A COHOMOLOGICAL INTERPRETATION OF ARCHIMEDEAN ZETA INTEGRALS FOR $\mathrm{GL}_3 \times \mathrm{GL}_2$

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Abstract. This article is a survey on the author's preprint [HN1], where the author constructs a p-adic Asai L-functions for irreducible cohomological cuspidal autmorphic representations of GL_2 over CM fields.

1. Introduction

1.1. **Motivation.** This article is a report of the author's talk at the conference "Automorphic forms, Automorphic representations, Galois representations, and its related topics", which was held at RIMS, Kyoto university between 25th to 29th, January, 2021.

The author's talk was based on our preprint [HN1], where authors study a detailed relationship between cup product pairings and zeta integrals for Rankin-Selberg L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$. Namely, this work deduces a rationality results for the critical values of Rankin-Selberg L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$.

The study of rationality of critical values of Rankin-Selberg L-functions is firstly developed by Manin [Man72] and Shimura [Shi76], [Shi77]. Mahnkopf [Mah05] generalized their work for $\mathrm{GL}_{n+1} \times \mathrm{GL}_n (n \geq 1)$, based on the generalized modular symbol method by Kazhdan, Mazur and Schmidt ([KMS00]). Mahnkopf gives a relation between cup product pairings for certain symmetric spaces and zeta integral for Rankin-Selberg L-functions, however he did not introduce a notion of periods and his final formula contains unspecified constant. Raghuram and Shahidi formulated a notion of Whittaker periods in [RS08, Definition/Proposition 3.3], and as a continuation of his work, Raghuram ([Rag16]) gives a rationality result for Rankin-Selberg L-functions with respect to their Whittaker periods. However, there still exists an unspecified constant $p_\infty^{\epsilon,\eta}(\mu, \lambda)$ in Raghuram's rationality statement in [Rag16, Theorem1.1 and 2.50]. See [Rag16, (2.49)] for the definition of $p_\infty^{\epsilon,\eta}(\mu, \lambda)$ and its detail. Here we use the same notation with [Rag16]. The analysis of the constant $p_\infty^{\epsilon,\eta}(\mu, \lambda)$ is one of fundamental problem in the generalized modular symbol method. B. Sun ([Sun17], [KS13] if $n = 2$) proved that the constant $p_\infty^{\epsilon,\eta}(\mu, \lambda)$ is non-zero, however no further property about the constant is known so far. One of the motivation of Hara and the author's work in [HN1] is to compare Raghuram's rationality result with Deligne's conjecture on rationality of critical values of L-functions for Rankin-Selberg motives by giving a precise evaluation of this unspecified constant $p_\infty^{\epsilon,\eta}(\mu, \lambda)$. Acutually we prove that $p_\infty^{\epsilon,\eta}(\mu, \lambda)$ is given by the product of a Γ-factor and an explicit easy constant.

There is another motivation in [HN1] which is coming from the construction of p-adic Rankin-Selberg L-functions due to Januszewski [Jan]. One of important expected properties of p-adic L-functions is Kummer type congruence, which is called as the Manin congruences in [Jan]. The Kummer type congruence yields that congruences between different critical values, however, the work of [Jan] is based on the rationality results of Mahnkopf [Mah05] and Raghuram [Rag16], and hence the interpolation formula in [Jan] still contains an unspecified constant $p_\infty^{\epsilon,\eta}(\mu, \lambda)$. Since the constants depend on the critical values, the study of Kummer type congruence is still difficult problem so far. In [HN1], we give a precise relation between cup product pairings with critical values, which makes possible to formulate the congruences between critical values with respect to Raghuram-Shahidi's Whittaker periods.

