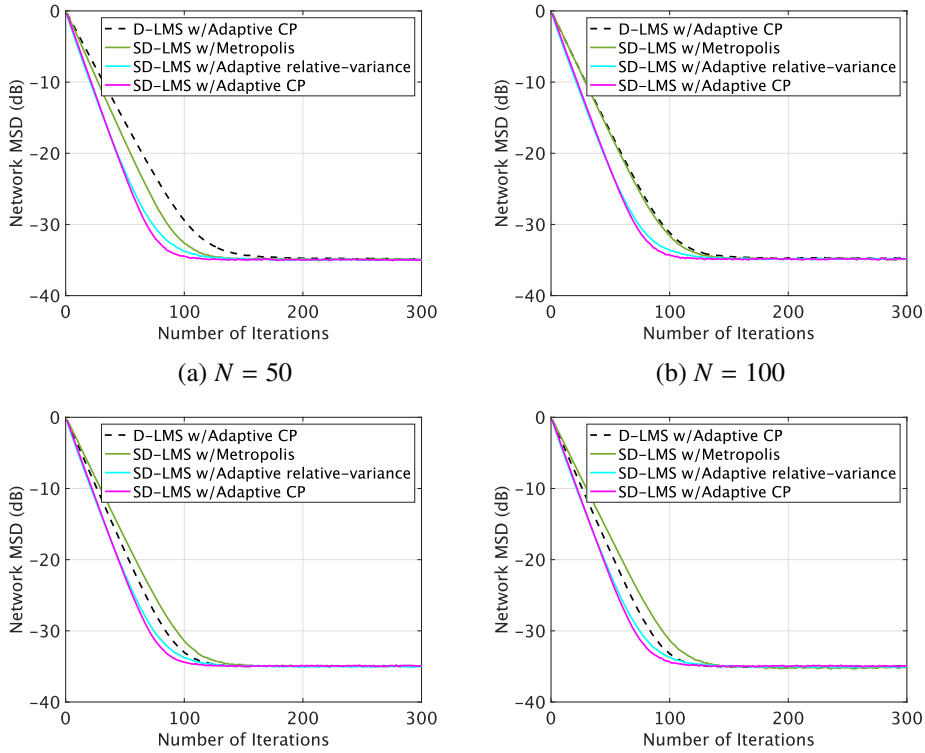
Figure 4.1: Network topologies.

communication cost of the adaptive CPO rule becomes twice because it requires exchanging $\bar{\phi}_l^{(i-1)}$ with neighboring nodes to calculate the additional coefficients but the order is the same as those of the other rules. As mentioned in Sect. 3.3, the communication time typically dominates the execution time in applications such as wireless sensor networks, one may conclude that the required time is comparable.

4.2 Simulation Results

4.2.1 Evaluation of Adaptive CP Rule

In this section, we verify the performance of SD-LMS with the proposed adaptive CP rule via computer simulations. All the simulation results were obtained by using MATLAB. In order to compare the performance in networks of different sizes, we have generated Erdős-Rényi random networks with $N = 50, 100, 500,$ and 1000, as shown in Figs. 4.1 (a)–(d). The average degree of each network was fixed to 6. Note that for the adaptive rules, the results in this subsection employed (3.2.18) for the estimation of γ_l^2. We have used node-independent step-size parameters $\mu_k = \mu$, forgetting factors $\nu_k = \nu$, and initial values $(\gamma_{lk}^2)^{(i)} = (\gamma^2)^{(i)}$, for all k, l. The initial values $(\gamma^2)^{(-1)}$ were set to be 0.1 and the forgetting factor was fixed to be $\nu = 0.05$. The size of the measurement vector was set to be $M = 20$. We



