# A summary on Zeta-functions of root systems and Poincaré polynomials of Weyl groups

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## 1 Introduction

Witten zeta-functions were introduced as partition functions of quantum gauge theories and are expressed as

$$\zeta_W(s;G)=\sum_{\psi}\frac{1}{(\dim\psi)^s}, \tag{1.1}$$

where $\psi$ runs over all finite dimensional irreducible representations of a connected compact semisimple Lie group $G$ [20, 21]. Some of these zeta-functions are explicitly given as the following multiple Dirichlet series:

$$\sum_{m=1}^{\infty}\frac{1}{m^s}=\zeta(s), \tag{1.2}$$

$$\sum_{m,n=1}^{\infty}\frac{2^s}{m^sn^s(m+n)^s}, \tag{1.3}$$

$$\sum_{m,n=1}^{\infty}\frac{6^s}{m^sn^s(m+n)^s(m+2n)^s}. \tag{1.4}$$

In [2–6, 8–10, 13] we consider multivariable analog of the above zeta-functions and call them zeta-functions of root systems and studied their special values at integers and established value relations among them. For example, (1.3) is generalized as

$$\zeta_2(s_{12},s_{23},s_{13};A_2)=\sum_{m,n=1}^{\infty}\frac{1}{m^{s_{12}}n^{s_{23}}(m+n)^{s_{13}}}, \tag{1.5}$$

and a special value is given as

$$\zeta_2(2,2,2;A_2)=\frac{1}{6}(-1)^3\frac{1}{3780}\frac{(2\pi i)^{2+2+2}}{2!2!2!}=\frac{\pi^6}{2835}, \tag{1.6}$$

where $\frac{1}{3780}$ is given by multiple analog of Bernoulli numbers. Then the next question arises naturally: What about functional relations? In the case of Euler-Zagier multiple zeta-functions, only harmonic products are known as functional relations on the whole space: For $s_1, s_2 \in \mathbb{C}$,

$$\zeta_{EZ,2}(s_1,s_2)+\zeta_{EZ,2}(s_2,s_1)=\zeta(s_1+s_2)-\zeta(s_1)\zeta(s_2). \tag{1.7}$$

If we admit the restriction of the domain, we also have another type of functional relation [7,16]. As for the multiple zeta-functions of root systems, it is known that there are some functional relations. One of such relations is given in [5,17,19]. For $k_{12}, k_{13} \in \mathbb{N}$ and $s_{23} \in \mathbb{C}$,

$$\begin{aligned}&\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2).\\&=2\sum_{j_2=0}^{[k_{12}/2]}(-1)^{k_{12}}\binom{k_{12}+k_{13}-1-2j_2}{k_{13}-1}\zeta(2j_2)\zeta(k_{12}+k_{13}+s_{23}-2j_2)\\&\quad+2\sum_{j_3=0}^{[k_{13}/2]}(-1)^{k_{13}}\binom{k_{12}+k_{13}-1-2j_3}{k_{12}-1}\zeta(2j_3)\zeta(k_{12}+k_{13}+s_{23}-2j_3).\end{aligned} \tag{1.8}$$

In particular, for $k_{12}=k_{13}=s_{23}=3$, we have

$$(1-1+1)\zeta_2(3,3,3;A_2)=-40\zeta(0)\zeta(9)-12\zeta(2)\zeta(7). \tag{1.9}$$

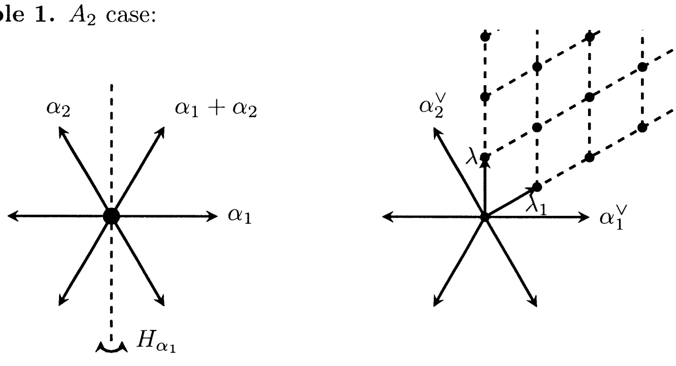
Our main purpose is to generalize this formula, that is, we understand the left-hand side by a group theoretic interpretation and the right-hand side by the Poincaré polynomials. For the details, see the forthcoming paper [14].

## 2 Zeta-Functions of Root Systems

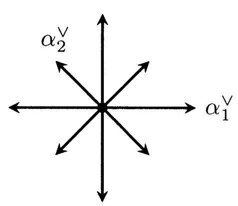
### 2.1 Root Systems

Let $V$ be an $r$ dimensional real vector space with inner product $\langle\cdot,\cdot\rangle$ and $\Delta\subset V$ be a root system. Let $\sigma_\alpha$ be the reflection with respect to the hyperplane $H_\alpha$ orthogonal to $\alpha\in\Delta$ and $W$ be the Weyl group, which is generated by all reflections $\sigma_\alpha$. Let $\alpha^\vee$ be the coroot of $\alpha$, which is equal to $2\alpha/\langle\alpha,\alpha\rangle$ and $\Delta_+$ be the set of all positive roots. Let $\{\alpha_1,\ldots,\alpha_r\}$ be the fundamental roots of $\Delta$, which consists of a basis such that $\alpha=c_1\alpha_1+\cdots+c_r\alpha_r\in\Delta_+$ with all $c_i \geq 0$. Let $P_{++} = \bigoplus \mathbb{Z}_{\geq 1}\lambda_i$ be the set of all strictly dominant weights, where $\{\lambda_1, \ldots, \lambda_r\}$ is a dual basis of $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$. For the geometric meaning of these symbols, see the following example [1].

