On prime vs. prime power pairs

Yuta Suzuki

1 Introduction

In his famous address at the 5th International Congress of Mathematicians, Landau [11] listed four problems in prime number theory, which are sometimes called Landau's problems. These problems are:

1. Does the function u^2+1 represent infinitely many primes for integers u?
2. Does the equation $m=p+p'$ have for any even $m>2$ a solution in primes?
3. Does the equation $2=p-p'$ have infinitely many solutions in primes?
4. Does at least one prime exist between n^2 and $(n+1)^2$ for any positive integer n?

The present note is related to the first three problems from Landau's list.

Landau's third problem is well-known as the twin prime problem. Let

$$\Psi(X,h)=\sum_{n\le X}\Lambda(n)\Lambda(n+h), \tag{1}$$

where h is a positive integer and $\Lambda(n)$ is the von Mangoldt function. This function $\Psi(X,h)$ counts the number of twin prime pairs, i.e. prime pairs (p,p') satisfying the twin prime equation

$$p'=p+h, \tag{2}$$

which slightly generalizes the twin prime problem. Although Landau confessed that his problems seem unattackable at the state of science at his time, Hardy and Littlewood introduced a new method, which is called now the circle method, and gave some important attacks against problems on prime numbers. By applying their method formally, Hardy and Littlewood found an hypothetical asymptotic formula

$$\Psi(X,h)=\mathfrak{S}(h)X+(\text{Error}) \tag{3}$$

for even h, where $\mathfrak{S}(h)$ is the singular series for the twin prime problem defined by

$$\mathfrak{S}(h)=\prod_{p|h}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid h}\left(1-\frac{1}{(p-1)^2}\right).$$

In this note, we call this type of hypothetical asymptotic formula *the Hardy-Littlewood asymptotic formula.* Note that the Bateman-Horn conjecture [3] gives a much wider picture on the distribution of prime numbers. Since $\mathfrak{S}(h) \gg 1$, the Hardy-Littlewood asymptotic formula (3) gives a positive answer to the twin prime problem. Unfortunately, any rigorous proof of (3) seems quite far from our current state of science. However, some average behavior of $\Psi(X,h)$ have been obtained by many researchers. As for the twin prime problem, Mikawa [13] or Perelli and Pintz [17] obtained the current best result:

Theorem A (Mikawa [13], Perelli and Pintz [17]). *Let $X, H, A \geq 2$, and $\varepsilon > 0$. Assume*

$$X^{1/3+\varepsilon} \leq H \leq X.$$

Then we have

$$\Psi(X,h) = \mathfrak{S}(h)X + O(XL^{-A})$$

for all but $\ll HL^{-A}$ even numbers $h \in [1, H]$.

Since the original twin prime problem is the case $h = 2$, we are interested in restricting h to some small neighborhood of $h = 2$. Namely, our goal is to obtain the result under the situation "the larger X with the smaller h". In this note, we consider this kind of average results for the Hardy-Littlewood asymptotic formulas.

We next consider Landau's first problem. Let

$$\Psi_k(X,h) = \sum_{n^k \leq X} \Lambda(n^k + h), \tag{4}$$

where $k \geq 2$ is a positive integer. This function counts the number of pairs (n^k, p) satisfying the equation

$$p = n^k + h, \tag{5}$$

which generalizes Landau's first problem. Note that if the polynomial $X^k + h \in \mathbb{Q}[X]$ is reducible, then the equation (5) has only a finite number of solutions. Thus we introduce

$$\mathbf{Irr}_k = \left\{ h \in \mathbb{N} \mid X^k + h \text{ is irreducible over } \mathbb{Q} \right\}.$$

As for this equation, the Hardy-Littlewood asymptotic formula is given by

$$\Psi_k(X,h) = \mathfrak{S}_k(h)X^{1/k} + (\text{Error}) \tag{6}$$

for $h \in \mathbf{Irr}_k$, where the singular series $\mathfrak{S}_k(h)$ is given by

$$\mathfrak{S}_k(h) = \prod_p \left(1 - \frac{r_k(h,p) - 1}{p - 1}\right),$$

$$r_k(h,p) = \left|\left\{ x \ (\mathrm{mod}\ p) \mid x^k + h \equiv 0 \ (\mathrm{mod}\ p) \right\}\right|.$$

