On discrete universality of Hurwitz zeta functions

見正　秀彦　(Hidehiko Mishou)
東京電機大学

1 Introduction

In 1910s, Bohr initiated the investigation of value distribution of the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\quad \text{for } \sigma>1,$$

where $s=\sigma+it$ denotes a complex variable and the symbol p denotes a prime number as usual. First he [2] showed that the set

$$\left\{\zeta(\sigma+it)\in\mathbb{C}\;\middle|\;\sigma>1,\ t\in\mathbb{R}\right\}$$

is dense in the set $\mathbb{C}$ of all complex numbers. Later Bohr and Courant [3] showed that for any fixed $1/2<\sigma_0<1$ the set

$$\{\zeta(\sigma_0+it)\in\mathbb{C}\mid t\in\mathbb{R}\}$$

is dense in $\mathbb{C}$. In 1975, Voronin [13] extended this denseness result to the infinite dimensional space, that is, the functional space and obtained the remarkable universality theorem. To state it in modern form which was established by Bagchi [1], we define a probability measure on $\mathbb{R}$. Let μ be the Lebesgue measure on the set $\mathbb{R}$ of all real numbers. For $T>0$ define

$$\nu_T(\cdots)=\frac{1}{T}\mu\left\{\tau\in[0,T]\ :\cdots\right\},$$

where in place of dots we write some conditions satisfied by a real number τ.

Theorem 1 (Voronin, [13]). *Let K be a compact subset in the strip $\frac{1}{2}<\sigma<1$ with connected complement and $f(s)$ be a non-vanishing and continuous function on K which is analytic in the interior of K. Then for any small positive number ε we have*

$$\liminf_{T\to\infty}\nu_T\left(\max_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right)>0.$$

This theorem asserts roughly that any analytic function can be approximated uniformly by suitable vertical translation of $\zeta(s)$.

These results have been developed into various directions by several mathematicians. Here we will describe one of such derivative studies, discrete value distribution of zeta functions.

The first result in this direction was obtained by Voronin [12]. He showed that for any fixed $\delta > 0$, $\frac{1}{2} < \sigma_0 \leq 1$ and $N \in \mathbb{N}$, the set

$$\left\{(\zeta(\sigma_0 + i\delta n), \ldots, \zeta^{(N-1)}(\sigma_0 + i\delta n)) \in \mathbb{C}^N \mid n \in \mathbb{N}\right\}$$

is dense in $\mathbb{C}^N$. This means that the multi-dimensional denseness result of the Riemann zeta function holds for the arithmetic progression $\{\delta n \mid n \in \mathbb{N}\}$. In 1980, Reich [11] established the discrete universality theorem for Dedekind zeta functions with respect to arithmetic progressions. Later, Dubickas and Laurinčikas [4] established the discrete universality theorem for the Riemann zeta function with respect to the sequence $\{\delta n^\eta \mid n \in \mathbb{N}\}$, where η is a positive real number with $\eta < 1$.

In the following, we treat only one sequence Γ of real numbers which is deeply related to the Riemann zeta function $\zeta(s)$ itself. As usual, $\rho = \beta + i\gamma$ denotes a non-trivial zero of $\zeta(s)$. For $x > 1$, E. Landau [10] established the following formula.

$$\sum_{\substack{\rho=\beta+i\gamma \\ 0<\gamma\leq T}} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(\log T), \tag{1}$$

where $\Lambda(x)$ is the extended von Mangoldt function

$$\Lambda(x) = \begin{cases} \log p & (x = p^k,\ k \geq 1), \\ 0 & (\text{otherwise}), \end{cases}$$

and the error term depends on x. Now we assume the Riemann hypothesis, which asserts that

$$\beta = \frac{1}{2} \tag{2}$$

for all non-trivial zeros ρ. Combining (1), (2) and the zero density estimate

$$N(T) := \sharp\left\{\rho = \beta + i\gamma \mid 0 < \gamma \leq T\right\} = \frac{1}{2\pi} T \log T + O(T),$$

we have

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} x^{i\delta\gamma} \longrightarrow 0 \quad \text{as} \quad T \to \infty,$$

for any $x > 1$ and a positive constant δ. This implies that the set Γ of all positive imaginary parts of non-trivial zeros of the Riemann zeta function is uniformly distributed modulo 1. From this, we could show that the set

$$\left\{\zeta(\sigma + i\delta\gamma) \in \mathbb{C} \mid \sigma > 1,\ \gamma \in \Gamma\right\}$$

is dense in $\mathbb{C}$ for any positive number δ. Recently, Garunkštis and Laurinčikas [6] obtained the next result