1.2. **Statement of the main theorem.** To state the main result, we introduce some notations in detail. Put $\Sigma_{\mathbf{Q}}$ as the set of all places of $\mathbf{Q}$, and denote by ∞ its unique infinite place. For $n = 2$ or 3, let $\pi^{(n)} = \bigotimes'_{v \in \Sigma_{\mathbf{Q}}} \pi_v^{(n)}$ denote a cohomological irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{Q}_{\mathbf{A}})$. Since $\pi^{(2)}$ is cohomological, the archimedean part $\pi_\infty^{(2)}$ of $\pi^{(2)}$ is a discrete series representation D_{ν_2, l_2} of $\mathrm{GL}_2(\mathbf{R})$ of weight $l_2 \geq 1$, and (the archimedean part of) the central character $\omega_{\pi^{(2)}}$ of $\pi^{(2)}$ satisfies $\omega_{\pi^{(2)},\infty}(t) = t^{2\nu_2}$ for every $t \in \mathbf{R}^\times$ with $t > 0$. Since $\pi^{(3)}$ is also cohomological, the archimedean part $\pi_\infty^{(3)}$ of $\pi^{(3)}$ is a generalized principal series representation of $\mathrm{GL}_3(\mathbf{R})$ parabolically induced from the direct product of a discrete series representation D_{ν_3, l_3} of $\mathrm{GL}_2(\mathbf{R})$ ($l_3 \geq 2$, l_3 is even) and a character $\chi_{\nu_3}^\delta : \mathbf{R}^\times \to \mathbf{C}^\times$ such that $\chi_{\nu_3}^\delta(t) = \mathrm{sgn}(t)^\delta |t|_\infty^{\nu_3}$. We always have to assume that the inequality $(1 \leq) \, l_2 < l_3$ holds, since our method is based upon the modular symbol method adopted in [Mah05]. We also normalize the parameter ν_n as $\nu_n = -\frac{l_n}{2} + \frac{n-1}{2}$. The for each $m \in \mathbf{Z}$, $L(m + \frac{1}{2}, \pi^{(3)} \times \pi^{(2)})$ is a critical value if and only if

$$\begin{cases} \frac{l_3}{2} - 1 \leq m \leq \frac{l_3}{2} + l_2 - 2, & (l_2 \leq \frac{l_3}{2}), \\ \frac{l_3}{2} - 1 \leq m \leq \frac{l_3}{2} + l_2 - 2, & (\frac{l_3}{2} < l_2 < l_3). \end{cases}$$

We call these m critical points for $\pi^{(3)}$ and $\pi^{(2)}$. We are interested in algebraicity of the critical values $L(m + \frac{1}{2}, \pi^{(3)} \times \pi^{(2)})$.

Our strategy to study the critical values $L(m + \frac{1}{2}, \pi^{(3)} \times \pi^{(2)})$ is the generalized modular symbol method in [KMS00], [Mah05], [Rag16] and [Jan]. So we introduce a setting on symmetric spaces and its cohomology groups. Let $\mathcal{K}_n$ be an open compact subgroup of $\mathrm{GL}_n(\widehat{\mathbf{Z}})$ and $Y_{\mathcal{K}_n}^{(n)}$ the corresponding symmetric space $\mathrm{GL}_n(\mathbf{Q}) \backslash \mathrm{GL}_n(\mathbf{Q}_{\mathbf{A}}) / \mathbf{R}_{>0}^\times \mathrm{SO}(n) \mathcal{K}_n$. Suppose that $\mathcal{K}_2 = \mathrm{GL}_2(\widehat{\mathbf{Z}})$ and that $\mathcal{K}_3$ is the mirabolic subgroup of $\mathrm{GL}_3(\widehat{\mathbf{Z}})$ consisting of matrices whose bottom rows are congruent to $(0, 0, 1)$ modulo an ideal $\mathfrak{N}$ of $\widehat{\mathbf{Z}}$. We also assume that $\pi^{(2)}$ has a $\mathcal{K}_2$-fixed vector and that $\mathfrak{N}$ is minimal among ideals of $\widehat{\mathbf{Z}}$ such that $\pi^{(3)}$ has a $\mathcal{K}_3$-fixed vector.