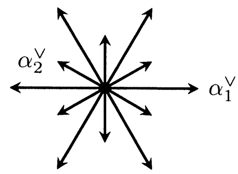
**Example 1.** $A_2$ case:




**Example 2.** $C_2$ case:




$G_2$:




## 2.2 Zeta-Functions of Root Systems

**Definition 1** (Zeta-functions of root systems [3], multivariable Lerch analog)**.** For a root system $\Delta$ and for $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^{|\Delta_+|}$ and $\mathbf{y} \in V$, define

$$\zeta_r(\mathbf{s}, \mathbf{y}; \Delta) = \sum_{\lambda \in P_{++}} e^{2\pi i \langle \mathbf{y}, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}, \tag{2.1}$$

**Example 3.** We obtain the corresponding zeta-functions by formally replacing $\alpha_1^\vee$ and $\alpha_2^\vee$ by $m$ and $n$ appearing in positive coroots. For example, in the root systems of rank 2, we have

$$\zeta_2(\mathbf{s}, \mathbf{y}; A_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i (m y_1 + n y_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3}}, \tag{2.2}$$

$$\zeta_2(\mathbf{s}, \mathbf{y}; C_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i (m y_1 + n y_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}}, \tag{2.3}$$

$$\zeta_2(\mathbf{s}, \mathbf{y}; G_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i (m y_1 + n y_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4} (m+3n)^{s_5} (2m+3n)^{s_6}}. \tag{2.4}$$

Here and hereafter if the root system $\Delta$ is of type $X_r$, we write $\zeta_r(\mathbf{s},\mathbf{y};X_r)$ instead of $\zeta_r(\mathbf{s},\mathbf{y};\Delta)$ for short.

# 3 Special Zeta-Values (Review)

We extend $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}$ to $(s_\alpha)_{\alpha\in\Delta}$ by $s_\alpha=s_{-\alpha}$ and define $(w\mathbf{s})_\alpha=s_{w^{-1}\alpha}$. Then we have the following.

**Theorem 1** (value relations [3,5]). *For* $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

$$\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};\Delta)=(-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr), \qquad (3.1)$$

*where* $P(\mathbf{k},\mathbf{y};\Delta)$ *is a multiple periodic Bernoulli function, which will be defined below.*

**Theorem 2** (special values [3,5]). *For* $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in(2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k}=\mathbf{k}$ *for all* $w\in W$,

$$\zeta_r(\mathbf{k},\mathbf{0};\Delta)=\frac{(-1)^{|\Delta_+|}}{|W|}P(\mathbf{k},\mathbf{0};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr)\in\mathbb{Q}\pi^{\sum_{\alpha\in\Delta_+}k_\alpha}. \qquad (3.2)$$

**Example 4.**

$$\begin{aligned}\zeta(2)&=\frac{-1}{2}\frac{1}{6}\frac{(2\pi i)^2}{2!}=\frac{\pi^2}{6}.\\ \zeta_2((2,4,4,2),\mathbf{0};C_2)&=\sum_{m,n=1}^{\infty}\frac{1}{m^2n^4(m+n)^4(m+2n)^2}\\ &=\frac{(-1)^4}{2^2 2!}\frac{53}{1513512000}\Bigl(\frac{(2\pi i)^2}{2!}\Bigr)^2\Bigl(\frac{(2\pi i)^4}{4!}\Bigr)^2=\frac{53}{6810804000}\pi^{12}.\end{aligned} \qquad (3.3)$$

# 4 Multiple Periodic Bernoulli Functions (Review)

Let $\mathscr{V}$ be the set of all bases $\mathbf{V}\subset\Delta_+$ and $\mathbf{V}^*=\{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$ be the dual basis of $\mathbf{V}^\vee=\{\beta^\vee\}_{\beta\in\mathbf{V}}$. Let $Q^\vee=\bigoplus_{i=1}^r\mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $L(\mathbf{V}^\vee)=\bigoplus_{\beta\in\mathbf{V}}\mathbb{Z}\beta^\vee$. Note that $|Q^\vee/L(\mathbf{V}^\vee)|<\infty$. Fix a certain $\phi\in V$ and define a multiple generalization of the fractional part of real numbers as

$$\{\mathbf{y}\}_{\mathbf{V},\beta}=\begin{cases}\{\langle\mathbf{y},\mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi,\mu_\beta^{\mathbf{V}}\rangle>0),\\ 1-\{-\langle\mathbf{y},\mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi,\mu_\beta^{\mathbf{V}}\rangle<0).\end{cases} \qquad (4.1)$$

**Definition 2** (generating functions [3,5]). For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$,

$$F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{V}\in\mathscr{V}} \Bigl( \prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \Bigr) \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \Bigl( \prod_{\beta\in\mathbf{V}} \frac{t_\beta \exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \Bigr). \tag{4.2}$$

**Definition 3** (multiple periodic Bernoulli functions [3,5]).

$$F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k},\mathbf{y};\Delta) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}. \tag{4.3}$$

*Remark.* The $A_1$ case reduces to the classical generating function:

$$F(t,y) = \frac{te^{t\{y\}}}{e^t-1} = \sum_{k=0}^{\infty} B_k(\{y\})\frac{t^k}{k!}. \tag{4.4}$$

## 5 Functional Relations

Let $I$ be a subset of $\{1,\ldots,r\}$. We will see that this determines which variables are complex. Let $\Delta_I$ be the subroot system of $\Delta$ with the fundamental roots $\{\alpha_i\}_{i\in I}$ and $W^I$ be the minimal coset representatives of $W/W_I$ with the Weyl group $W_I$ of $\Delta_I$, that is, $W = W_I W^I$.