(a) $N = 50$ (b) $N = 100$

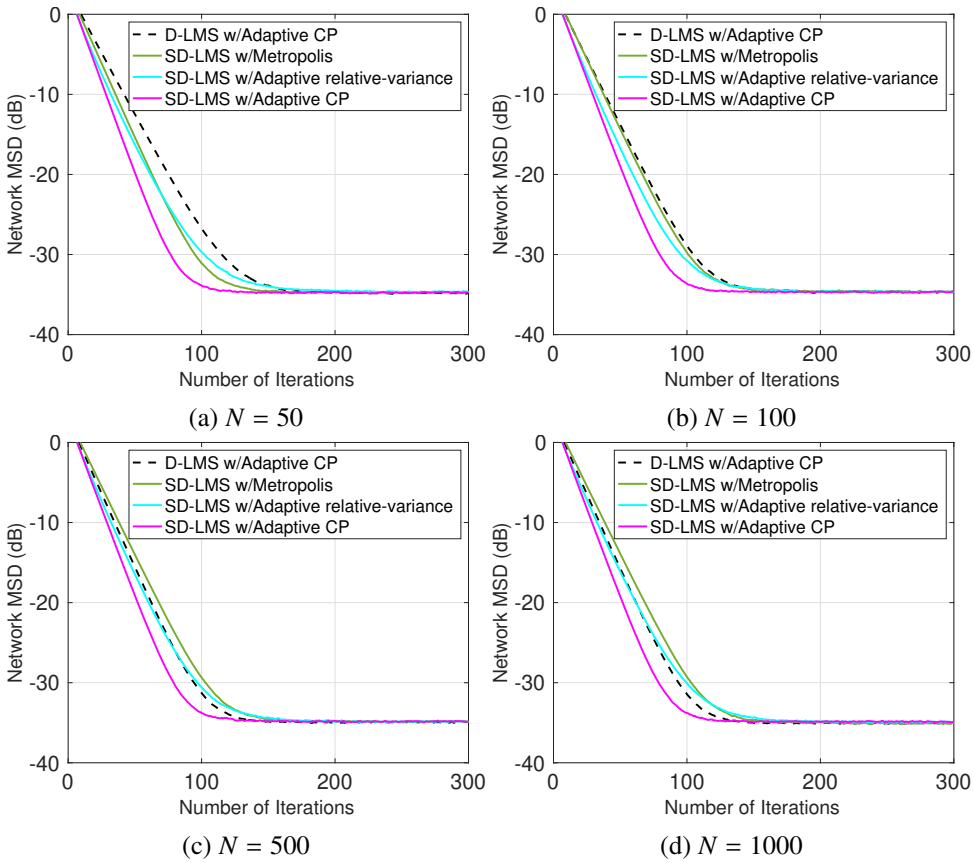
(c) $N = 500$ (d) $N = 1000$

Figure 4.2: Network MSD learning curves for 1-sparse.

have investigated instantaneous network MSD $\mathrm{MSD}^{\mathrm{nw,in}}$ of the proposed method and conventional methods.

All the simulation results were obtained by averaging over 20 independent trials. The measurement noise power σ_k^2 was independently generated by uniform distribution over $[0.01, 0.02]$. The measurement vectors $\{\boldsymbol{u}_k^{(i)}\}$ had the same structures as (3.4.1) except that α_k is set to 0 for all k. The indices of the non-zero elements of unknown vector were randomly selected in each trial.

We compare the convergence performance of SD-LMS with that of D-LMS, both of which employ the proposed adaptive CP rule (4.1.11). Also, we have evaluated the performance of SD-LMS using conventional weights, static Metropolis rule a_{lk}^{met} (2.2.13) and adaptive RV rule $a_{lk}^{\mathrm{rv}(i)}$ (3.2.19). The unknown vector was assumed to be 1-sparse whose non-zero element is equal to 1. We have fixed the regularization parameter as $\lambda = 0.0005$ for Metropolis rule and $\lambda = 0.00025$ for adaptive relative-variance rule and adaptive CP rule, with which the steady-state performance of each rule has become the best among our trials. We have controlled the step-size parameters of the algorithms so that the steady-state performance be-