Since $\pi^{(n)}$ is cohomological, there exist certain local system $\mathcal{L}^{(n)}(\mathbf{C})$ with coefficient $\mathbf{C}$ such that $\pi^{(n)}$-isotypic component of the cuspidal cohomology group $H^\bullet_{\mathrm{cusp}}(Y_{\mathcal{K}_n}^{(n)}, \mathcal{L}^{(n)}(\mathbf{C}))$ is non-zero. Let E be a sufficiently large number field so that E contains the field of rationality of $\pi^{(2)}$ and $\pi^{(3)}$ in [Clo90, Section 3]. Put $\mathrm{b}_n = 1, 2$ for $n = 2, 3$ respectively. We can define cohomology classes $\eta^\pm_{\pi^{(2)}}$ and $\eta_{\pi^{(3)}}$ in $\pi_{\mathrm{fin}}^{(n)}$-isotypic component of $H^{\mathrm{b}_n}_{\mathrm{cusp}}(Y_{\mathcal{K}_n}^{(n)}, \mathcal{L}^{(n)}(E))$ for $n = 2, 3$ respectively. Raghuram and Shahidi define their periods $\Omega_{\pi^{(3)}}$ and $\Omega^\pm_{\pi^{(2)}}$ attached to $\pi^{(3)}$ and $\pi^{(2)}$ in [RS08] as the ratios of $\eta_{\pi^{(3)}}$ and $\eta^\pm_{\pi^{(2)}}$ to the images of appropriate cusp forms of $\pi^{(3)}$ and $\pi^{(2)}$ under the Eichler–Shimura maps $\delta^{(3)}$ and $\delta^{(2)}$ respectively. The definition of periods $\Omega_{\pi^{(3)}}$ and $\Omega^\pm_{\pi^{(2)}}$ depends on a normalization of Whittaker functions associated with cusp forms. We adapt a suitable normalization of Whittaker functions so that the local zeta integrals of the Whittaker functions give local L-factors. In particular, at archimedian place, we use an explicit formula of archimedean zeta integrals in [HIM]. It is also necessary to write down Eichler-Shimura maps for GL_n to clarify the relation between periods and zeta integrals. If $n = 2$, the explicit description is well-known ([Shi71, Section 8]). In [HN1], we give an explicit description of Eichler-Shimura map for GL_3. See Section 2 for more details on these cohomology classes.

Recall the cohomological interpretation of zeta integrals, which is given in [KMS00] and [Mah05]. Let us consider the natural projection

$$\mathrm{p}_n : \mathcal{Y}_{\mathcal{K}_n}^{(n)} := \mathrm{GL}_n(\mathbf{Q}) \backslash \mathrm{GL}_n(\mathbf{Q}_{\mathbf{A}}) / \mathrm{SO}(n) \mathcal{K}_n \longrightarrow Y_{\mathcal{K}_n}^{(n)}.$$

Consider GL_2 as a subgroup of GL_3 via the embedding $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$. Then the branching rule for irreducible algebraic representations of GL_3 and GL_2 induces maps ∇^{n_m}, which is indexed by critical points m, from local systems on $\mathcal{Y}^{(3)}_{\mathcal{K}_3}$ to local systems on $\mathcal{Y}^{(2)}_{\mathcal{K}_2}$. See [HN1, Section 3] for the explicit branching rule and the maps ∇^{n_m}.

Let $I(m, \pi^{(3)}, \pi^{(2),\pm})$ be the cup product of $\nabla^{n_m}\iota^*\mathrm{p}_3^*\eta_{\pi^{(3)}}$ and $\mathrm{p}_2^*\eta^{\pm}_{\pi^{(2)}}$. The generalized modular symbol method yields $I(m, \pi^{(3)}, \pi^{(2)})$ is described by a certain zeta integrals for Rankin-Selberg L-functions. By our detailed construction on Eichler-Shimura maps and the branching rule, we found the following explicit formula for $I(m, \pi^{(3)}, \pi^{(2)})$, which is the main theorem in [HN1]:

Theorem 1.1. (i) *Suppose that $(-1)^{\delta+\frac{l_3}{2}} = \pm(-1)^m$ holds. Then we have*

$$I(m, \pi^{(3)}, \pi^{(2),\pm}) = 2^{-2}(-1)^{\delta}\sqrt{-1}^{\frac{l_3}{2}-m-1}\binom{\frac{l_3}{2}-1}{l_3-2-m}\binom{\frac{l_3}{2}-1}{m-l_2+1}\frac{L(m+\frac{1}{2}, \pi^{(3)}\times\pi^{(2)})}{\Omega_{\pi^{(3)}}\Omega^{\pm}_{\pi^{(2)}}}.$$

Here $\binom{a}{b}$ denotes the binomial coefficient. Furthermore the right-hand side gives an element in E.

(ii) *If $(-1)^{\delta+\frac{l_3}{2}} = \pm(-1)^m$ does not hold, then we have $I(m, \pi^{(3)}, \pi^{(2),\pm}) = 0$.*

The above theorem deduced from the evaluation of the unspecified constant $p^{\epsilon,\eta}_{\infty}(\mu, \lambda)$ in [Rag16, (2.49)] and hence Theorem 1.1 refines the rationality result in [Rag16, Theorem 1.1 (2), Theorem 2.50]. The study of $p^{\epsilon,\eta}_{\infty}(\mu, \lambda)$ is done by a detailed analysis of Eichler-Shimura maps for GL_3 and an explicit formula for archimedean Rankin-Selberg zeta integral of $\mathrm{GL}_3 \times \mathrm{GL}_2$ due to Hirano-Ishii-Miyazaki ([HIM]). In [HN1], we found an appropriate form of Eichler-Shimura maps for GL_3 for the study of cup product pairings. In this article, we briefly explain our description of Eichler-Shimura maps for GL_3 in Section 2.

We are also interested in the motivic nature of periods $\Omega_{\pi^{(3)}}, \Omega^{\pm}_{\pi^{(2)}}$ in Theorem 1.1. From the definition and the usual modular symbol method, it immediately follows that $\Omega^{\pm}_{\pi^{(2)}}$ are given by Deligne's periods in [Del79]. By using Theorem 1.1 and assuming the conjectural motive attached to $\pi^{(3)}$, we found an explicit relation between $\Omega_{\pi^{(3)}}$ and Deligne's periods. See Section 3 for the motivic interpretations of the Raghuram-Shahidi's Whittaker periods $\Omega_{\pi^{(3)}}$.

1.3. **Organization of this article.** All the details about Theorem 1.1 can be found in [HN1]. Hence, in this article, we focus on Eichler-Shimura maps for GL_3, since it is one of main ingredients in [HN1] and it might be useful beyond the scope of [HN1]. We describe Eichler-Shimura maps in Section 2. The proof of Theorem 1.1 is done by detailed study of Eichler-Shimura maps. The detail of the proof can be found in [HN1, Section 7]. We also discuss the motivic background of Raghuram-Shahidi's Whittaker periods $\Omega_{\pi^{(3)}}$ in Section 3. In particular, see Corollary 3.2 for the result.

1.4. **Convention.** In this article, if we say L-functions, it implies "complete" L-functions. For instance, for a pure motive $\mathcal{M}$, $L(s, \mathcal{M})$ is the product of Γ-factor of $\mathcal{M}$ and local L-factor of $\mathcal{M}$ at finite places. We sometimes write $L(s, \mathcal{M}) = L_{\infty}(s, \mathcal{M})L_{\mathrm{fin}}(s, \mathcal{M})$, where $L_{\infty}(s, \mathcal{M})$ denotes the Γ-factor of $\mathcal{M}$ and $L_{\mathrm{fin}}(s, \mathcal{M})$ denotes the product of local L-factor of $\mathcal{M}$ at finite places. The similar notion is applied to automorphic L-functions.