The average result for this problem is obtained recently by [1, 2, 8]. We note that as for the "conjugate" equation

$$N = p + n^k,$$

some results were obtained earlier by [14, 18, 19], and it seems straightforward to apply these earlier work to the function $\Psi_k(X,h)$ and give the same result as in [2] or even better results than those of [1, 8]. We have to mention that the interesting method used in [2] is completely different from the earlier work. Namely, Baier and Zhao showed that Linnik's dispersion method is sometimes applicable to our problem, which is originally attacked by the circle method in earlier work. As a result of these work, the current best result is:

Theorem B (Perelli and Zaccagnini [19]). *Let $X, H, A \geq 2$, and $\varepsilon > 0$. Assume*

$$X^{1-1/k+\varepsilon} \leq H \leq X.$$

Then we have

$$\Psi_k(X,h) = \mathfrak{S}_k(h)X^{1/k} + O(X^{1/k}L^{-A})$$

for all but $\ll HL^{-A}$ integers $h \in [1,H] \cap \mathbf{Irr}_k$.

In this note, we consider a kind of mixture of the above two problems. Namely, we consider the "prime vs. prime power" pairs (p^k, p') satisfying the equation

$$p' = p^k + h \tag{7}$$

which can be regarded as a mixture of equations (2) and (5). We introduce the sets

$$\mathbb{H}_k^{\mathrm{local}} = \{\, h \in \mathbb{N} \mid \forall p\colon \mathrm{prime},\ (p-1)|k \Rightarrow h \not\equiv -1 \pmod{p} \,\},$$
$$\mathbb{H}_k = \mathbb{H}_k^{\mathrm{local}} \cap \mathbf{Irr}_k.$$

As for this equation (7), the counting function is given by

$$\Psi_k^*(X,h) = \sum_{n^k \leq X} \Lambda(n)\Lambda(n^k + h),$$

and the Hardy-Littlewood asymptotic formula takes the form

$$\Psi_k^*(X,h) = \mathfrak{S}_k^*(h)X^{1/k} + (\mathrm{Error}) \tag{8}$$

for $h \in \mathbb{H}_k$, where

$$\mathfrak{S}_k^*(h) = \prod_{p|h}\left(1 + \frac{1}{p-1}\right)\prod_{p\nmid h}\left(1 - \frac{(w_k(h,p)-1)p+1}{(p-1)^2}\right),$$

$$w_k(h,p) = \left|\{\, x \pmod{p} \mid x^k + h \equiv 0 \pmod{p},\ (x,p) = 1 \,\}\right|. \tag{9}$$

As for the equation (7), Liu and Zhan [12] obtained a result for the case $k = 2$, and Bauer [4] generalized their result to general k:

Theorem C (Bauer [4]). *Let $X, H, A \geq 2$, and $\varepsilon > 0$. Assume*

$$X^{1-1/2k+\varepsilon} \leq H \leq X.$$

Then we have

$$\Psi_k^*(X,h) = \mathfrak{S}_k^*(h)X^{1/k} + O(X^{1/k}L^{-A})$$

for all but $\ll HL^{-A}$ integers $h \in [1,H] \cap \mathbb{H}_k$.

We remark that the results in [4, 12] are stated with the conjugate equation

$$N = p^k + p'. \tag{10}$$

The aim of this note is to improve this result of Bauer. In particular, we have

Theorem 1. *Let $X, H, A \geq 2$, and $\varepsilon > 0$. Assume*

$$X^{1-1/k+\varepsilon} \leq H \leq X.$$

Then we have

$$\Psi_k^*(X,h) = \mathfrak{S}_k^*(h)X^{1/k} + O(X^{1/k}L^{-A})$$

for all but $\ll HL^{-A}$ integers $h \in [1,H] \cap \mathbb{H}_k$.

As it can be easily predicted, our method is also applicable to the conjugate equation (10). Moreover, our method gives a minor variant of the proof of Theorem B, i.e. our method is applicable to somewhat broader context than the method in [19]. Although our method gives an improvement of Theorem C, it has some disadvantage compared with [4, 12, 19]. Briefly speaking, our method can not be applied to the restricted counting function. See the last section of the preprint [20].

