Transcendence of zeros of certain weakly holomorphic modular forms

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1 Introduction

The Eisenstein series $E_k(z)$ is perhaps the easiest example of modular forms. In 1970, F.K.C. Rankin and Swinnerton-Dyer [10] showed that all zeros of the Eisenstein series $E_k(z)$ in the standard fundamental domain for $SL_2(\mathbb{Z})$ lie on the arc for all weight $k \geq 4$. Simlar results for other modular forms were proved [9, 11, 12].

In 2008, Duke and Jenkins [2] constructed a canonical basis $f_{k,m}$ for the space of weakly holomorphic modular forms for level 1. Let Δ be the Ramanujan Δ function and j be weight 0 modular function which is known simply as the j-function. The basis $f_{k,m}$ is defined by

$$f_{k,m} = \Delta^{\ell} E_{k'} F_{k,D}(j)$$

where $k = 12\ell + k'$, $k' \in \{0, 4, 6, 8, 10, 14\}$ and $F_{k,D}(x)$ is a monic polynomial in x of degree $D = \ell + m$. The basis $f_{k,m}$ have the following form

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}).$$

They considered $f_{k,m}$ as a two-parameter family of weakly holomorphic modular forms that is a canonical basis for the space and proved almost all of the basis elements have all of their zeros on a lower boundary of the standard fundamental domain for $SL_2(\mathbb{Z})$. We consider the locations of the zeros for weakly holomorphic modular forms $g_{k,m}$ of level 1 defined by

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z) \Delta(z)^j,$$

where $a_j \in \mathbb{R}$.

2 Definitions and statements of results

Let $k \in 2\mathbb{Z}$, N be a prime number or 1, and $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod{N}\}$. Put $\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}$, and $q = e^{2\pi i z}$ for $z \in \mathbb{H}$.

A holomorphic function f on $\mathbb{H}$ is a weakly holomorphic modular form of weight k with respect to $\Gamma_0(N)$ if f satisfies the following two conditions:

- $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.
- $f(z) = \sum_{n \geq n_0} a(n) q^n$ and $\dfrac{1}{z^k} f\left(-\dfrac{1}{z}\right) = \sum_{n \geq n_1} b(n) q^{\frac{n}{N}}$ with $a(n_0) \neq 0$ and $b(n_1) \neq 0$.

We define f is holomorphic if $n_0 \geq 0$ and $n_1 \geq 0$, a cusp form if $n_0 \geq 1$ and $n_1 \geq 1$. We denote the space of holomorphic modular form of weight k on $\Gamma_0(N)$ by $M_k(N)$, the space of weakly holomorphic modular forms by $M_k^!(N)$. Put $M_k = M_k(1)$ and $M_k^! = M_k^!(1)$ in this paper.

Duke and Jenkins considered an explicit basis of $M_k^!$ which is indexed by the order of the pole at ∞ in [2]. Let $k = 12\ell + k'$ where $\ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. For any integer $m \geq -\ell$, there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!$ which has an expansion

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}). \tag{1}$$

We note that there exists a unique $f_{k,m} \in M_k^!$ with the expansion (1). For any $f = \sum a(n) q^n \in M_k^!$, we can write

$$f = \sum_{n_0 \leq n \leq \ell} a(n) f_{k,-n}$$

when we know first few Fourier coefficients of f. Therefore we see that $\{f_{k,m}\}_{m \geq -\ell}$ form a natural basis of $M_k^!$.

We define three modular forms to construct the basis $\{f_{k,m}\}_{m \geq -\ell}$. Bernoulli numbers B_k and $\sigma_{k-1}(n)$ are each defined by

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!}, \qquad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Then the Ramanujan Δ function, Eisenstein series E_k and j function are each defined by

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \quad (k \geq 4), \; E_0 = 1,$$

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + \sum_{n\geq 1}c(n)q^n.$$

Their weights are each 12, k, 0 and orders at ∞ are each 1, 0, -1. The function $f_{k,m}$ is constructed by

$$f_{k,m} = \Delta^{\ell}E_{k'}F_{k,D}(j)$$

where $F_{k,D}(x)$ is a monic polynomial in x of degree $D = \ell + m$.