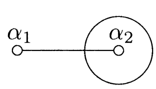
**Theorem 3** (functional relations). *For* $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+}$ *with* $s_\alpha \in \mathbb{C}$ $(\alpha\in\Delta_{I+})$ *and* $s_\alpha = k_\alpha \in \mathbb{Z}_{\geq 2}$ $(\alpha\in\Delta_+\setminus\Delta_{I+})$, *we have*

$$\sum_{w\in W^I} \Bigl( \prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \Bigr) \zeta_r(w^{-1}\mathbf{s}, w^{-1}\mathbf{y};\Delta) = (-1)^{|\Delta_+\setminus\Delta_{I+}|} \Bigl( \prod_{\alpha\in\Delta_+\setminus\Delta_{I+}} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \Bigr) \sum_{\lambda\in P_{I++}} \Bigl( \prod_{\alpha\in\Delta_{I+}} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}} \Bigr) P(\mathbf{k},\mathbf{y},\lambda;I;\Delta), \tag{5.1}$$

*where* $P(\mathbf{k},\mathbf{y},\lambda;I;\Delta)$ *is a multiple periodic Bernoulli function associated with* $I$, *which will be defined below.*

It should be noted that generally, the right-hand side consists of sum of several zeta-functions of lower rank.

**Example 5.** In the root system of type $A_2$, we choose $I = \{2\}$, which we express as the following diagram



(5.2)

where the circled node belongs to $I$. Then we have

$$\begin{aligned}
&\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2)\\
&=(-1)^2\left(\frac{(2\pi i)^{k_{12}}}{k_{12}!}\frac{(2\pi i)^{k_{13}}}{k_{13}!}\right)\\
&\quad\times\sum_{m=1}^{\infty}\frac{1}{m^{s_{23}}}\left(\frac{b_0}{m^{k_{12}+k_{13}}}+\frac{b_2}{m^{k_{12}+k_{13}-2}}+\cdots+\frac{b_j}{m^{k_{12}+k_{13}-2j}}\right),
\end{aligned}\tag{5.3}$$

where $j=\max\{[k_{12}/2],[k_{13}/2]\}$ and $b_0,\ldots,b_j$ are certain real numbers. It should be noted that the right-hand side consists of sum of several Riemann zeta-functions.

To define a multiple periodic Bernoulli function associated with $I$, we need some definitions. Let $\mathscr{V}_I$ be the set of all bases of the form $\mathbf{V}=\mathbf{V}_I\cup\{\alpha_i\mid i\in I\}$ with $\mathbf{V}_I=\{\gamma_1,\ldots,\gamma_d\}\subset\Delta_+\setminus\Delta_{I+}$ and $p_{\mathbf{V}_I^\perp}$ be the projection defined by

$$p_{\mathbf{V}_I^\perp}(v)=v-\sum_{\gamma\in\mathbf{V}_I}\mu_\gamma^{\mathbf{V}}\langle\gamma^\vee,v\rangle\tag{5.4}$$

for $v\in V$.

Then we obtain the following:

**Theorem and Definition 4** (generating function). *For* $\mathbf{t}_I=(t_\alpha)_{\alpha\in\Delta_+\setminus\Delta_{I+}}$ *and* $\lambda\in P_I$*,*

$$\begin{aligned}
F(\mathbf{t}_I,\mathbf{y},\lambda;I;\Delta)&=\sum_{\mathbf{V}\in\mathscr{V}_I}\Bigg(\prod_{\gamma\in\Delta_+\setminus\Delta_{I+}\cup\mathbf{V}_I}\frac{t_\gamma}{t_\gamma-\sum_{\beta\in\mathbf{V}_I}t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle-2\pi\sqrt{-1}\langle\gamma^\vee,p_{\mathbf{V}_I^\perp}(\lambda)\rangle}\Bigg)\\
&\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}\exp(2\pi\sqrt{-1}\langle\mathbf{y}+q,p_{\mathbf{V}_I^\perp}(\lambda)\rangle)\Bigg(\prod_{\beta\in\mathbf{V}_I}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Bigg)\\
&=\sum_{\mathbf{k}\in\mathbb{N}_0^{|\Delta_+\setminus\Delta_{I+}|}}P(\mathbf{k},\mathbf{y},\lambda;I;\Delta)\prod_{\alpha\in\Delta_+\setminus\Delta_{I+}}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}.
\end{aligned}\tag{5.5}$$

In particular, if $I=\emptyset$, $F(\mathbf{t}_I,\mathbf{y},\lambda;I;\Delta)$ reduces to the generating function for value relations:

$$\begin{aligned}
F(\mathbf{t}_\emptyset,\mathbf{y},\lambda;\emptyset;\Delta)=F(\mathbf{t},\mathbf{y};\Delta)&=\sum_{\mathbf{V}\in\mathscr{V}}\Bigg(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}}\frac{t_\gamma}{t_\gamma-\sum_{\beta\in\mathbf{V}}t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle}\Bigg)\\
&\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}\Bigg(\prod_{\beta\in\mathbf{V}}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Bigg).
\end{aligned}\tag{5.6}$$

*Remark.* In the proof of this theorem, we use the results in [12].