Our method is inspired by the work [4, 15, 16, 17]. In particular, the idea of Mikawa [15] or its variant of Mikawa and Peneva [16] gives our strategy for the treatment of the minor arcs. In these work [15, 16], the minor arc estimates are reduced in an efficient way to some Vinogradov-type estimates for sums over prime numbers. In our case, we shall reduce the minor arcs estimate for the equation (7) to the minor arc estimate for the twin prime equation (2) which is given by Mikawa [13] or by Perelli and Pintz [17]. See Sections 6 and 7. We also remark that the origin of the technique in this note can be traced back to Brüdern and Watt [7].

Since the details of the proof was given in the preprint [20], we describe our method by using the particular case $k = 2$ as an example in this note.

2 Notation

Throughout the letters α, η denote real numbers, $X, Y, H, U, V, P, Q, R, A, B, \varepsilon$ denote positive real numbers, m, n, d, h, u, N denote integers, p denotes a prime number, and $L = \log X$. For any real number α, let $e(\alpha) = e^{2\pi i\alpha}$. The arithmetic function $\varphi(n)$ denotes the Euler totient function, $\Lambda(n)$ denotes the von Mangoldt function, $\mu(n)$ denotes the Möbius function, and $\tau(n)$ is the number of divisors of n. The letters a, q denote positive integers satisfying $(a,q) = 1$ and the expressions

$$\sum_{a \ (\mathrm{mod}\ q)}^{*}, \quad \coprod_{a \ (\mathrm{mod}\ q)}^{*}$$

denote a sum and a disjoint sum over all reduced residues a (mod q) respectively.

We use the following trigonometric polynomials:

$$S_1(\alpha) = \sum_{n\le 3X} \Lambda(n)e(n\alpha), \quad V_1(\eta) = \sum_{n\le 3X} e(n\eta),$$

$$S_2(\alpha) = \sum_{n^2\le X} \Lambda(n)e(n^2\alpha), \quad V_2(\eta) = \frac{1}{2}\sum_{n\le X} n^{-1/2}e(n\eta),$$

We introduce the following complete exponential sums

$$C_k(a,q) = \sum_{m \pmod q}^{*} e\left(\frac{am^k}{q}\right), \quad A_k(n,q) = \sum_{a \pmod q}^{*} \overline{C_k(a,q)}e\left(-\frac{an}{q}\right)$$

for $k = 1, 2$. Note that if $(a,q) = 1$, then the exponential sum $C_1(q,a)$ is reduced to the Möbius function $\mu(q)$. Then we introduce the remainder terms

$$R_k(\eta,a,q) = S_k\left(\frac{a}{q}+\eta\right) - \frac{C_k(a,q)}{\varphi(q)}V_k(\eta)$$

and the truncated singular series

$$\mathfrak{S}_2^*(h,P) = \sum_{q\le P} \frac{\mu(q)A_2(h,q)}{\varphi(q)^2}.$$

We assume $B \ge B_0(A)$, where $B_0(A)$ is some positive constant depends only on A. The implicit constants may depend on A, B, ε.

3 The Farey dissection

As usual, we can deduce Theorem 1 with $k = 2$ from the following L^2-estimate:

Theorem 2. *Let $X, H, A, B \ge 2$, $U \ge 0$, $\varepsilon > 0$, and $P = L^B$. Assume*

$$X^{1/2+\varepsilon} \le H \le X, \quad 0 \le U \le X.$$

Then for sufficiently large $B \ge B_0(A)$, we have

$$\sum_{U<h\le U+H} \left|\Psi_2^*(X,h) - \mathfrak{S}_2^*(h,P)X^{1/2}\right|^2 \ll HXL^{-4A}, \tag{11}$$

where the implicit constant depends on A, B, ε.

We can assume, without loss of generality, that U and H are positive integers and that $H \le X^{4/5}$, which makes the proof of Theorem 3 simpler.