Theorem 2 (Garunkštis and Laurinčikas [6]). *Suppose that the Riemann hypothesis holds. Let* $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ *be the positive imaginary parts of non-trivial zeros of* $\zeta(s)$. *Let a set* K *and a function* $f(s)$ *be as in Theorem 1. Then for any small positive number* ε *we have*

$$\liminf_{N\to\infty} \frac{1}{N+1} \sharp \left\{ 1 \leq k \leq N \;\middle|\; \max_{s\in K} |\zeta(s+i\gamma_k) - f(s)| < \varepsilon \right\} > 0.$$

They proved this theorem using the explicit version of Landau's formula (1) due to Gonek [8] and [9]. From more stronger formula due to Fujii [5], the author established the following joint discrete universality theorem for Dirichlet L-functions.

Theorem 3 (Mishou, Palanga Conference in 2016, Lithuania). *Assume that the Riemann hypothesis holds. Let* δ *be a positive constant satisfying* $\delta \leq 1$. *Let* $\chi_1, \cdots, \chi_r$ *be pairwise non-equivalent Dirichlet characters. For each* $1 \leq j \leq r$, *let* K_j *be a compact subset in* $\frac{1}{2} < \sigma < 1$ *with connected complement and* $f_j(s)$ *be a non-vanishing and continuous function on* K_j *which is analytic in the interior of* K_j. *Then for any small positive number* ε *we have*

$$\liminf_{T\to\infty} \frac{1}{N(T)} \sharp \left\{ 0 < \gamma \leq T \;\middle|\; \max_{1\leq j\leq r} \max_{s\in K_j} |L(s+i\delta\gamma, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

For a real number α with $0 < \alpha \leq 1$, the Huriwitz zeta function is defined by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}$$

for $\sigma > 1$. Now we state our main result, which is the discrete universality theorem for Hurwitz zeta functions.

Theorem 4. *Assume that the Riemann hypothesis holds. Let* δ *be a positive constant satisfying* $\delta \leq 1$. *Let* $0 < \alpha < 1$ *be a real number which is rational without* $1/2$ *or transcendental. Let* K *be a compact subset in* $\frac{1}{2} < \sigma < 1$ *with connected complement and* $f(s)$ *be a continuous function on* K *which is analytic in the interior of* K. *Then for any small positive number* ε *we have*

$$\liminf_{T\to\infty} \frac{1}{N(T)} \sharp \left\{ 0 < \gamma \leq T \;\middle|\; \max_{s\in K} |\zeta(s+i\delta\gamma, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

2 Outline of the proof of Theorem 4

In this section we sketch the proof of Theorem 4. First we consider the case that α is a rational number $\frac{a}{q}$ without $1/2$. Then the Hurwitz zeta function is represented as a sum of Dirichlet L-functions

$$\zeta(s, \frac{a}{q}) = \frac{q^s}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi}(a) L(s,\chi).$$

From this expression and Theorem 3 we could easily obtain the discrete universality for $\zeta(s, \frac{a}{q})$.

Next we consider the case that α is a real transcendental number. In this case, the set of Dirichlet exponents $\{\log(m+\alpha) \mid m \geq 0\}$ of $\zeta(s,\alpha)$ is linearly independent over $\mathbb{Q}$. Also, as we stated in §1, the set

$$\Gamma = \{\text{positive imaginary parts of non-trivial zeros of } \zeta(s)\}$$

is uniformly distributed modulo 1. From these two properties, we have the following lemma.

Lemma 1. *Let δ be a positive real number. For any $N \in \mathbb{N}$, the set*

$$\left\{\left(-\frac{\delta\log\alpha}{2\pi}\gamma, -\frac{\delta\log(1+\alpha)}{2\pi}\gamma, \dots, -\frac{\delta\log(N+\alpha)}{2\pi}\gamma\right) \in \mathbb{R}^{N+1} \,\middle|\, \gamma \in \Gamma\right\}$$

is uniformly distributed in $[0,1]^{N+1}$ modulo 1. Namely, for $T > 0$, $0 < \eta < 1$, a sequence $\{\theta_m\}$ of real numbers with $0 \leq \theta_m < 1$ define a subset $A_{N,\eta}(T)$ of Γ by

$$A_{N,\eta}(T) = \left\{0 < \gamma < T \,\middle|\, \left\|-\frac{\delta\log(m+\alpha)}{2\pi}\gamma - \theta_m\right\| \leq \eta \quad (0 \leq m \leq N)\right\},$$

where $\|x\| = \min_{n\in\mathbb{Z}}|x-n|$. Then we have

$$\frac{\sharp A_{N,\eta}(T)}{N(T)} = (2\eta)^{N+1} \quad (T \to \infty). \tag{3}$$

Here we remark that Lemma 1 holds for all positive δ. Next we prepare the denseness lemma obtained by Gonek [7].