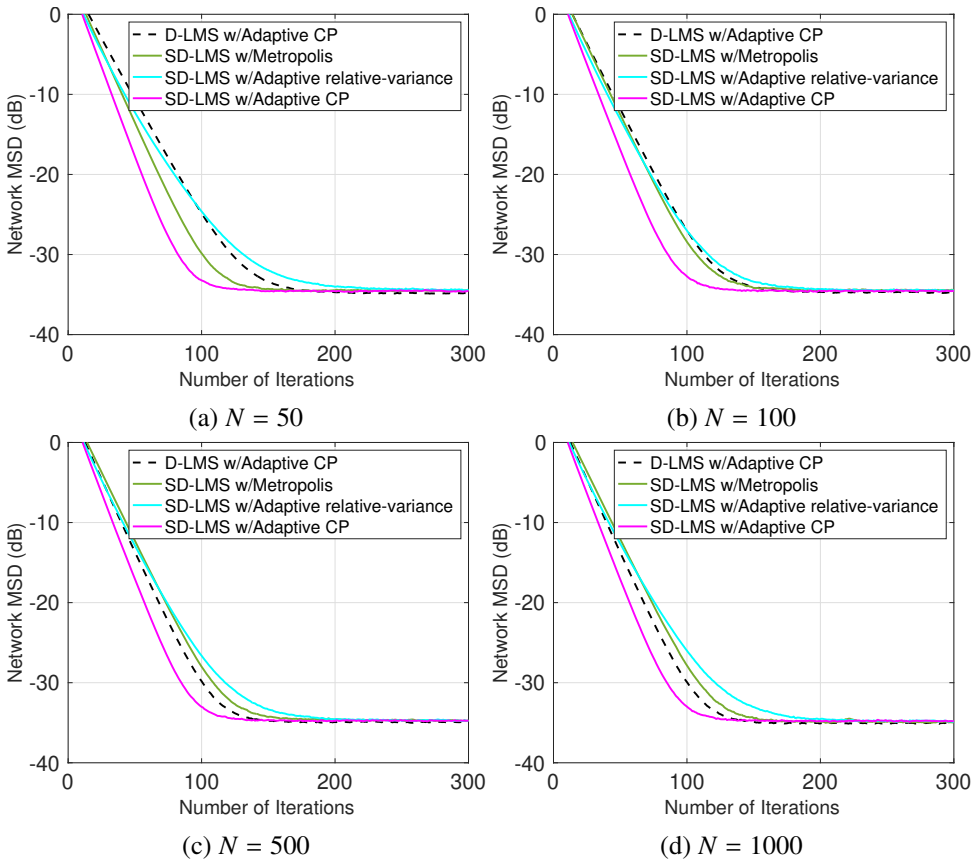
(a) $N = 50$ (b) $N = 100$

(c) $N = 500$ (d) $N = 1000$

Figure 4.3: Network MSD learning curves for 2-sparse.

came comparable for all algorithms. The parameter ϵ in the regularization function (2.2.39) was set to be $\epsilon = 0.01$. Figs. 4.2 (a)–(d) show the learning curves for the networks of different sizes. The figures demonstrated that SD-LMS with the proposed adaptive CP rule achieved faster convergence than D-LMS and than SD-LMS with the conventional weights regardless of the network size. It should be noted here that we do not see so much difference in the convergence rate among different network sizes. This may be because a larger network has more measurements, which will be beneficial for faster convergence, while a larger network requires more steps for average consensus, so that one can understand both types of the influences have been canceled. We further evaluated the performance when the unknown vector was 2-sparse and 3-sparse. Fig. 4.3 shows the learning curves for 2-sparse and Fig. 4.4 is those for 3-sparse. The figures indicate that the validity of the proposed method is not specific to 1-sparse.

Finally, we verify the impact of the choice of ϵ on the convergence performance of SD-LMS using the proposed adaptive CP rule (4.1.11). Figs. 4.5 (a)–(d) show