For the group $SL_2(\mathbb{Z})$, we use a fundamental domain in the upper half-plane bounded by the lines $\Re(z) = -\frac{1}{2}$ and $\Re(z) = \frac{1}{2}$, the circles of radius 1 centered at $z = 0$. We include the boundary on the left half of this fundamental domain. The cusps of this fundamental domain can be taken to be at ∞.

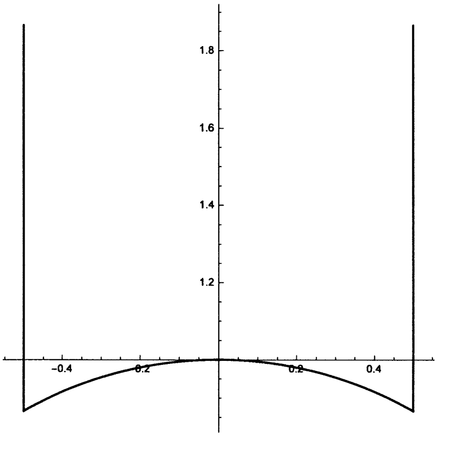

Figure 1: A fundamental domain for $SL_2(\mathbb{Z})$.

The description of the zeros of a weakly holomorphic modular form $f \in M_k^!$ on $\mathbb{H}$ is clearly equivalent to the description of the zeros of f on $\mathcal{F}$. Thus, for the remainder of this paper, when we speak of a zero z_0 of $f \in M_k^!$, we assume $z_0 \in \mathcal{F}$.

We define four constants by $\delta_1 = 0.432207, \delta_2 = 0.024975, \delta_3 = 0.004807$ and $\delta_4 = 0.257348$. Then we define $\gamma(j)$ and $A_{k'}$ by

$$\gamma(j) = \begin{cases} \delta_3^j \delta_1^{\ell-j} & \text{if } 1 \le j \le \ell, \\ \delta_2^j \delta_3^\ell & \text{if } \ell+1 \le j \le \ell+m. \end{cases} \qquad A_{k'} = \begin{cases} 2.76009 & \text{if } k'=0, \\ 0.684214 & \text{if } k'=4, \\ 0.950549 & \text{if } k'=6, \\ 0.184724 & \text{if } k'=8, \\ 0.258108 & \text{if } k'=10, \\ 0.075404 & \text{if } k'=14. \end{cases}$$

We note here

$$\begin{aligned} \left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)} \right| &\le \delta_1, \\ \left|\Delta\left(x+0.65i\right)\right| &\le \delta_2, \\ \left|\Delta\left(e^{i\theta}\right)\right| &\le \delta_3, \\ e^{-2\pi m(\sin\theta - 0.65)} &\le \delta_4 \end{aligned}$$

$$\text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{1}{\Delta\left(x+0.65i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right) - j\left(e^{i\theta}\right)} \right| dx \le A_{k'},$$

for $\theta \in [1.9, 2\pi/3]$ and $x \in [-1/2, 1/2]$. Then we have the following theorem.

Theorem 2.1. *Let $k = 12\ell + k'$, where $\ell \in \mathbb{Z}_{\ge 0}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. Let*

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z) \Delta(z)^j,$$

where $a_j \in \mathbb{R}, m \ge 0$ and $\ell + m \ge 1$. If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j| (3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'},$$

then all of the zeros of $g_{k,m}$ in the fundamental domain for $SL_2(\mathbb{Z})$ lie on the circle $|z| = 1$.

Besides, we consider transcendence of zeros of $g_{k,m}$. We have the following theorem.