# 6 Examples

## 6.1 $A_r$ Case

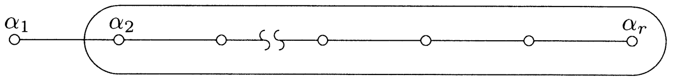
We use the following realization of the root system of type $A_r$:

$$\Delta_+ = \{e_i - e_j \mid 1 \leq i < j \leq r+1\} \subset \mathbb{R}^{r+1}, \qquad (\langle e_i, e_j \rangle = \delta_{ij}). \tag{6.1}$$

Then the zeta-function of type $A_r$ is expressed as

$$\zeta_r((s_{ij})_{1\leq i<j\leq r}, (y_i)_{1\leq i\leq r}; A_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\exp(2\pi\sqrt{-1}\sum_{1\leq i\leq r} m_i y_i)}{\prod_{1\leq i<j\leq r+1}(m_i + \cdots + m_{j-1})^{s_{ij}}}. \tag{6.2}$$

We choose $I = \{2, \ldots, r\}$ and $I^c = \{1\}$ as in the following Dynkin diagram.




(6.3)

Then we have the following theorem:

**Theorem 5** (generating function). *Put* $t_{e_1-e_i} = t_i$ *for* $2 \leq i \leq r+1$.

$$\begin{aligned} &F((t_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \\ &= \sum_{j=2}^{r+1} \prod_{i=2}^{j-1} \frac{t_i}{t_i - t_j + 2\pi\sqrt{-1}(m_i + \cdots + m_{j-1})} \prod_{i=j+1}^{r+1} \frac{t_i}{t_i - t_j - 2\pi\sqrt{-1}(m_j + \cdots + m_{i-1})} \\ &\quad \times \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_i y_i\Big)\Big) \frac{t_j \exp(t_j\{y_1\})}{e^{t_j} - 1}. \end{aligned} \tag{6.4}$$

**Theorem 6** (multiple periodic Bernoulli function).

$$\begin{aligned} &F((t_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \\ &= \sum_{k_2,\ldots,k_{r+1}\geq 0} P((k_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \frac{t_2^{k_2} \cdots t_{r+1}^{k_{r+1}}}{k_2! \cdots k_{r+1}!}, \end{aligned} \tag{6.5}$$

*where*

$$\begin{aligned} &P((k_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \\ &= k_2! \cdots k_{r+1}! \sum_{j=2}^{r+1} \Big(\prod_{\substack{i=2 \\ i\neq j}}^{r+1} \delta_{k_i\neq 0}\Big) \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_i y_i\Big)\Big) \\ &\quad \times \Big( \sum_{\substack{l_2,\ldots,l_{r+1}\geq 0 \\ l_2+\cdots+l_{r+1}=k_j}} \frac{B_{l_j}(\{y_1\})}{l_j!} \prod_{\substack{2\leq i\leq r+1 \\ i\neq j}} (-1)^{k_i-1} \binom{k_i + l_i - 1}{l_i} \Big(\frac{1}{2\pi\sqrt{-1}m_{ij}}\Big)^{k_i+l_i} \Big), \end{aligned} \tag{6.6}$$

*with*

$$m_{ij} = \begin{cases} m_i + \cdots + m_{j-1} & (i < j) \\ -(m_j + \cdots + m_{i-1}) & (i > j). \end{cases} \tag{6.7}$$

**Theorem 7.** *For $(s_{ij})_{1\le i<j\le r+1}$ with $s_{1j} = k_{1j}$ $(2 \le j \le r+1)$, we have*

$$\begin{aligned}
&\sum_{j=0}^{r}\Big(\prod_{i=1}^{j}(-1)^{k_{1,i+1}}\Big)\zeta_r((s_{(1\cdots j+1)pq})_{1\le p<q\le r+1}, (y_2-y_1,\dots,y_{j+1}-y_1,y_{j+1},\dots,y_r); A_r) \\
&= -\sum_{j=2}^{r+1}\sum_{\substack{l_2,\dots,l_{r+1}\ge 0\\ l_2+\cdots+l_{r+1}=k_{1,j}}}(-1)^{k_{1,2}+\cdots+k_{1,j-1}+l_{j+1}+\cdots+l_{r+1}}(2\pi\sqrt{-1})^{l_j}\frac{B_{l_j}(\{y_1\})}{l_j!} \\
&\quad\times\prod_{\substack{2\le i\le r+1\\ i\ne j}}\binom{k_{1,i}+l_i-1}{l_i}\zeta_{r-1}((s_{pq}+\delta_{p<j}\delta_{q=j}(k_{1,p}+l_p)+\delta_{p=j}\delta_{q>j}(k_{1,q}+l_q))_{2\le p<q\le r+1}, \\
&\qquad\qquad (y_2-y_1,\dots,y_{j-1}-y_1,y_j,\dots,y_r); A_{r-1}).
\end{aligned} \tag{6.8}$$

*Remark.* It should be noted that this is a special case. Generally, $\zeta_r(\mathbf{s},\mathbf{y};X_r)$'s are not necessarily described in terms of $\zeta_{r-1}(\mathbf{s},\mathbf{y};X_{r-1})$. It depends on the pair $(X_r, I)$. We need more general multiple zeta-functions, which may not be classified as zeta-functions of root systems.