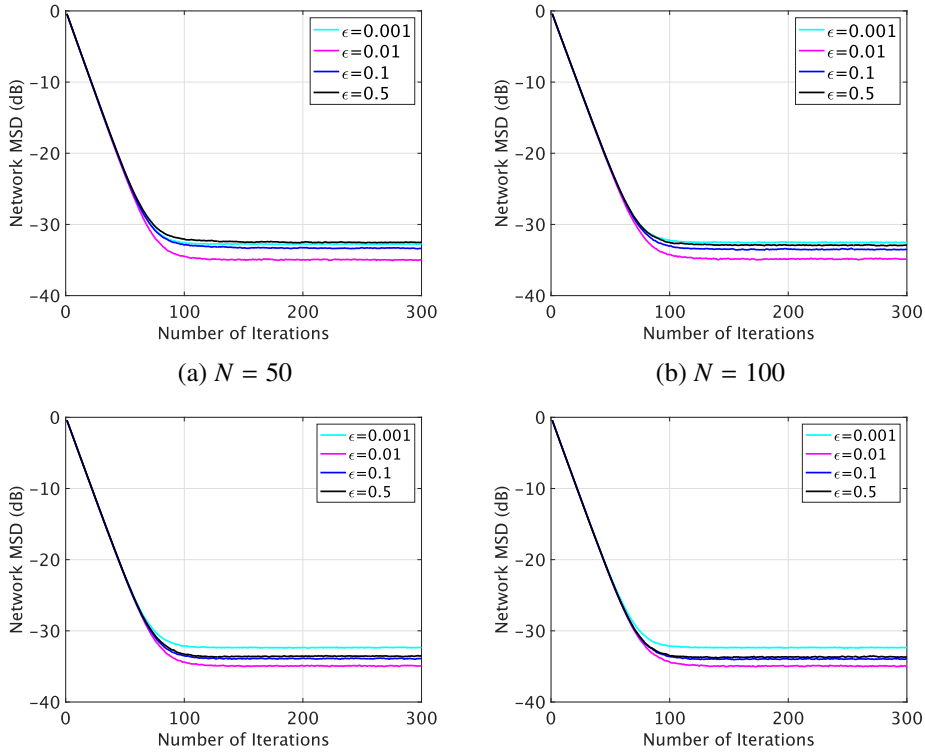
(a) $N = 50$ (b) $N = 100$

(c) $N = 500$ (d) $N = 1000$

Figure 4.4: Network MSD learning curves for 3-sparse.

the learning curves for the networks of different sizes with $\epsilon = 0.001, 0.01, 0.1, 0.5$. We have fixed step-size parameters $\mu = 0.098$ in Fig. 4.5 (a), $\mu = 0.1$ in Fig. 4.5 (b), $\mu = 0.108$ in Fig. 4.5 (c), and $\mu = 0.109$ in Fig. 4.5 (d). The rest of the parameters were the same as in Fig. 4.2. In our settings, $\epsilon = 0.01$ was the best regardless of the network size and the worst choice resulted in degradation of about 3dB.

We have also evaluated the optimal value of ϵ for the case with different sparsity and measurement noise variance. Fig. 4.6 demonstrates the learning curves for the network with $N = 50$ shown in Fig. 4.1 (a) when the unknown vector had lower sparsity, namely 10-sparse. Fig. 4.7 shows those when the measurement noise power σ_k^2 was larger, which was independently generated by uniform distribution over $[0.11, 0.12]$. The rest of the parameters in each experiment were the same as in Fig. 4.2. From the figures, we can see that the optimal value of ϵ in Fig. 4.6 was the same as that in Fig. 4.5 (a), while it was different in Fig. 4.7. Thus, the value of ϵ should be appropriately determined, especially depending on the signal-to-noise ratio of measurements.



(a) $N = 50$ (b) $N = 100$

(c) $N = 500$ (d) $N = 1000$

Figure 4.5: Network MSD learning curves for comparison of ϵ.

4.2.2 Evaluation of Adaptive CPO Rule

In order to compare the performance in networks of different density, we have generated Erdős-Rényi random networks with $N = 20$ where the mean degrees are $D = 12, 14, 16$, and 18, as shown in Figs. 4.8 (a)–(d). The step-size parameter, the forgetting factor, and the parameter in the regularization function were fixed to be $\mu = 0.05$, $\nu = 0.005$, and $\epsilon = 0.001$, respectively. In this subsection, the equation (4.1.9) was employed for the estimation of γ_l^2. The initial values $(\gamma^2)^{(-1)}$ were set to be 1. We have fixed the regularization parameter as $\lambda = 0.0005$ for $D = 12$ and 14, and $\lambda = 0.0004$ for $D = 16$ and 18, with which the steady-state performance has become the best among our trials. The parameter c that controls the balance of MSD (4.1.19) has been chosen as $c = 0.1, 1$, or 5. The unknown vector was with size $M = 100$ and the number of nonzero element was 1, where the index of the nonzero element switches every 1000 iterations in order to evaluate the tracking performance of the proposed and conventional methods. The measurement vectors $\{\boldsymbol{u}_k^{(i)}\}$ had the same structures as (3.4.1) except that α_k was set to 0 for all k. All simulation results were obtained by averaging 100 independent

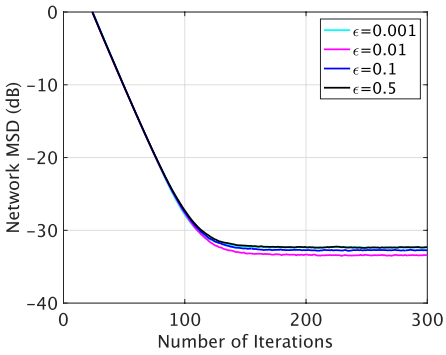


Figure 4.6: Network MSD learning curves with $N = 50$ when unknown vector has lower sparsity.