Theorem 2.2. *Let $k = 12\ell + k'$, where $\ell \in \mathbb{Z}_{\geq 0}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. Let*

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j,$$

where $a_j \in \mathbb{Q}, m \geq 0$ and $\ell + m \geq 1$. If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j|(3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'},$$

then all of zeros of $g_{k,m}$ in the fundamental domain for $SL_2(\mathbb{Z})$ are transcendental or equal to i or $\rho = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$.

3 Sketch of proof of Theorem 2.1

Applying the valence formula for $k = 12\ell + k'$, there are at most $\ell + m$ zeros on $\mathcal{F} - \{\rho, i\}$. Thus if $g_{k,m} \in M_k^!$ satisfies the hypotheses of Theorem 2.1, then to prove Theorem 2.1 it suffices to demonstrate that $g_{k,m}$ has $\ell + m$ simple zeros in $\{e^{i\theta} : \frac{\pi}{2} < \theta < \frac{2\pi}{3}\}$.

An easy argument [4, Proposition 2.1] shows that for any weakly holomorphic modular form f of weight k with real coefficients, the quantity $e^{ik\theta/2} f(e^{i\theta})$ is real for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. Thus, we approximate $e^{ik\theta/2} g_{k,m}(e^{i\theta})$ by an elementary function having the required number of zeros on the arc.

Suppose $\ell \geq 1$ and $m \geq 1$. Then we set

$$H(\theta) = e^{ik\theta/2} e^{-2\pi m \sin\theta} g_{k,m}\left(e^{i\theta}\right) = H_{0,m}(\theta) + \sum_{j=1}^{\ell+m} a_j e^{12ji\theta/2} \Delta\left(e^{i\theta}\right)^j H_{j,m}(\theta),$$

where $H_{j,m}(\theta) = e^{(k-12j)i\theta/2} e^{-2\pi m \sin\theta} f_{k-12j,m}\left(e^{i\theta}\right)$. We define the function $R_{j,m}(\theta)$ for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ by

$$H_{j,m}(\theta) = 2\cos\left(\frac{(k-12j)\theta}{2} - 2\pi m \cos\theta\right) + R_{j,m}(\theta).$$

We seek a bound for the function $R_{j,m}(\theta)$. Details for the computation of the numerical bounds is given as with [3, 5]. By the argument in [2],

$$|R_{j,m}(\theta)| = \left| e^{-2\pi m \sin\theta} \int_{-\frac{1}{2}+\alpha'}^{\frac{1}{2}+\alpha'} \frac{\Delta^{\ell-j}(z)}{\Delta^{1+\ell-j}(\tau)} \frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)} e^{-2\pi i m\tau} d\tau \right|.$$

When $1.9 \leq \theta \leq 2\pi/3$, we have

$$|R_{j,m}(\theta)| \leq \frac{e^{-\pi m(2\sin\theta - \tan(\theta/2))}}{(2\cos(\theta/2))^k} + e^{-2\pi m(\sin\theta - 0.65)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|G_j(x+0.65i, e^{i\theta})\right| dx,$$

where

$$G_j(\tau, z) = \frac{\Delta^{\ell-j}(z)}{\Delta^{1+\ell-j}(\tau)} \frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)}.$$

Looking at the first term, for $\theta \in [1.9, 2\pi/3]$ and $m \geq 0$, we have

$$\left| \frac{e^{-\pi m(2\sin\theta - \tan(\theta/2))}}{(2\cos(\theta/2))^k} \right| \leq 1.$$

Considering the exponential term $e^{-2\pi m(\sin\theta - 0.65)}$, it is bounded above by 0.257348 for $\theta \in [1.9, 2\pi/3]$. We set $\delta_4 = 0.257348$.