2. Eichler-Shimura maps for GL_3

Let $\pi^{(3)} = \otimes'_{v\in\Sigma_{\mathbf{Q}}}\pi^{(3)}_v$ be a cohomological irreducible cuspidal automorphic representation of $\mathrm{GL}_3(\mathbf{Q}_{\mathbf{A}})$. Recall that $\pi^{(3)}_{\infty}$ is a generalized principal series representation, which is parabolic induction from D_{ν_3,l_3} and $\chi^{\delta}_{\nu_3}$. In this section, we describe an explicit form of Eichler-Shimura map for $\pi^{(3)}$, which we used in [HN1].

We prepare some notations. Let $\mathcal{A}$ a (commutative) integral domain of characteristic 0. For non-negative $w \in \mathbf{Z}$, $\mathcal{A}[z_1, z_2, z_3]_w$ denotes the set of homogeneous polynomials of degree w of variables z_1, z_2, z_3 with a coefficient $\mathcal{A}$.

2.1. **Model of representations of** $\mathrm{O}(3)$**.** Set $\Lambda_3 = \{(\lambda, \delta) \mid \lambda \in \mathbf{Z}, \lambda \geq 0, \delta \in \mathbf{Z}/2\mathbf{Z}\}$ and let $\boldsymbol{\lambda} = (\lambda, \delta) \in \Lambda_3$. Define an action $\tau_{\boldsymbol{\lambda}}$ of $\mathrm{O}(3)$ on $\mathbf{C}[z_1, z_2, z_3]_\lambda$ by

$$\tau_{\boldsymbol{\lambda}}(u)P(z_1, z_2, z_3) = (\det u)^\delta P((z_1, z_2, z_3)u)$$

for each $u \in \mathrm{O}(3)$ and $P \in \mathbf{C}[z_1, z_2, z_3]_\lambda$. Define subspace $\mathcal{V}_{\boldsymbol{\lambda}}$ of $\mathbf{C}[z_1, z_2, z_3]_\lambda$ to be $\mathcal{V}_{\boldsymbol{\lambda}} = (z_1^2 + z_2^2 + z_3^2)\mathbf{C}[z_1, z_2, z_3]_{\lambda-2}$ for $\lambda \geq 2$ and $\mathcal{V}_{\boldsymbol{\lambda}} = 0$ otherwise. Since $\mathcal{V}_{\boldsymbol{\lambda}}$ is stable under the action $\tau_{\boldsymbol{\lambda}}$ of $\mathrm{O}(3)$, we put $V_{\boldsymbol{\lambda}} = \mathbf{C}[z_1, z_2, z_3]_\lambda/\mathcal{V}_{\boldsymbol{\lambda}}$ and we also denote by $\tau_{\boldsymbol{\lambda}}$ the action of $\mathrm{O}(3)$. Then each irreducible representation of $\mathrm{O}(3)$ is classified as $\tau_{\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \Lambda_3$.

We introduce an explicit basis of the representation space $V^{(3)}_{\boldsymbol{\lambda}}$. For every $\mu \in \mathbf{Z}$ with $0 \leq \mu \leq \lambda$, let $v^{(3),\boldsymbol{\lambda}}_{\pm\mu}$ be elements of $V^{(3)}_{\boldsymbol{\lambda}}$ defined as

$$v^{(3),\boldsymbol{\lambda}}_{\pm\mu} = (\pm z_1 + \sqrt{-1}z_2)^\mu z_3^{\lambda-\mu} \mod \mathcal{V}_{\boldsymbol{\lambda}} \qquad \text{(double sign in the same order)}.$$

Let $M_{\boldsymbol{\lambda}}(u) \in \mathrm{GL}_{2\lambda+1}(\mathbf{C})$ be the matrix representation of the action of $u \in \mathrm{O}(3)$ on $V^{(3)}_{\boldsymbol{\lambda}}$ with respect to the above basis, in particular we have

$$\begin{aligned}(2.1)\qquad &\begin{pmatrix}\tau^{(3)}_{\boldsymbol{\lambda}}(u)v^{(3),\boldsymbol{\lambda}}_{\lambda} & \tau^{(3)}_{\boldsymbol{\lambda}}(u)v^{(3),\boldsymbol{\lambda}}_{\lambda-1} & \ldots & \tau^{(3)}_{\boldsymbol{\lambda}}(u)v^{(3),\boldsymbol{\lambda}}_{-\lambda}\end{pmatrix}\\ &\qquad = \begin{pmatrix}v^{(3),\boldsymbol{\lambda}}_{\lambda} & v^{(3),\boldsymbol{\lambda}}_{\lambda-1} & \ldots & v^{(3),\boldsymbol{\lambda}}_{-\lambda}\end{pmatrix} M_\lambda(u).\end{aligned}$$