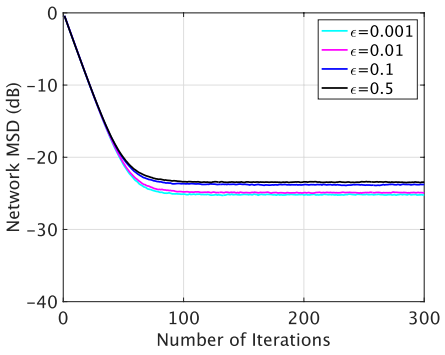


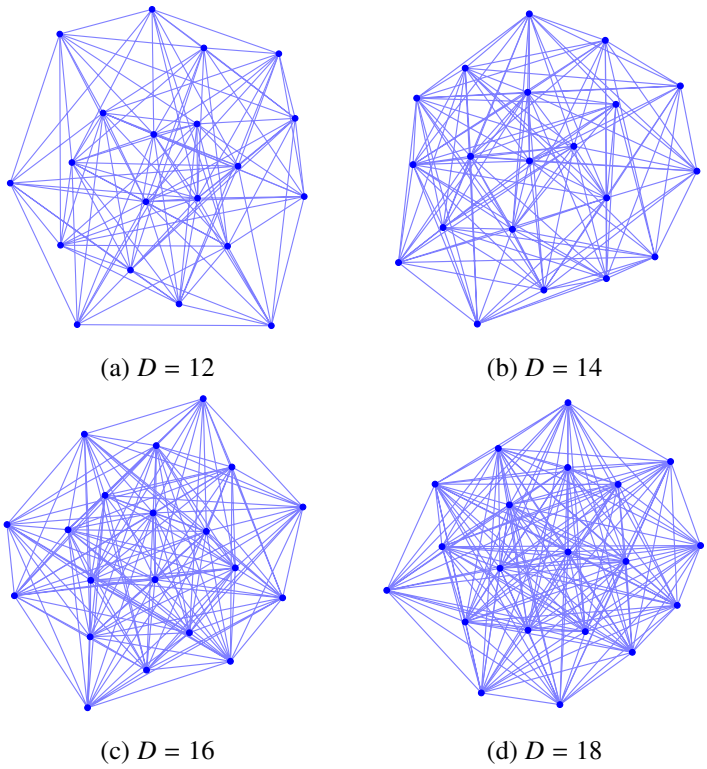
Figure 4.7: Network MSD learning curves with $N = 50$ when measurement noise variance is larger.

trials. The measurement noise power σ_k^2 was independently generated by uniform distribution over $[0.1, 0.2]$ in each trial. We compare learning curves of SD-LMS in terms of instant network MSD $\frac{1}{N}\sum_{k=1}^{N}\|\boldsymbol{\phi}_k^{(i)} - \boldsymbol{w}^{\mathrm{o}}\|^2$ using the proposed adaptive CPO rule (4.1.32) with that using the adaptive CP rule (4.1.11), static Metropolis rule (2.2.13), and adaptive relative-variance (RV) rule (3.2.19).

Figs. 4.9 (a)–(d) show the learning curves in the case of $D = 12, 14, 16$, and 18, respectively. In Fig. 4.9 (a), the adaptive RV rule converged slightly slower than the proposed adaptive CPO rule and CP rule but achieved the lowest error. However, as the density of the network increased, the algorithm with the proposed adaptive CPO rule achieved faster convergence and lower MSD under the suitable choice of c than that with all the other rules. Namely, the proposed adaptive CPO rule showed the best tracking performance to the change of the unknown vector.