We next seek a bound for $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|G_j(x+0.65i, e^{i\theta})\right| dx$. This integral is equal to

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)} \right|^{\ell-j} \left| \frac{1}{\Delta\left(x+0.65i\right)} \right| \left| \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right) - j\left(e^{i\theta}\right)} \right| dx.$$

First, we consider

$$\left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)} \right|^{\ell-j}.$$

We have

$$0.002691 \leq \left|\Delta\left(e^{i\theta}\right)\right| \leq 0.004807.$$

We set $\delta_3 = 0.004807$. We compute that

$$0.011122 \leq \left|\Delta\left(x+0.65i\right)\right| \leq 0.024975.$$

We set $\delta_2 = 0.024975$. Putting this together, we have, for $\ell \geq j$,

$$\left|\frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)}\right|^{\ell-j} \leq |0.432207|^{\ell-j}.$$

We set $\delta_1 = 0.432207$.

Next, we consider

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{\Delta\left(x+0.65i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right) - j\left(e^{i\theta}\right)}\right| dx.$$

We will break our path of integration into small pieces, and consider $j(\tau)$ in relation to $j(z)$ on each. We can bound the quotient by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{\Delta\left(x+0.65i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right) - j\left(e^{i\theta}\right)}\right| dx \leq A_{k'},$$

where

$$A_{k'} = \begin{cases} 2.76009 & \text{if } k' = 0, \\ 0.684214 & \text{if } k' = 4, \\ 0.950549 & \text{if } k' = 6, \\ 0.184724 & \text{if } k' = 8, \\ 0.258108 & \text{if } k' = 10, \\ 0.075404 & \text{if } k' = 14. \end{cases}$$

Putting all of these pieces together, we see that

$$|R_{j,m}(\theta)| \leq 1 + \delta_4^m \delta_1^{\ell-j} A_{k'}$$

for $1 \leq j \leq \ell$ and

$$|R_{j+\ell,m}(\theta)| \leq 1 + \delta_4^m \left|\frac{\delta_2}{\Delta\left(e^{i\theta}\right)}\right|^j A_{k'}$$

for $1 \leq j \leq m$.

Similarly, for $\theta \in [\pi/2, 1.9)$, we can bound $|R_{j,m}(\theta)|$. We note that the bound of $|R_{j+\ell,m}(\theta)|$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$ is larger than for $\frac{\pi}{2} \leq \theta < 1.9$ since $|R_{j+\ell,m}(\theta)| > 1$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$. Therefore we also use the bound of $|R_{j+\ell,m}(\theta)|$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$ when $\frac{\pi}{2} \leq \theta < 1.9$.

$\left|H(\theta) - 2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)\right|$ is bounded above by

$$\begin{aligned} &|R_{0,m}(\theta)| + \sum_{j=1}^{\ell} |a_j| \left(2 + |R_{j,m}(\theta)|\right) \left|\Delta\left(e^{i\theta}\right)\right|^j \\ &\qquad + \sum_{j=1}^{m} |a_{j+l}| \left(2 + |R_{j+\ell,m}(\theta)|\right) \left|\Delta\left(e^{i\theta}\right)\right|^{j+\ell} \\ \leq\ & 1 + \delta_4^m \delta_1^\ell A_{k'} + \sum_{j=1}^{\ell+m} |a_j| \left(3\delta_3^j + \delta_4^m \gamma^j(j) A_{k'}\right). \end{aligned}$$

Now suppose

$$\sum_{j=1}^{\ell+m} |a_j|(3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'}.$$

Then we have

$$\left|H(\theta) - 2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)\right| < 2.$$

This inequality is enough to prove the theorem. To see this, note that as θ increases from $\pi/2$ to $2\pi/3$, the quantity

$$\frac{k\theta}{2} - 2\pi m \cos\theta$$

increases from $\pi\left(3\ell + k'/4\right)$ to $\pi\left(3\ell + k'/3 + D\right)$, where $D = \ell + m$, hitting $D + 1$ distinct consecutive integer multiples of π (this is independent of the choice of k'). A short computation shows that if $D \geq |\ell|$, then the quantity $\frac{k\theta}{2} - 2\pi m \cos\theta$ is strictly increasing on this interval. Thus, there are exactly $D + 1$ values of θ in the interval $[\pi/2, 2\pi/3]$ where the function

$$2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)$$

has absolute value 2, alternating between $+2$ and -2 as θ increases. Then real-valued function $H(\theta)$ must have at least D distinct zeros as θ moves through the interval $(\pi/2, 2\pi/3)$. This accounts for all D nontrivial zeros of $g_{k,m}$.