Note that the minimal $\mathrm{O}(3)$-type of $\pi^{(3)}_\infty$ is $\tau_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = (l_3 + 1, \delta)$. Let

$$\Lambda_3^{\mathrm{coh}} = \{(\lambda_3, \delta) \mid \lambda_3 \in 1 + 2\mathbf{Z}, \lambda_3 \geq 3, \delta \in \mathbf{Z}/2\mathbf{Z}\}.$$

We also note that $\pi^{(3)}$ is cohomological if and only if $\boldsymbol{\lambda} \in \Lambda_3^{\mathrm{coh}}$.

2.2. **Model of finite dimensional representations of** GL_3**.** Set $\boldsymbol{w} = (w_1^+, w_1^-, w_2)$ with $w_1^+, w_1^-, w_2 \in \mathbf{Z}$ and $w_1^+, w_1^- \geq 0$. Define $\mathcal{A}[X, Y, Z; A, B, C]_{w_1^+,w_1^-}$ to be the set of homogenous polynomials of degree w_1^+ in variables X, Y, Z and of degree w_1^- in variables A, B, C. Define an action $\varrho^{(3)}_{\boldsymbol{w}}$ of $\mathrm{GL}_3(\mathcal{A})$ on $\mathcal{A}[X, Y, Z; A, B, C]_{w_1^+,w_1^-}$ as follows:

$$\varrho^{(3)}_{\boldsymbol{w}}(g)P(X, Y, Z; A, B, C) = (\det g)^{w_2}P((X, Y, Z)g; (A, B, C){}^{\mathrm{t}}g^{-1})$$
$$\text{for } P \in \mathcal{A}[X, Y, Z; A, B, C]_{w_1^+,w_1^-} \text{ and } g \in \mathrm{GL}_3(\mathcal{A}).$$

We also define a differential operator $\iota_{w_1^+,w_1^-}$ to be

$$\begin{aligned}\iota_{w_1^+,w_1^-} =&\frac{\partial^2}{\partial X\partial A} + \frac{\partial^2}{\partial Y\partial B} + \frac{\partial^2}{\partial Z\partial C} :\\ &\mathcal{A}[X, Y, Z; A, B, C]_{w_1^+,w_1^-} \to \mathcal{A}[X, Y, Z; A, B, C]_{w_1^+-1,w_1^--1}.\end{aligned}$$

We note that $\iota_{w_1^+,w_1^-}$ is equivariant with respect to the actions $\varrho^{(3)}_{\boldsymbol{w}}$ and $\varrho^{(3)}_{\boldsymbol{w}-(1,1,0)}$ of $\mathrm{GL}_3(\mathcal{A})$. Hence define $L^{(3)}(\boldsymbol{w}; \mathcal{A})$ as the kernel of $\iota_{w_1^+,w_1^-}$. It is known that $L^{(3)}(\boldsymbol{w}; \mathcal{A})$ is an irreducible algebraic representation of $\mathrm{GL}_3(\mathcal{A})$ when $\mathcal{A}$ is a field ([FH13, Section 13.2]).