4.3 Conclusion

This chapter has considered achieving faster convergence of SD-LMS for the adaptive sparse signal processing in networks. We have applied loopy CP to SD-LMS and shown that the proposed algorithm results in SD-LMS that uses CP rule determined by the parameters of CP. We have further optimized the parameters in terms of the steady-state MSD of SD-LMS and extended the weight to an adaptive version, adaptive CP rule. We have further improved the approximation of the optimization problem to achieve better convergence performance and robustness, and optimized the parameters. Moreover, we have shown an adaptive implementation for tracking the optimal coefficients, which has been named adaptive CPO rule. Simulation results demonstrated that the proposed method achieves faster convergence than the original SD-LMS. We have also numerically verified that the

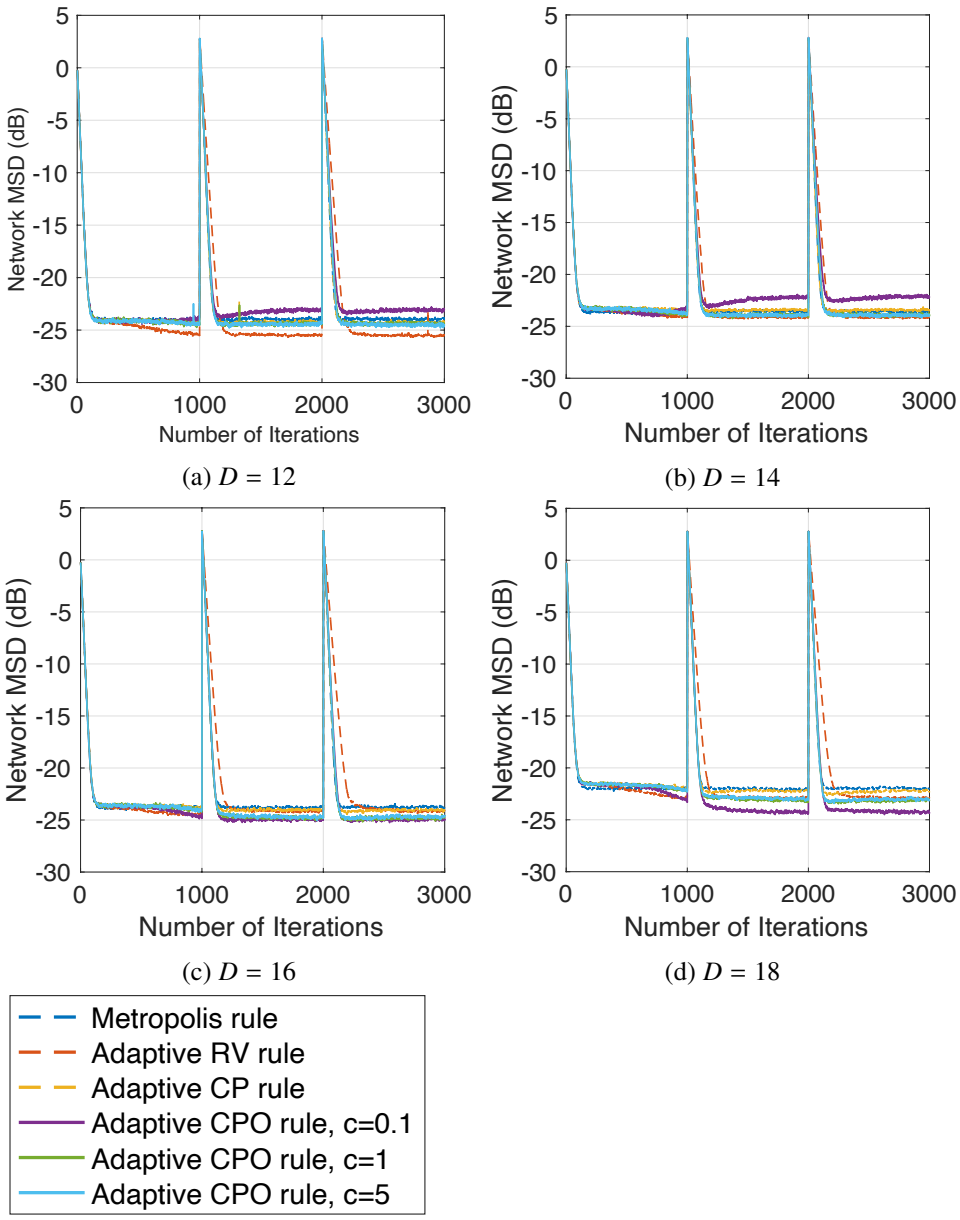


(a) $D = 12$ (b) $D = 14$

(c) $D = 16$ (d) $D = 18$

Figure 4.8: Network topologies.

optimal value of ϵ in the regularization function with respect to the convergence performance of the proposed adaptive CP rule significantly depended not on the sparsity of unknown vector but on the signal-to-noise ratio of measurements. The algorithm with the proposed adaptive CPO rule has shown better tracking performance for the change of the unknown vector, especially in dense networks, under the suitable choice of the parameter c, at the cost of a slightly higher computational complexity.