By the orthogonality of additive characters we have

$$\Psi_2^*(X,h) = \int_0^1 S_1(\alpha)\overline{S_2(\alpha)}e(-h\alpha)d\alpha + O(X^{1/2}L^{-3A}) \tag{12}$$

for any $h \le U + H \le 2X$. We use the Farey dissection given by

$$P = L^B, \quad Q = H^{1/2}, \quad R = XP^{-5}, \quad I = \left[Q^{-1}, 1 + Q^{-1}\right],$$
$$\mathfrak{M}_{a,q} = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right], \quad \mathfrak{M}'_{a,q} = \left[\frac{a}{q} - \frac{1}{qR}, \frac{a}{q} + \frac{1}{qR}\right],$$
$$\mathfrak{M} = \coprod_{q \le P} \ \coprod_{a \pmod q}^{*} \mathfrak{M}'_{a,q}, \quad \mathfrak{m} = I \setminus \mathfrak{M}.$$

Then by the integral expression (12), we have

$$\begin{aligned}
&\sum_{U<h\le U+H} \left|\Psi_2^*(X,h) - \mathfrak{S}_2^*(h,P)X^{1/2}\right|^2 \\
\ll &\sum_{U<h\le U+H} \left|\int_{\mathfrak{M}} S_1(\alpha)\overline{S_2(\alpha)}e(-h\alpha)d\alpha - \mathfrak{S}_2^*(h,P)X^{1/2}\right|^2 \\
&\quad + \sum_{U<h\le U+H} \left|\int_{\mathfrak{m}} S_1(\alpha)\overline{S_2(\alpha)}e(-h\alpha)d\alpha\right|^2 + HXL^{-6A} \\
= &\sum_{\mathfrak{M}} + \sum_{\mathfrak{m}} + HXL^{-6A}, \text{ say.}
\end{aligned}$$

4 Preliminary lemmas

We first approximate trigonometric polynomials $S_k(\alpha)$ in a standard way.

Lemma 1. *We have*

$$S_k\left(\frac{a}{q} + \eta\right) = \frac{C_k(a,q)}{\varphi(q)}V_k(\eta) + O\left(q(1+|\eta|X)X^{1/k}P^{-16}\right)$$

for $k = 1, 2$.

Proof. This follows from the Siegel-Walfisz theorem [9, Corollary 5.29]. □

We next recall some basic facts on the complete exponential sums. For the detailed proofs and discussions, see Section 4 and 5 of [5].

Lemma 2 ([5, Lemma 4.3 (b)]). *Suppose that* $(q_1, q_2) = 1$. *Then*

$$A_2(h, q_1q_2) = A_2(h, q_1)A_2(h, q_2).$$

Lemma 3 ([5, Lemma 4.4 (a)]). *For any prime* p, *we have*

$$A_2(h, p) = p \cdot w_2(h, p) - \varphi(p),$$

where $w_2(h, p)$ *is given by* (9).

5 The major arcs

In this section, we shall evaluate the integral over the major arcs. We have

$$\int_{\mathfrak{M}} = \sum_{q\le P} \sideset{}{^*}\sum_{a \pmod q} e\left(-\frac{ah}{q}\right) \int_{|\eta|\le 1/qR} S_1\left(\frac{a}{q}+\eta\right) \overline{S_2\left(\frac{a}{q}+\eta\right)} e(-h\eta)d\eta,$$

which we denote by

$$= \sum_{q\le P} \sideset{}{^*}\sum_{a \pmod q} e\left(-\frac{ah}{q}\right) J_{a,q}(h).$$

We approximate each integral $J_{a,q}(h)$ by decomposing into the following parts:

$$J_{a,q}(h) = A_{a,q}(h) + B_{a,q}(h) + C_{a,q}(h) + I_{a,q}(h),$$

where

$$\begin{aligned}
A_{a,q}(h) &= \int_{|\eta|\le 1/qR} S_1\left(\frac{a}{q}+\eta\right) \overline{R_2(\eta,a,q)} e(-h\eta)d\eta,\\
B_{a,q}(h) &= \frac{\overline{C_2(a,q)}}{\varphi(q)} \int_{|\eta|\le 1/qR} R_1(\eta,a,q)\overline{V_2(\eta)} e(-h\eta)d\eta,\\
C_{a,q}(h) &= -\frac{\mu(q)\overline{C_2(a,q)}}{\varphi(q)^2} \int_{1/qR<|\eta|\le 1/2} V_1(\eta)\overline{V_2(\eta)} e(-h\eta)d\eta,\\
I_{a,q}(h) &= \frac{\mu(q)\overline{C_2(a,q)}}{\varphi(q)^2} \int_{|\eta|\le 1/2} V_1(\eta)\overline{V_2(\eta)} e(-h\eta)d\eta.
\end{aligned}$$