*Remark.* Other special cases are $(B_r,\{2,\dots,r\})$, $(C_r,\{2,\dots,r\})$.

**Example 6.** Set $r=2$, $(y_1,y_2)=(0,0)$. For $s_{23}\in\mathbb{C}$,

$$\begin{aligned}
&\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2) \\
&= 2\sum_{j_2=0}^{[k_{12}/2]}(-1)^{k_{12}}\binom{k_{12}+k_{13}-1-2j_2}{k_{13}-1}\zeta(2j_2)\zeta(k_{12}+k_{13}+s_{23}-2j_2) \\
&\quad + 2\sum_{j_3=0}^{[k_{13}/2]}(-1)^{k_{12}}\binom{k_{12}+k_{13}-1-2j_3}{k_{12}-1}\zeta(2j_3)\zeta(k_{12}+k_{13}+s_{23}-2j_3).
\end{aligned} \tag{6.9}$$

**Example 7.** Set $r=3$, $(y_1,y_2,y_3)=(0,0,0)$. For $(s_{23},s_{24},s_{34})\in\mathbb{C}^3$,

$$\begin{aligned}
&\zeta_3(k_{12},k_{13},k_{14},s_{23},s_{24},s_{34};A_3)+(-1)^{k_{12}+k_{13}}\zeta_3(s_{23},k_{12},s_{24},k_{13},s_{34},k_{14};A_3) \\
&+(-1)^{k_{12}}\zeta_3(k_{12},s_{23},s_{24},k_{13},k_{14},s_{34};A_3)+(-1)^{k_{12}+k_{13}+k_{14}}\zeta_3(s_{23},s_{24},k_{12},s_{34},k_{13},k_{14};A_3) \\
&= 2\sum_{j_2=0}^{[k_{12}/2]}\sum_{\substack{l_3,l_4\ge 0\\ l_3+l_4=k_{12}-2j_2}}(-1)^{k_{12}}\binom{k_{13}+l_3-1}{l_3}\binom{k_{14}+l_4-1}{l_4} \\
&\quad\times\zeta(2j_2)\zeta_2(s_{23}+k_{13}+l_3,s_{24}+k_{14}+l_4,s_{34};A_2) \\
&+ 2\sum_{j_3=0}^{[k_{13}/2]}\sum_{\substack{l_2,l_4\ge 0\\ l_2+l_4=k_{13}-2j_3}}(-1)^{k_{12}+l_4}\binom{k_{12}+l_2-1}{l_2}\binom{k_{14}+l_4-1}{l_4} \\
&\quad\times\zeta(2j_3)\zeta_2(s_{23}+k_{12}+l_2,s_{24},s_{34}+k_{14}+l_4;A_2)
\end{aligned} \tag{6.10}$$

$$+2\sum_{j_4=0}^{[k_{14}/2]}\sum_{\substack{l_2,l_3\geq 0\\ l_2+l_3=k_{14}-2j_4}}(-1)^{k_{12}+k_{13}}\binom{k_{12}+l_2-1}{l_2}\binom{k_{13}+l_3-1}{l_3}$$
$$\times\zeta(2j_4)\zeta_2(s_{23},s_{24}+k_{12}+l_2,s_{34}+k_{13}+l_3;A_2).$$

## 6.2 Various Expressions

In particular, if $k_{12}=k_{13}=k_{14}=s_{23}=s_{24}=s_{34}=2$,

$$\begin{aligned}4\zeta_3(2,2,2,2,2,2;A_3)&=2\zeta(2)\{2\zeta_2(4,4,2;A_2)+\zeta_2(4,2,4;A_2)\}\\&\quad-6\zeta_2(6,4,2;A_2)-6\zeta_2(6,2,4;A_2)-8\zeta_2(5,5,2;A_2)\\&\quad+4\zeta_2(5,2,5;A_2)-6\zeta_2(4,6,2;A_2).\end{aligned}\tag{6.11}$$

On the other hand, we obtained already in [2, Eq. (4.28)]

$$\begin{aligned}4\zeta_3(2,2,2,2,2,2;A_3)&=8\zeta(2)\{\zeta_2(4,4,2;A_2)+\zeta_2(3,5,2;A_2)\}\\&\quad-12\zeta_2(6,4,2;A_2)+12\zeta_2(5,5,2;A_2)-6\zeta_2(4,6,2;A_2).\end{aligned}\tag{6.12}$$

*Remark.* These two expressions are transformed into each other by use of partial fraction decompositions.

*Remark.* (Open Problem) However in general $A_r$ cases, we have two different expressions of the right-hand side and we do not know whether these two expressions are transformed into each other by use of partial fraction decompositions. Thus these expressions may give new value relations.