Lemma 2. *Let a set K and a function $f(s)$ be as in Theorem 4. For any $\varepsilon > 0$, there exists a sequence $\{\theta_m\}$ with $0 \leq \theta_m < 1$ and $N_0 > 0$ such that if $N > N_0$ we have*

$$\max_{s\in K}\left|f(s) - \sum_{m\leq N}\frac{e(\theta_m)}{(m+\alpha)^s}\right| < \varepsilon,$$

where $e(x) = e^{2\pi ix}$.

Lemma 3. *Assume that δ be a positive real number with $\delta \leq 1$. For $T > 0$ and $z > 0$, define a subset $B_z(T)$ of Γ by*

$$B_z(T) = \left\{0 < \gamma < T \,\middle|\, \max_{s\in K}\left|\zeta(s+i\delta\gamma,\alpha) - \sum_{m\leq z}\frac{1}{(m+\alpha)^{s+i\delta\gamma}}\right| < \varepsilon\right\}. \tag{4}$$

For any $\varepsilon > 0$ and any $\varepsilon' > 0$, there exists $z_0 > 0$ such that if $z > z_0$ we have

$$\lim_{T\to\infty}\frac{\sharp B_z(T)}{N(T)} > 1 - \varepsilon'. \tag{5}$$

This lemma implies that for almost all γ the attached Hurwitz zeta function $\zeta(s+i\delta\gamma,\alpha)$ is uniformly approximated by truncated Dirichlet polynomials. To prove Lemma 3, we need the following explict version of Landau formula due to Fujii [5].

Lemma 4 (Fujii [5]). *Assume that the Riemann hypothesis holds. For $x > 1$ and $T > T_0$, we have*

$$\sum_{0<\gamma\le T} x^{\frac{1}{2}+i\gamma} = -\frac{T}{2\pi}\Lambda(x) + \sqrt{x}\cdot M(x,T)$$
$$+ O(x\log(2x)) + O\left(\log x \min\left(x, \frac{x}{\langle x\rangle}\right)\right) + O\left(x\sqrt{\frac{\log T}{\log\log T}}\right)$$
$$+ O\left(x^{\frac{1}{2}+\frac{1}{\log\log T}}\cdot \log(2x)\cdot \frac{\log T}{\log\log T}\right) + O\left(x\log(2x)\log\log(3x)\right),$$

where $\langle x\rangle$ is the distance from x to the neartest prime power other than x itself and

$$M(x,T) = \frac{1}{2\pi}\int_1^T x^{it}\log\left(\frac{t}{2\pi}\right)dt.$$

To prove Lemma 3, we consider the second mean sum

$$I := \sum_{0<\gamma<T}\iint_K \left|\zeta(s+i\delta\gamma,\alpha) - \sum_{m\le z}\frac{1}{(m+\alpha)^{s+i\delta\gamma}}\right|^2 d\sigma dt.$$

Applying the approximate functional equation of the Hurwitz zeta function and Lemma 4, we have

$$I \ll N(T)z^{1-2\sigma_1+\varepsilon} + N(T)T^{1-2\sigma_1+\varepsilon}, \tag{6}$$

where $\sigma_1 = \min_{s\in K}\Re s > \frac{1}{2}$. This implies Lemma 3. Here we remark that to obtain estimate (6), we need the restriction $\delta \le 1$. If $\delta > 1$, the error terms arise from Lemma 4 become too large.

As in the proof of Lemma 3, we could obtain the next lemma.

Lemma 5. *Suppose that δ be a positive real number with $\delta \le 1$. Let K be a compact subset of the strip $\frac{1}{2} < \sigma < 1$. Let $\sigma_1 > \frac{1}{2}$ satisfying $K \subset \{s\in\mathbb{C} \mid \sigma > \sigma_1\}$. Let $\varepsilon > 0$. Then there exists a large positive integer $N_1 = N_1(K,\sigma_1,\varepsilon)$ depending on K, σ_1 and ε satisfying the following:*
Fix any positive integer $N > N_1$. For any $T > 0$, $0 < \eta < 1$ and a sequence $\{\theta_m\}$ of real numbers with $0 \le \theta_m < 1$, define a subset $A_{N,\eta}(T)$ of Γ as in Lemma 1. For any $z > N$ define a subset $C_{N,\eta}(T)$ of $A_{N,\eta}(T)$ by

$$C_{N,\eta}(T) := \left\{\gamma\in A_{N,\eta}(T) \;\middle|\; \max_{s\in K}\left|\sum_{N<m\le z}\frac{1}{(m+\alpha)^{s+i\delta\gamma}}\right| \ll_K N^{1-2\sigma_1}.\right\}.$$