We shall prove the estimates

$$A_{a,q}(h), B_{a,q}(h), C_{a,q}(h) \ll X^{1/2}P^{-2}L^{-2A}, \tag{13}$$

and the asymptotic formula

$$\sum_{q\le P} \sideset{}{^*}\sum_{a \pmod q} e\left(-\frac{ah}{q}\right) I_{a,q}(h) = \mathfrak{S}_2^*(h,P)X^{1/2} + O(X^{1/2}L^{-2A}). \tag{14}$$

We start with $A_{a,q}(h)$. By using $S_1(\alpha) \ll X$ and Lemma 1, we have

$$A_{a,q}(h) \ll X \int_{|\eta|\le 1/qR} |R_2(\eta,a,q)|\, d\eta \ll X^{1/2}P^{-2}L^{-2A}.$$

This proves (13) for $A_{a,q}(h)$. The integral $B_{a,q}(h)$ can be estimated similarly.

We next estimate the integral $C_{a,q}(h)$. Note that for $|\eta| \le 1/2$, we have $V_1(\eta) \ll |\eta|^{-1}$. Thus we have by the Cauchy-Schwarz inequaliity

$$\begin{aligned}
C_{a,q}(h) &\ll \frac{1}{\varphi(q)} \left(\int_{qR<|\eta|\le 1/2} \frac{d\eta}{|\eta|^2}\right)^{1/2} \left(\int_0^1 |V_2(\eta)|^2 d\eta\right)^{1/2}\\
&\ll \frac{(qR)^{1/2}}{\varphi(q)} \left(\sum_{n\le X} \frac{1}{n}\right)^{1/2} \ll X^{1/2}P^{-2}L^{-2A}.
\end{aligned}$$

This proves (13) for $C_{a,q}(h)$.

Finally we prove the asymptotic formula (14). Clearly

$$\sum_{q\le P}\sideset{}{^*}\sum_{a \pmod q} e\left(-\frac{ah}{q}\right) I_{a,q}(h) = \mathfrak{S}_2^*(h,P)\int_{|\eta|\le 1/2} V_1(\eta)\overline{V_2(\eta)}e(-h\eta)d\eta.$$

By the orthogonality of additive characters, we have

$$\int_{|\eta|\le 1/2} V_1(\eta)\overline{V_2(\eta)}e(-h\eta)d\eta = X^{1/2} + O(1).$$

Since Lemma 2 and 3 implies $\mathfrak{S}_2^*(h,P) \ll L^2$, we obtain (14).

By (13) and (14), we arrive at

$$\sum_{\mathfrak{M}} \ll HXL^{-4A}. \tag{15}$$

This completes the evaluation of the major arcs.

6 The minor arcs

The remaining task is to estimate the integral over the minor arcs. As we mentioned before, we shall reduce our minor arc estimate to the corresponding estimate for the twin prime problem. This minor arc estimate was obtained by Mikawa [13] or by Perelli and Pintz [17]. Their result can be stated as:

Theorem 3. *Let $0 \le U \le X$, $H \le V \ll X$ and assume the above setting. Then*

$$\sum_{U<h\le U+V}\left|\int_{\mathfrak{m}} |S_1(\alpha)|^2 e(h\alpha)d\alpha\right|^2 \ll VX^2L^{-32A}$$

for sufficiently large $B \ge B_0(A)$.

Since our Farey dissection is given in the same manner as Perelli and Pintz [17] used, it is more direct to apply the proof of Perelli and Pintz [17]. Note that the admissible range of H obtained in [13, 17] is $X^{1/3+\varepsilon} \le H \ll X$, which is much stronger than we need here.

As for the reduction of our minor arc estimate to Theorem 3, we use the idea of Mikawa [15]. First we expand the square and make cancellations along n. Then we have

$$\begin{aligned}\sum_{\mathfrak{m}} &= \int_{\mathfrak{m}}\int_{\mathfrak{m}} S_1(\alpha)\overline{S_2(\alpha)S_1(\beta)}S_2(\beta)\sum_{X<h\le X+H} e(-h(\alpha-\beta))d\alpha d\beta \\ &\ll \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S_1(\alpha)S_2(\alpha)S_1(\beta)S_2(\beta)| \min\left(H, \frac{1}{\|\alpha-\beta\|}\right) d\alpha d\beta.\end{aligned}$$