Recall that $\pi^{(3)}_\infty$ has the minimal $\mathrm{O}(3)$-type $\tau_{\boldsymbol{\lambda}}$. Then later we will only use the case that $\boldsymbol{w} = \boldsymbol{w}_{\boldsymbol{\lambda}} := (\frac{l_3}{2} - 1, \frac{l_3}{2} - 1, \frac{l_3}{2} - 1)$ for even $l_3 \geq 2$. In this case, assuming that $\mathcal{A}$ is a field, then the dimension $\dim_{\mathcal{A}} L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}};\mathcal{A})$ of $L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}};\mathcal{A})$ is give by the following formula:

$$\dim_{\mathcal{A}} L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}};\mathcal{A}) = \left(\frac{l_3}{2}\right)^3.$$

2.3. **Lie algebras.** For the construction of Eichler-Shimura isomorphism, we also need to write down the adjoint action of O(3) on differential forms on $Y^{(3)}_{\mathcal{K}_3}$. So we have to prepare some explicit formulas about Lie algebras of $\mathrm{GL}_3(\mathbf{R})$.

Let $\mathcal{P}_3 = \mathfrak{gl}_3(\mathbf{R})/\mathrm{Lie}(K_3)$ and define $\mathcal{P}_{3,\mathbf{C}}$ to be its complexification. We consider the adjoint action of SO(3) on $\mathcal{P}_{3,\mathbf{C}}$. It is immediately checked that $\mathcal{P}_{3,\mathbf{C}}$ is isomorphic to $\tau^{(3)}_{(2,0)}$ as SO(3)-modules. For $i = 2$ and 3, the wedge product $\bigwedge^i \mathcal{P}_{3,\mathbf{C}}$ is of dimension 10, and we readily observe that its $\mathbf{C}$-linear dual $\bigwedge^i \mathcal{P}^*_{3,\mathbf{C}}$ is isomorphic to $\tau^{(3),\vee}_{(1,0)} \oplus \tau^{(3),\vee}_{(3,0)}$ as SO(3)-modules (here $\tau^{(3),\vee}_{\boldsymbol{\lambda}}$ denotes the contragredient of $\tau^{(3)}_{\boldsymbol{\lambda}}$). Furthermore, we have the following lemma:

Lemma 2.1. *Suppose that $i = 2$ or 3. Then there exist explicit elements $\boldsymbol{\omega}^i_j$ $(-3 \leq j \leq 3)$ in $\bigwedge^i \mathcal{P}^*_{3,\mathbf{C}}$ such that*

$$\wedge^i \mathrm{Ad}^*(u) \begin{pmatrix} \boldsymbol{\omega}^i_3 & \ldots & \boldsymbol{\omega}^i_{-3} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega}^i_3 & \ldots & \boldsymbol{\omega}^i_{-3} \end{pmatrix} {}^{\mathrm{t}}M_{(3,0)}(u)^{-1}$$

for each $u \in \mathrm{SO}(3)$.

For each $g \in \mathrm{GL}_3(\mathbf{R})$, by Iwasawa decomposition, we write g as follows

$$g = \begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot k \qquad \text{for } k \in \mathbf{R}^\times \mathrm{SO}(3).$$

Consider x_1, x_2, x_3, y_1, y_2 as a coordinates of $Y^{(3)}_{\mathcal{K}_3,\infty} := \mathrm{GL}_3(\mathbf{Q})\backslash \mathrm{GL}_3(\mathbf{R})/\mathbf{R}^\times \mathrm{SO}(3)$. Then we can write down the differential i-form $(i = 2, 3)$, which is determined by $\boldsymbol{\omega}^i_j$ $(-3 \leq j \leq 3)$ as follows:

Lemma 2.2. *Set $\varsigma_2 = \mathrm{d}y_1$, $\varsigma_1 = \mathrm{d}y_2$, $\varsigma_0 = \mathrm{d}x_1$, $\varsigma_{-1} = \mathrm{d}x_2$ and $\varsigma_{-2} = \mathrm{d}x_3$. For $i = 2$ and 3, define $\boldsymbol{\varsigma}^i$ as*

$$\boldsymbol{\varsigma}^2 = \begin{pmatrix} \varsigma_{2,1} & \varsigma_{2,0} & \varsigma_{2,-1} & \varsigma_{2,-2} & \varsigma_{1,0} & \varsigma_{1,-1} & \varsigma_{1,-2} & \varsigma_{0,-1} & \varsigma_{0,-2} & \varsigma_{-1,-2} \end{pmatrix},$$