Then for all T sufficiently large we have

$$\frac{\sharp C_{N,\eta}(T)}{N(T)} > \frac{1}{2}(2\eta)^{N+1}. \tag{7}$$

Now we prove Theorem 4. Let a set K and a function $f(s)$ be as in Theorem 4. Let $\varepsilon > 0$. Lemma 2 implies that there exist a large positive integer N_0 and a sequence $\{\theta_m\}$ of real numbers with $0 \leq \theta_m \leq 1$ such that for any $N > N_0$ we have

$$\max_{s \in K} \left| f(s) - \sum_{m \leq N} \frac{e(\theta_m)}{(m+\alpha)^s} \right| < \varepsilon. \tag{8}$$

Now we fix $N > \max\{N_0, N_1\}$ satisfying

$$C_K N^{1-2\sigma_1} < \varepsilon,$$

where C_K is the O-constant in Lemma 5. By the definition of the subset $A_{N,\eta}(T)$ in Lemma 1 and the continuity of Dirichlet polynomials, we can choose a sufficiently small positive real number η such that

$$\max_{s \in K} \left| \sum_{m \leq N} \frac{e(\theta_m)}{(m+\alpha)^s} - \sum_{m \leq N} \frac{1}{(m+\alpha)^{s+i\delta\gamma}} \right| < \varepsilon \tag{9}$$

holds for all $\gamma \in A_{N,\eta}(T)$. In Lemma 3, we take $\varepsilon' = \frac{1}{4}(2\eta)^{N+1}$ and fix $z > \max\{z_0, N\}$. From (5) and (7), we have,

$$\frac{\sharp(B_z(T) \cap C_{N,\eta}(T))}{N(T)} > \frac{1}{2}(2\eta)^{N+1} - \frac{1}{4}(2\eta)^{N+1} = \frac{1}{4}(2\eta)^{N+1}$$

This means that $B_z(T) \cap C_{N,\eta}(T)$ has a positive lower density. For all $\gamma \in B_z(T) \cap C_{N,\eta}(T)$, we have (9), (4) in Lemma 3 and

$$\max_{s \in K} \left| \sum_{N < m \leq z} \frac{1}{(m+\alpha)^{s+i\delta\gamma}} \right| \ll_K N^{1-2\sigma_1} < \varepsilon.$$

Combining these estimates and (8), we obtain Theorem 4. □

3 A Conjecture

In Theorem 3 and Theorem 4, the restriction

$$\delta \leq 1$$

arises from the technical reason. On the other hand, as we stated in §1, the denseness result of the set

$$\{\zeta(\sigma + i\delta\gamma) \in \mathbb{C} \mid \sigma > 1, \ \gamma \in \Gamma\}$$

holds for any positive δ. Therefore it is expected that these universality theorems also hold for $\delta > 1$. Especially, if the discrete universality theorem of $\zeta(s)$ holds for $\delta = 2$, we have an interesting result on value-distribution of multiple zeta functions.

Let u and v be complex variables. The Euler - Zagier double sum is defined by

$$Z_2(u,v) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^u(m+n)^v}$$

for $\Re u > 1$ and $\Re v > 1$ and is meromorphically continued to the whole complex space $\mathbb{C}^2$. The function $Z_2(u,v)$ is one of the most classical multiple zeta functions. The study of multiple zeta functions have been extensively developed by many mathematicians on zeta-values and analytic continuation. Meanwhile, there are a few results on value distribution of the multiple zeta functions. By the definition of $\zeta(s)$ and $\zeta_2(u,v)$, we have the relation

$$\zeta(u)\zeta(v) = \zeta(u+v) + Z_2(u,v) + Z_2(v,u).$$

Now we assume that the Riemann hypothesis holds. If we put $u = v = \rho = \frac{1}{2} + i\gamma$, we have

$$Z_2(\rho,\rho) = -\frac{1}{2}\zeta(1+2\gamma i).$$

This relation predicts the next conjecture

Conjecture 1. The set

$$\left\{ Zeta_2(\rho,\rho) \in \mathbb{C} \mid \zeta(\rho) = 0 \right\}$$

is dense in $\mathbb{C}$.

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Tokyo Denki University,
5 Senju-Asahi-Cho, Adachi-ku, Tokyo, 120-8551, Japan
E-mail address: h_mishou@mail.dendai.ac.jp