By the arithmetic and geometric mean inequality, we have

$$|S_1(\alpha)S_2(\alpha)S_1(\beta)S_2(\beta)| \ll |S_1(\alpha)S_2(\beta)|^2 + |S_1(\beta)S_2(\alpha)|^2 .$$

Therefore we have

$$\begin{aligned}\sum_{\mathfrak{m}} &\ll \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S_1(\alpha)|^2 |S_2(\beta)|^2 \min\left(H, \frac{1}{\|\alpha-\beta\|}\right) d\alpha d\beta \\ &\ll \int_{-1/2}^{1/2}\int_{\mathfrak{m}} |S_1(\alpha)|^2 |S_2(\alpha+\beta)|^2 \min\left(H, \frac{1}{|\beta|}\right) d\alpha d\beta.\end{aligned}$$

We next expand the square $|S_2(\alpha+\beta)|^2$, and interchange the order of summation and integration. Then we have

$$\begin{aligned}\sum_{\mathfrak{m}} \ll &\sum_{m_1^2, m_2^2 \le X}\sum \Lambda(m_1)\Lambda(m_2) \\ &\times \int_{-1/2}^{1/2}\int_{\mathfrak{m}} |S_1(\alpha)|^2 \min\left(H, \frac{1}{|\beta|}\right) e((m_1^2-m_2^2)(\alpha+\beta)) d\alpha d\beta \\ \ll &\sum_{m_1^2, m_2^2 \le X}\sum \Lambda(m_1)\Lambda(m_2) \left|\int_{-1/2}^{1/2} \min\left(H, \frac{1}{|\beta|}\right) e((m_1^2-m_2^2)\beta) d\beta\right| \\ &\times \left|\int_{\mathfrak{m}} |S_1(\alpha)|^2 e((m_1^2-m_2^2)\alpha) d\alpha\right| .\end{aligned}$$

We introduce a new variable $u = m_1^2 - m_2^2$. Clearly the range of this variable u is restricted by the condition $|u| \le X$. Then we can rewrite the above sum as

$$\ll \sum_{|u|\le X} J(u)I(u) \left|\int_{\mathfrak{m}} |S_1(\alpha)|^2 e(u\alpha) d\alpha\right| , \tag{16}$$

where

$$J(u) := \sum_{\substack{m_1^2, m_2^2 \le X \\ m_1^2 - m_2^2 = u}}\sum \Lambda(m_1)\Lambda(m_2), \quad I(u) := \left|\int_{-1/2}^{1/2} \min\left(H, \frac{1}{|\beta|}\right) e(u\beta) d\beta\right| .$$

Since the terms for u and $-u$ contribute the same amount, we can rewrite the sum (16) as

$$\ll \sum_{0\le u\le X} J(u)I(u) \left|\int_{\mathfrak{m}} |S_1(\alpha)|^2 e(u\alpha) d\alpha\right| . \tag{17}$$

We estimate two coefficients $J(u)$ and $I(u)$. Clearly,

$$J(0) = \sum_{m^2 \le X} \Lambda(m)^2 \ll X^{1/2}L. \tag{18}$$

On the other hand, the equation

$$u = m_1^2 - m_2^2$$

can be factorized into

$$u = (m_1 - m_2)(m_1 + m_2).$$

Therefore we have

$$J(u) \ll \tau(u) L^2 \tag{19}$$

for $u \geq 1$. For the integral $I(u)$, we have a trivial estimate:

$$I(u) \ll \int_{-1/2}^{1/2} \min\left(H, \frac{1}{|\beta|}\right) d\beta = \int_{|\beta| \leq 1/H} + \int_{1/H < |\beta| \leq 1/2} \ll L.$$

On the other hand, the oscillation of $e(u\beta)$ gives

$$I(u) \ll H \left| \int_{|\beta| \leq 1/H} e(u\beta) d\beta \right| + \left| \int_{1/H < \beta \leq 1/2} \frac{e(u\beta)}{\beta} d\beta \right| \ll \frac{H}{u}$$

for $u \geq 1$. Therefore we obtained

$$I(u) \ll \min\left(L, \frac{H}{u}\right). \tag{20}$$

Substituting (18), (19), and (20) into (17), we arrived at

$$\begin{aligned} \sum_{\mathfrak{m}} &\ll X^{1/2} L^2 \int_{\mathfrak{m}} |S_1(\alpha)|^2 \, d\alpha \\ &\quad + L^2 \sum_{u \leq X} \tau(u) \min\left(L, \frac{H}{u}\right) \left| \int_{\mathfrak{m}} |S_1(\alpha)|^2 e(u\alpha) d\alpha \right|. \end{aligned}$$