4 Proof of Theorem 2.2

For the proof of Theorem 2.2, we use the following lemma of Schneider.

Lemma 4.1. *[8, Corollary 3.4] If $z \in \mathbb{H}$ and $j(z)$ is algebraic, then either z is transcendental or z is imaginary quadratic, i.e. $\mathbb{Q}(z)$ is a degree 2 extension of $\mathbb{Q}$, with $z \notin \mathbb{R}$.*

We can prove Theorem 2.2 as with [6]. We have the following lemma.

Lemma 4.2. *[6, Lemma 2.2] Let $a, b, c \in \mathbb{Z}$ such that $a > 0$, $\gcd(a, b, c) = 1$, and $D = b^2 - 4ac < 0$. If $z \in \mathbb{H}$ is a root of the polynomial $ax^2 + bx + c$, then the lattice $[1, z]$ is a proper fractional ideal of the order $\mathfrak{O} = [1, az]$ of $K = \mathbb{Q}(\sqrt{D})$. Moreover,*

$$\mathfrak{O} = \begin{cases} \frac{i\sqrt{D}}{2} & \text{if } D \equiv 0 \pmod{4}, \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

We find that the order $\mathfrak{O}$ does not depend on z, but instead on the discriminant D of the reduced integer polynomial that has z as a root. Recall, if Λ is a lattice of C we define $j(\Lambda) = j(z)$, where $z \in \mathbb{H}$ and $\Lambda = [1, z]$. The choice of $z \in \mathbb{H}$ is well defined. By Lemma 4.2, we see that we can map a point $z \in \mathbb{H}$ to the proper fractional ideal $\Lambda = [1, z]$ of $\mathfrak{O}$, where $j([1, z]) = j(z)$.

The following lemma follows from [1, Theorem 11.1 and Proposition 13.2], and is the last result we need before the proof of Theorem 2.2.

Lemma 4.3. *[6, Lemma 2.3] If $\mathfrak{A}$ is a proper fractional ideal of an order $\mathfrak{O}$ of an imaginary quadratic field K, then $j(\mathfrak{A})$ is an algebraic over $\mathbb{Q}$. If $\mathfrak{B}$ is any other proper fractional ideal of $\mathfrak{O}$, then $K(j(\mathfrak{A})) = K(j(\mathfrak{B}))$ and $j(\mathfrak{A})$ and $j(\mathfrak{B})$ are conjugate over K. Furthermore, the degree of $j(\mathfrak{A})$ is the class number of $\mathfrak{O}$.*

Let $g_{k,m}(z)$ satisfy the assumption of Theorem 2.2. Then we can write

$$\begin{aligned} g_{k,m}(z) &= f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j \\ &= \Delta(z)^\ell E_{k'}(z) F_{k,L}(j(z)), \end{aligned}$$

where $F_{k,L}(j(z))$ is a monic polynomial in $j(z)$ of degree $L = \ell + m$ with rational number coefficients. By Kohnen [7], the only possible zeros of $E_{k'}(z)$

are i and ρ. Also, we see from the valence formula that $\Delta(z)$ is never zero on $\mathbb{H}$. Thus, the only zeros of $g_{k,m}(z)$ in $\mathcal{F}$ other than i, ρ are the zeros of $F_{k,L}(j(z))$.

Suppose $z_0 \in \mathcal{F}$ such that $F_{k,L}(j(z_0)) = 0$. Since $F_{k,L}(x)$ is a polynomial with rational number coefficients, $j(z_0)$ is algebraic. Thus from Lemma 4.1, z_0 is either transcendental or imaginary quadratic.

If z_0 is imaginary quadratic, then z_0 is a root of a polynomial $P(x) = ax^2 + bx + c$, where $\gcd(a, b, c) = 1, a > 0$, and the discriminant $D_0 = b^2 - 4ac < 0$. Let $K = \mathbb{Q}(\sqrt{D_0})$.