## 6.3 $B_r$ Case

**Theorem 8** (generating function for $B_r$ case with $I^c=\{1\}$). *We use the following realization:*

$$\Delta_+=\{e_i\pm e_j\mid 1\leq i<j\leq r\}\cup\{e_j\mid 1\leq j\leq r\}.\tag{6.13}$$

*Put* $t_{e_1\pm e_i}=t_{\pm i}$ *for* $2\leq i\leq r$ *and* $t_{e_1}=t_1$.

$$F(t_1,(t_{\pm i})_{2\leq i\leq r},(y_j)_{1\leq j\leq r},(m_i)_{2\leq i\leq r};\{2,\ldots,r\};B_r)$$
$$=\sum_{j=2}^{r}\prod_{2\leq i<j}\frac{t_{-i}}{t_{-i}-t_{-j}+2\pi\sqrt{-1}(m_i+\cdots+m_{j-1})}\prod_{j<i\leq r}\frac{t_{-i}}{t_{-i}-t_{-j}-2\pi\sqrt{-1}(m_j+\cdots+m_{i-1})}$$
$$\times\prod_{2\leq i\leq j}\frac{t_{+i}}{t_{+i}-t_{-j}-2\pi\sqrt{-1}(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r)}$$
$$\times\prod_{j<i\leq r}\frac{t_{+i}}{t_{+i}-t_{-j}-2\pi\sqrt{-1}(m_j+\cdots+m_{i-1}+2(m_i+\cdots+m_{r-1})+m_r)}$$
$$\times\frac{t_1}{t_1-2t_{-j}-2\pi\sqrt{-1}(2(m_j+\cdots+m_{r-1})+m_r)}$$

$$\times \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i-y_1)+\sum_{i=j}^{r} m_i y_i\Big)\Big)\frac{t_{-j}\exp(t_{-j}\{y_1\})}{e^{t_{-j}}-1}$$

$$+\sum_{j=2}^{r}\prod_{2\le i\le j}\frac{t_{-i}}{t_{-i}-t_{+j}+2\pi\sqrt{-1}(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r)}$$

$$\times\prod_{j<i\le r}\frac{t_{-i}}{t_{-i}-t_{+j}+2\pi\sqrt{-1}(m_j+\cdots+m_{i-1}+2(m_i+\cdots+m_{r-1})+m_r)}$$

$$\times\prod_{2\le i<j}\frac{t_{+i}}{t_{+i}-t_{+j}-2\pi\sqrt{-1}(m_i+\cdots+m_{j-1})}\prod_{j<i\le r}\frac{t_{+i}}{t_{+i}-t_{+j}+2\pi\sqrt{-1}(m_j+\cdots+m_{i-1})}$$

$$\times\frac{t_1}{t_1-2t_{+j}+2\pi\sqrt{-1}(2(m_j+\cdots+m_{r-1})+m_r)}$$

$$\times \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i-y_1)+\sum_{i=j}^{r-1} m_i(y_i-2y_1)+m_r(y_r-y_1)\Big)\Big)\frac{t_{+j}\exp(t_{+j}\{y_1\})}{e^{t_{+j}}-1}$$

$$+\prod_{2\le i\le r}\frac{t_{-i}}{t_{-i}-t_1+\pi\sqrt{-1}(2(m_i+\cdots+m_{r-1})+m_r)}$$

$$\times\prod_{2\le i\le r}\frac{t_{+i}}{t_{+i}-t_1-\pi\sqrt{-1}(2(m_i+\cdots+m_{r-1})+m_r)}$$

$$\times\frac{1}{2}\Big(\exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{r-1} m_i(y_i-y_1)+m_r(y_r-\frac{1}{2}y_1)\Big)\Big)\frac{t_1\exp(t_1\{\frac{1}{2}y_1\})}{e^{t_1}-1}$$

$$+\exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{r-1} m_i(y_i-(y_1+1))+m_r(y_r-\frac{1}{2}(y_1+1))\Big)\Big)\frac{t_1\exp(t_1\{\frac{1}{2}(y_1+1)\})}{e^{t_1}-1}\Big)$$

Note that by expanding this expression, we see that we obtain functional relations among $\zeta_r(\cdot;B_r)$ and $\zeta_{r-1}(\cdot;B_{r-1})$ similar to those in the case of type $A_r$ obtained in Theorem 7.

## 6.4 $X_r$ with $|I|=1$ Case

In the case $|I|=1$, we will see that the sum of some $\zeta_r(\cdot;X_r)$ is expressed in terms of Lerch zeta-functions. Let $\phi(u,s)$ be the Lerch zeta-function defined by

$$\phi(u,s)=\sum_{n=1}^{\infty}\frac{e^{2\pi\sqrt{-1}un}}{n^s}. \tag{6.14}$$

**Theorem 9.** *Let $s_\alpha=k_\alpha\in\mathbb{Z}_{\ge 2}$ for $\alpha\in\Delta_+\setminus\{\alpha_i\}$ and $s_{\alpha_i}\in\mathbb{C}$. Let $|\mathbf{k}|=\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}}k_\alpha$. Let $X_i=\{\nu=\{\langle q,\mu^{\mathbf{V}}_{\alpha_i}\rangle\}\mid \mathbf{V}\in\mathscr{V}_I, q\in Q^\vee/L(\mathbf{V}^\vee)\}\subset\mathbb{Q}$.*

$$\sum_{w\in W^I}\Big(\prod_{\alpha\in\Delta_{w^{-1}}}(-1)^{-k_\alpha}\Big)\zeta_r(w^{-1}\mathbf{s},0;\Delta)$$

$$=(-1)^{|\Delta_+|-1}\Big(\prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\Big)\sum_{\nu\in X_i}\sum_{j=0}^{|\mathbf{k}|}\frac{b_{\mathbf{k}\nu j}}{(2\pi\sqrt{-1})^j}\phi(\nu,s_{\alpha_i}+j), \tag{6.15}$$