By Parseval's identity, we have

$$X^{1/2} L^2 \int_{\mathfrak{m}} |S_1(\alpha)|^2 \, d\alpha \ll X^{1/2} L^2 \int_0^1 |S_1(\alpha)|^2 \, d\alpha \ll X^{3/2} L^3.$$

Therefore we obtained

$$\sum_{\mathfrak{m}} \ll L^2 \sum_H + L^3 \sup_{H < U \leq X} \sum_U + X^{3/2} L^3, \tag{21}$$

where

$$\sum_H := L \sum_{u \leq H} \tau(u) \left| \int_{\mathfrak{m}} |S_1(\alpha)|^2 e(u\alpha) d\alpha \right|,$$

$$\sum_U := H \sum_{U < u \leq 2U} \frac{\tau(u)}{u} \left| \int_{\mathfrak{m}} |S_1(\alpha)|^2 e(u\alpha) d\alpha \right|.$$

Now we estimate the sum $\sum_H$. We first use the Cauchy-Schwarz inequality in order to make L^2-moment. Then we have

$$L^2 \sum_H \ll H^{1/2} L^5 \left(\sum_{u \le H} \left| \int_{\mathfrak{m}} |S_1(\alpha)|^2 e(u\alpha) d\alpha \right|^2 \right)^{1/2}. \tag{22}$$

Applying Theorem 3 with $U = 0$ and $V = H$, we have

$$L^2 \sum_H \ll HXL^{-4A}. \tag{23}$$

Finally, we estimate the sum $\sum_U$. As before, we start with the Cauchy-Schwarz inequality. Then

$$L^3 \sum_U \ll HU^{-1/2} L^5 \left(\sum_{U < u \le 2U} \left| \int_{\mathfrak{m}} |S_1(\alpha)|^2 e(u\alpha) d\alpha \right|^2 \right)^{1/2}.$$

Applying Theorem 3 with $V = U$, we have

$$L^3 \sum_U \ll HXL^{-4A}. \tag{24}$$

Summing up all of the above estimates, we arrived at

$$\sum_{\mathfrak{m}} \ll HXL^{-4A} + X^{3/2} L^3 \ll HXL^{-4A} \tag{25}$$

since $X^{1/2+\varepsilon} \le H$. This completes the estimate for the minor arcs and the proof of Theorem 2.

7 Completion of the proof

We need to approximate the truncated series $\mathfrak{S}_2^*(h, P)$ by the full series $\mathfrak{S}_2^*(h)$.

Lemma 4. *Assume $X^\varepsilon \le H \le X$. Then we have*

$$\mathfrak{S}_2^*(h, P) = \mathfrak{S}_2^*(h) + O\left(L^{-A}\right)$$

for all but $\ll HL^{-A}$ integers $h \in [1, H] \cap \mathbb{H}_2$.

Proof. This can be proven by the method of Kawada [10, Corollary 1]. □

We can now prove Theorem 1. By Theorem 2 with $U = 0$, we have

$$\begin{aligned} &\# \left\{ h \in [1, H] \cap \mathbb{H}_2 \;\middle|\; \left| \Psi_2^*(X, h) - \mathfrak{S}_2^*(h, P) X^{1/2} \right| > X^{1/2} L^{-A} \right\} \\ &\qquad \ll \frac{HXL^{-4A}}{XL^{-2A}} \ll HL^{-A}. \end{aligned}$$

Therefore we have

$$\Psi_2^*(X,h) = \mathfrak{S}_2^*(h,P)X^{1/2} + O\left(X^{1/2}L^{-A}\right)$$

for all but $\ll HL^{-A}$ integers $h \in [1,H] \cap \mathbb{H}_2$. Now Lemma 4 implies that

$$\Psi_2^*(X,h) = \mathfrak{S}_2^*(h)X^{1/2} + O\left(X^{1/2}L^{-A}\right)$$

with $\ll HL^{-A}$ additional exceptions. This completes the proof of Theorem 1 for the case $k=2$.

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Yuta Suzuki
Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan
e-mail: m14021y@math.nagoya-u.ac.jp