We consider the order $\mathfrak{D} = [1, az_0]$ of K. From Lemma 4.2, the lattice $[1, z_0]$ is a proper fractional ideal of $\mathfrak{D}$, and the order $\mathfrak{D}$ has the form

$$\mathfrak{D} = \begin{cases} \left[1, \frac{i\sqrt{D_0}}{2}\right] & \text{if } D_0 \equiv 0 \pmod 4, \\ \left[1, \frac{1+\sqrt{D_0}}{2}\right] & \text{if } D_0 \equiv 1 \pmod 4. \end{cases}$$

Thus by Lemma 4.3, if $\mathfrak{A}$ is any other proper fractional ideal of $\mathfrak{D}$, $j(z_0) = j([1, z_0])$ and $j(\mathfrak{A})$ are conjugate.

We consider the point $z_1 \in \mathbb{C}$ defined by

$$z_1 = \begin{cases} \frac{i\sqrt{|D_0|}}{2} & \text{if } D_0 \equiv 0 \pmod 4, \\ \frac{1+i\sqrt{|D_0|}}{2} & \text{if } D_0 \equiv 1 \pmod 4. \end{cases}$$

Then $z_1 \in \mathcal{F}$ and we have $[1, z_1] = \mathfrak{D}$. Thus by definition $[1, z_1]$ is a proper fractional ideal of $\mathfrak{D}$, and so $j(z_0)$ and $j(z_1)$ are conjugate.

We take an automorphism σ of $K(j(\mathfrak{D}))$ such that $\sigma(j(z_0)) = j(z_1)$. Since σ acts as the identity on $\mathbb{Q}$ and $F_{k,L}$ is a polynomial with rational number coefficients, we have that

$$\begin{aligned} 0 &= \sigma(0) \\ &= \sigma(F_{k,L}(j(z_0))) \\ &= F_{k,L}(\sigma(j(z_0))) \\ &= F_{k,L}(j(z_1)). \end{aligned}$$

Thus z_1 is also a zero of $F_{k,L}$ and hence a zero of $g_{k,m}$. Since $z_1 \in \mathcal{F}$, by Theorem 2.1 we have that z_1 must lie on the arc of the unit circle given by

$$\left\{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3} \right\}.$$

Suppose $D_0 \equiv 0 \pmod 4$, so that $D_0 = -4n$ for some positive integer n. Then $z_1 = i\sqrt{n}$, but since z_1 must lie on the unit circle we must have $n = 1$. Thus, $D_0 = -4$. Since $z_0 \in \mathbb{H}$, we have by the quadratic formula that

$$z_0 = \frac{-b + 2i}{2a}.$$

But $z_0 \in \mathcal{F}$, and so $\Im(z_0) \geq \frac{\sqrt{3}}{2}$. Thus $a = 1$, and so

$$z_0 = -\frac{b}{2} + i.$$

But again by Theorem 2.1 we have that z_0 must lie on the unit circle, so $b = 0$ and $z_0 = i$.

.If $D_0 \equiv 1 \pmod 4$, then $D_0 = -4n + 1$ for some positive integer n. Hence,

$$z_1 = \frac{-1 + i\sqrt{4n - 1}}{2},$$

and thus $|z_1|^2 = n$. Again, since z_1 must lie on the unit circle we must have $n = 1$. Therefore $D_0 = -3$. Since $z_0 \in \mathbb{H}$, we have that

$$z_0 = \frac{-b + i\sqrt{3}}{2a}$$

by the quadratic formula. And again since $z_0 \in \mathcal{F}$, we have $a = 1$ so that

$$z_0 = -\frac{b}{2} + i\frac{\sqrt{3}}{2}.$$

But again by Theorem 2.1 we have that z_0 must lie on the unit circle, so $b = 1$ and $z_0 = \rho$. Thus, we completed Theorem 2.2.

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