*where* $b_{\mathbf{k}\nu j} \in \mathbb{Q}$ *is given by*

$$\sum_{\mathbf{k}\in\mathbb{N}_0^{|\Delta^*|}} \sum_{\nu\in X_i} \sum_{j=0}^{|\mathbf{k}|} b_{\mathbf{k}\nu j} x^j y^\nu \prod_{\alpha\in\Delta^*} \frac{t_\alpha^{k_\alpha}}{k_\alpha!} = \sum_{\mathbf{V}\in\mathscr{V}_I} \prod_{\gamma\in\Delta^*\setminus\mathbf{V}_I} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}_I} t_\beta \langle\gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle - \langle\gamma^\vee, \mu_{\alpha_i}^{\mathbf{V}}\rangle / x}$$
$$\times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} y^{\{\langle q, \mu_{\alpha_i}^{\mathbf{V}}\rangle\}} \prod_{\gamma\in\mathbf{V}_I} \frac{t_\gamma \exp(t_\gamma \{q\}_{\mathbf{V},\gamma})}{e^{t_\gamma} - 1}. \tag{6.16}$$

## 7 A Remarkable Theorem

It is natural that from functional relations we obtain value relations; we have only to substitute integers into variables. However it is remarkable that the converse holds, that is, the generating function for $I = \emptyset$ knows "everything." The following theorem tells that $F(\mathbf{t}_I, \mathbf{y}, \lambda; I; \Delta)$ for general $I$ can be deduced from the case $I = \emptyset$.

**Theorem 10** (Remarkable Theorem). *Let* $I \subset \{1, \dots, r\}$. *For* $\lambda \in P_{I++}$, *we have*

$$F(\mathbf{t}_I, \mathbf{y}, \lambda; I; \Delta) = \operatorname*{Res}_{\substack{t_\alpha = 2\pi\sqrt{-1}\langle\alpha^\vee, \lambda\rangle \\ \alpha\in\Delta_{I+}}} \Bigl( \prod_{\alpha\in\Delta_{I+}} \frac{1}{t_\alpha} \Bigr) F(\mathbf{t}, \mathbf{y}; \Delta). \tag{7.1}$$

## 8 Poincaré Polynomials and Special Zeta-Values

For $\mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in (\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ satisfying $w^{-1}\mathbf{k} = \mathbf{k}$ for all $w \in W^I$, the left-hand side of (5.1) is

$$\sum_{w\in W^I} \Bigl( \prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \Bigr) \zeta_r(w^{-1}\mathbf{k}, \mathbf{0}; \Delta) = \Bigl( \sum_{w\in W^I} \prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \Bigr) \zeta_r(\mathbf{k}, \mathbf{0}; \Delta). \tag{8.1}$$

From this expression, we notice that the coefficient of $\zeta_r(\mathbf{k}, \mathbf{0}; \Delta)$ coincides with the special value $W^I(((-1)^{k_\alpha})_{\alpha\in\Delta_+})$ of the Poincaré polynomial for $W^I$, where the Poincaré polynomials due to Macdonald are defined as follows [15]: For indeterminates $\mathbf{u} = (u_\alpha)_{\alpha\in\Delta_+}$ and for $X \subset W$

$$X(\mathbf{u}) = \sum_{w\in X} \prod_{\alpha\in\Delta_+\cap w\Delta_-} u_\alpha. \tag{8.2}$$

Since generally it is very difficult to calculate special values of these Poincaré polynomials, we need their simple descriptions.

### 8.1 Poincaré polynomials

It is known [15] that if $u_\alpha = u$ for all $\alpha \in \Delta_+$,

$$W^I(\mathbf{u}) = \frac{W(\mathbf{u})}{W_I(\mathbf{u})}, \tag{8.3}$$

with

$$W(\mathbf{u})=\prod_{i=1}^{r}\frac{u^{d_i}-1}{u-1},\qquad W_I(\mathbf{u})=\prod_{i\in I}\frac{u^{d'_i}-1}{u-1},\tag{8.4}$$

where $d_i$ and $d'_i$ are the degrees of the Weyl groups $W$ and $W_I$, and these degrees are given as in the following table.

| Type | $\{d_1,\ldots,d_r\}$ | Type | $\{d_1,\ldots,d_r\}$ |
|---|---|---|---|
| $A_r$ | $2,3,4,\ldots,r+1$ | $E_7$ | $2,6,8,10,12,14,18$ |
| $B_r,C_r$ | $2,4,\ldots,2r$ | $E_8$ | $2,8,12,14,18,20,24,30$ |
| $D_r$ | $2,4,\ldots,2r-2,r$ | $F_4$ | $2,6,8,12$ |
| $E_6$ | $2,5,6,8,9,12$ | $G_2$ | $2,6$ |

From these facts, we see that if $u_\alpha=u$ for all $\alpha\in\Delta_+$,

$$W^I(\mathbf{u})=\sum_{w\in W^I}\prod_{\alpha\in\Delta_+\cap w\Delta_-}u_\alpha=\frac{\prod_{i=1}^{r}(u^{d_i}-1)/(u-1)}{\prod_{i\in I}(u^{d'_i}-1)/(u-1)}.\tag{8.5}$$

## 8.2 Case 1 (all even)

Consider the case $u_\alpha=(-1)^{k_\alpha}=1$ for all $\alpha\in\Delta_+$. Then by l'Hôpital's rule, we obtain

$$W^I(1)=|W^I|=\frac{\prod_{i=1}^{r}d_i}{\prod_{i\in I}d'_i}\in\mathbb{Z}_{\geq 1}.\tag{8.6}$$