(a) $D = 12$

(b) $D = 14$

(c) $D = 16$

(d) $D = 18$

Figure 4.9: Network MSD learning curves for networks with different density.

Part II

New Representation of Scalable GPR

Scalable Estimation on Linear and Nonlinear Regression Models via Decentralized Processing: Adaptive LMS Filter and Gaussian Process Regression

Chapter 5: Scalable GPR with Sketching

Ayano NAKAI

Abstract

Chapter 5 discusses scalable approximation of GPR. It focuses on a recently proposed aggregation method named nested pointwise aggregation of experts (NPAE), and provides its reinterpretation in terms of sketching. On the basis of the sketching-based reinterpretation, it proposes an extended framework of NPAE, called nested aggregation of experts using inducing points (NAE-IP), which offers a more flexible trade off between accuracy and computational cost. Efficiency of the proposed method is investigated via numerical simulations.

Part III

Conclusion and Future Directions

6: Conclusion

6.1 Summary

In this thesis, we have studied how to enhance performance of decentralized algorithms derived by approximate divisions of corresponding centralized processing. Specifically, we have coped with two scalable estimation problems via decentralized processing, whose subjects have been introduced in Chapter 1, namely, in-network adaptive least-mean-square (LMS) filter and Gaussian process regression (GPR). The former's target is tracking an unknown deterministic vector by using the measurements at nodes in the network obtained from the linear regression model. Diffusion LMS (D-LMS) is the fully distributed LMS algorithm in such networks that enhances scalability with respect to network size. The latter adopts a nonlinear regression model, so that it achieves more flexible estimation. However, it requires high computational complexity, and therefore aggregation methods have been proposed as approximation methods of GPR on the basis of partial decentralization.

In Part I, we have aimed to improve convergence performance of the fully distributed D-LMS and its sparsity-aware extension, sparse diffusion LMS (SD-LMS), by focusing on ways of cooperation among nodes that compensate for the performance deterioration caused by approximate divisions of the centralized problems. The cooperation method in both D-LMS and SD-LMS can be regarded as an example of average consensus protocol but the protocol is known to require a lot of iterations, especially in large networks. We have thus employed another method for the cooperation based on message propagation, i.e., consensus propagation (CP), which can be categorized into two schemes, exact CP and loopy CP. We have proposed a novel fully distributed LMS, CP-LMS, by applying exact CP, although the algorithm requires extraction of tree structure of the original network to achieve fast convergence. We have also proposed Loopy CP-LMS (LCP-LMS) by applying loopy CP to D-LMS, which can eliminate the extraction of tree structure but the required number of iteration for convergence has not been known. The special case where the iteration of loopy CP is limited to one has improved interpretability of the proposed algorithm and enabled optimization of constants involved in loopy CP. The resulting algorithm has become novel combination weights of D-LMS named (static) CP rule. We have optimized the constants in terms of the steady-state network mean-squared-error (MSD) and derived an adaptive implementation named adaptive CP rule. We have also applied the special case to SD-LMS and

derived two kinds of adaptive combination weights under some assumptions. One is the adaptive CP rule that is the same as for D-LMS and another is adaptive CP with Optimization (CPO) rule. The theoretical performance has been analyzed via the framework for D-LMS and SD-LMS. Efficiency of the proposed methods has been shown via computer simulations.

In Part II, we have aimed to improve performance of the aggregation methods where covariance information of data is lost by the decentralization. Among the aggregation methods proposed so far, we have focused on one of them, nested pointwise aggregation of experts (NPAE), that uses the richest covariance information among them and generalized the prediction process via sketching. The proposed generalization has yielded a more flexible approximation of GPR than NPAE, which has been named nested aggregation of experts using inducing points (NAE-IP). The proposed NAE-IP can control the computational complexity and the performance of the algorithm by choices of the inducing points. Simulation results on synthetic and real data have shown that NAE-IP can achieve lower complexity than the original NPAE and better predictive error than the conventional methods.