**Example 8** ($A_2$ with $I=\{2\}$). In this case, $\Delta$ is of type $A_2$ and hence $d_1=2,d_2=3$ and $\Delta_I$ is of type $A_1$ and hence $d'_1=2$. Put $s_{ij}=k_{ij}=2m$ (even). Then the left-hand side of (5.1) is directly calculated as

$$\begin{aligned}&1\cdot\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2)\\&=(1+(-1)^{k_{12}}+(-1)^{k_{12}+k_{13}})\zeta_2(2m,2m,2m;A_2)\\&=3\cdot\zeta_2(2m,2m,2m;A_2).\end{aligned}\tag{8.7}$$

On the other hand this coefficient is calculated via Poincaré polynomials as

$$W^I(1)=\frac{d_1d_2}{d'_1}=3.\tag{8.8}$$

## 8.3 Case 2 (all odd)

Consider the case $u_\alpha=(-1)^{k_\alpha}=-1$ for all $\alpha\in\Delta_+$. Let $K=\{i\mid 1\leq i\leq r,d_i\in 2\mathbb{Z}\}$, $K_I=\{i\mid i\in I,d'_i\in 2\mathbb{Z}\}$. Then

$$W^I(-1)=\begin{cases}\dfrac{\prod_{i\in K}d_i}{\prod_{i\in K_I}d'_i}\in\mathbb{Z}_{\geq 1} & (|K|=|K_I|)\\ 0 & (|K|\neq|K_I|).\end{cases}\tag{8.9}$$

The following is a table of several examples where $W^I(-1)$ survives.

| Type of $\Delta$ | Type of $\Delta_I$ | $W^I(-1)$ |
|---|---|---|
| $A_{2m}$ | $A_{2m-1}$ | $2\cdot 4\cdots 2m/2\cdot 4\cdots 2m=1$ |
| $A_3$ | $A_1^2$ | $2\cdot 4/2\cdot 2=2$ |
| $D_{2m+1}$ | $D_{2m}$ | $2\cdot 4\cdots 4m/2\cdot 4\cdots (4m-2)\cdot 2m=2$ |
| $E_6$ | $D_4$ | $2\cdot 6\cdot 8\cdot 12/2\cdot 4\cdot 6\cdot 4=6$ |

**Example 9** ($A_2$ with $I=\{2\}$). In this case, $\Delta$ is of type $A_2$ and $\Delta_I$ is of type $A_1$ as in the previous example. Put $s_{ij}=k_{ij}=2n+1$ (odd). Then the left-hand side of (5.1) is directly calculated as

$$\begin{aligned}
&1\cdot\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2)\\
&=(1+(-1)^{k_{12}}+(-1)^{k_{12}+k_{13}})\zeta_2(2m,2m,2m;A_2)\\
&=1\cdot\zeta_2(2m,2m,2m;A_2).
\end{aligned}\tag{8.10}$$

On the other hand this coefficient is obtained from the above table as

$$W^I(-1)=1.\tag{8.11}$$

## 8.4 Case 3 (Mixture)

Let $\Delta_1$ be the set of all long roots and $\Delta_2$, that of all short roots. Assume $k_\alpha$ are odd for $\alpha\in\Delta_1$ and $k_\beta$ are even for $\beta\in\Delta_2$, and hence $u_\alpha=-1$ for $\alpha\in\Delta_1$ and $u_\beta=1$ for $\beta\in\Delta_2$.

**Lemma 11.** *Let* $\mathbf{u}=(u,1)$. *Then we have*

$$W^I(\mathbf{u})=\frac{W(u,1)}{W_I(u,1)}=\frac{|W_J|W(\Delta_1)(u)}{|W_{I\cap J}|W(\Delta_1\cap\Delta_I)(u)}.\tag{8.12}$$

The following is a table of some examples, where $W^I(-1,1)$ survives.

| Type of $\Delta$ | Type of $\Delta_I$ | $W^I(-1,1)$ |
|---|---|---|
| $B_{2k+1}$ | $B_{2k}$ | $2\cdot 2\cdot 4\cdots 4k/2\cdot 2\cdot 4\cdots (4k-2)\cdot 2k=2$ |
| $C_{2k+1}$ | $C_{2k}$ | $2\cdot 2\cdot 4\cdots 4k/2\cdot 2\cdot 4\cdots (4k-2)\cdot 2k=2$ |
| $G_2$ | $A_1$ | $2\cdot 2/2=2$ |

**Example 10.** Let $\Delta$ be of type $G_2$, and $\Delta_I$ be of type $A_1$. Let $p=u=v$ be even and $s=q=r$, odd. Then the left-hand side of (5.1) is directly calculated as

$$\begin{aligned}
&\zeta_2(p,s,q,r,u,v;G_2)+(-1)^p\zeta_2(p,q,s,r,v,u;G_2)+(-1)^{p+q}\zeta_2(v,q,r,s,p,u;G_2)\\
&\quad+(-1)^{p+q+v}\zeta_2(v,r,q,s,u,p;G_2)+(-1)^{p+q+r+v}\zeta_2(u,r,s,q,v,p;G_2)\\
&\quad+(-1)^{p+q+r+u+v}\zeta_2(u,s,r,q,p,v;G_2)\\
&=2\zeta_2(p,q,q,q,p,p;G_2).
\end{aligned}\tag{8.13}$$

On the other hand this coefficient is obtained from the above table as

$$W^I(-1,1)=2. \tag{8.14}$$

This recovers the result in [13].

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