This thesis is no more than having coped with the two specific problems for in-network processing and for general data processing. However, results in the thesis have shown that the improvements focusing on factors that affect performance degradation caused by decentralization enhance performance with a little increase or even decrease in complexity. As for other decentralization problems, the improvements are expected to be realized while maintaining scalability by properly identifying the parts that can affect the performance degradation and reconsidering the parts in a different context.

6.2 Future Directions

The decentralized processing that the thesis have focused on is expected to further improve the performance and to extend the range of applicable applications. This section points out the future directions, thus concluding the thesis.

Reconsideration of cost function in D-LMS

In Part I, we have derived proposed algorithms by applying CP to the combination step of D-LMS. This was motivated by the fact that the combination step has a tight relation to average consensus. One can also reconsider minimizing estimation error with a constraint for consensus among nodes. If one chose Gaussian belief propagation as a distributed solution of the problem, one may derive another distributed algorithm for the same problem settings as D-LMS. Note that it requires

taking time-dependent potentials into accounts. Moreover, it may be possible to obtain interpretation of the constants β_k involved in the proposed methods in relation to weights for controlling balance between the error and the constraint.

Analysis of convergence time using non-backtracking operator

For the proposed method in Part I, we have optimized parameters in the fully distributed LMS by minimizing cost function in terms of steady-state error. On the other hand, in order to explicitly accelerate the convergence, another approach is conceivable that minimizes the convergence time of the algorithm. The convergence time for algorithms on the basis of average consensus protocol has been analyzed by using spectra of adjacency matrices but the analysis for CP has not had much progress. As discussed at the end of Sect. 2.2.3, we expect that algorithms on the basis of CP can be represented by the non-backtracking operator (Coja-Oghlan et al., 2009; Decelle et al., 2011; Krzakala et al., 2013) and that the spectrum of the operator (Bordenave et al., 2015) may enable the analysis of the convergence time. It should be a next challenge to solve an optimization problem in terms of the convergence time represented by using the spectrum of the non-backtracking operator.

Acceleration of iterative decentralized algorithms by deep unfolding

For iterative algorithms such as the centralized LMS filter, the required number of iterations for the convergence is also affected by some parameters such as step-size, the control of which largely affects the performance of the algorithms. Especially for the decentralized cases such as D-LMS, the available information is less than that for the centralized cases and the required number of iterations is generally large, so that the acceleration is an important matter. The control of the parameters of the iterative algorithms has been achieved by the aid of deep learning, which is known as deep unfolding (Gregor and LeCun, 2010; Balatsoukas-Stimming and Studer, 2019). Its application to decentralized algorithms also gathers attention recently (Kishida et al., 2020). Future directions include the extension of the deep unfolding approach to decentralized adaptive filters or online learning, where the parameters are needed to be learned in a decentralized manner as well.

Supervised optimization of sketching matrix

In Part II, we have introduced approximation methods for full GPR. Most of the aggregation methods divide covariance information $\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X})$ on the basis of the training data $\boldsymbol{X}$ and many of sparse GP methods lower the dimension of $\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X})$ on the basis of inducing points. These can be interpreted as unsupervised learning

that does not take the information of test data $\boldsymbol{X}^*$ into consideration. However, Proposition 2 has shown that optimal sketching matrix depends not only on the covariance $\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X})$ of the training data but also on the covariance $\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X}^*)$ related to the test data. This suggests that both training and test data should be incorporated into the design of approximation methods. Therefore, it seems beneficial to utilize the test data in optimization of a block-structured sketching matrix.

Fully distributed online GPR

Fully distributed GPR is also in demand from not only computational complexity but also application point of view, for example, in environmental monitoring by multiple unmanned aerial vehicles (Gu and Hu, 2012; Choi et al., 2014; Tiwari et al., 2018). The temporal variation of the phenomena should be also taken into account (Garg et al., 2012). In addition, the algorithm that can track the phenomena is desirable, namely, online GPR (Nguyen-Tuong et al., 2008; Hoang et al., 2019). The theoretical property must be discussed as well as to manage both scalability and predictive performance.

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