$$\boldsymbol{\varsigma}^3 = \begin{pmatrix} \varsigma_{2,1,0} & \varsigma_{2,1,-1} & \varsigma_{2,1,-2} & \varsigma_{2,0,-1} & \varsigma_{2,0,-2} & \varsigma_{2,-1,-2} & \varsigma_{1,0,-1} & \varsigma_{1,0,-2} & \varsigma_{1,-1,-2} & \varsigma_{0,-1,-2} \end{pmatrix},$$

where $\varsigma_{j,j'}$ and $\varsigma_{j,j',j''}$ abbreviate $\varsigma_j \wedge \varsigma_{j'}$ and $\varsigma_j \wedge \varsigma_{j'} \wedge \varsigma_{j''}$ respectively. Then the differential forms $\begin{pmatrix} \boldsymbol{\omega}^i_3 & \ldots & \boldsymbol{\omega}^i_{-3} \end{pmatrix}$ is described on $Y^{(3)}_{\mathcal{K}_3,\infty}$ as $\boldsymbol{\varsigma}^i Q^i$ for

$$Q^2 = \begin{pmatrix}
0 & \frac{1}{2y_1y_2} & 0 & 0 & 0 & -\frac{1}{2y_1y_2} & 0 \\
0 & 0 & -\frac{x_2+\sqrt{-1}y_2}{2y_1^2y_2} & 0 & -\frac{x_2-\sqrt{-1}y_2}{2y_1^2y_2} & 0 & 0 \\
0 & -\frac{\sqrt{-1}}{2y_1y_2} & 0 & 0 & 0 & -\frac{\sqrt{-1}}{2y_1y_2} & 0 \\
0 & 0 & \frac{1}{2y_1^2y_2} & 0 & \frac{1}{2y_1^2y_2} & 0 & 0 \\
-\frac{x_2+\sqrt{-1}y_2}{8y_1y_2^2} & 0 & \frac{x_2-5\sqrt{-1}y_2}{8y_1y_2^2} & 0 & \frac{x_2+5\sqrt{-1}y_2}{8y_1y_2^2} & 0 & -\frac{x_2-\sqrt{-1}y_2}{8y_1y_2^2} \\
0 & -\frac{\sqrt{-1}}{4y_2^2} & 0 & \frac{\sqrt{-1}}{2y_2^2} & 0 & -\frac{\sqrt{-1}}{4y_2^2} & 0 \\
\frac{1}{8y_1y_2^2} & 0 & -\frac{1}{8y_1y_2^2} & 0 & -\frac{1}{8y_1y_2^2} & 0 & \frac{1}{8y_1y_2^2} \\
-\frac{\sqrt{-1}x_2-y_2}{8y_1y_2^2} & 0 & \frac{3(\sqrt{-1}x_2+y_2)}{8y_1y_2^2} & 0 & -\frac{3(\sqrt{-1}x_2-y_2)}{8y_1y_2^2} & 0 & \frac{\sqrt{-1}x_2+y_2}{8y_1y_2^2} \\
0 & 0 & 0 & \frac{\sqrt{-1}}{y_1^2y_2} & 0 & 0 & 0 \\
-\frac{\sqrt{-1}}{8y_1y_2^2} & 0 & \frac{3\sqrt{-1}}{8y_1y_2^2} & 0 & -\frac{3\sqrt{-1}}{8y_1y_2^2} & 0 & \frac{\sqrt{-1}}{8y_1y_2^2}
\end{pmatrix},$$

$$Q^3 = \begin{pmatrix} \frac{x_2+\sqrt{-1}y_2}{24y_1^2y_2^2} & 0 & \frac{x_2-\sqrt{-1}y_2}{8y_1^2y_2^2} & 0 & \frac{x_2+\sqrt{-1}y_2}{8y_1^2y_2^2} & 0 & \frac{x_2-\sqrt{-1}y_2}{24y_1^2y_2^2} \\ 0 & 0 & 0 & \frac{\sqrt{-1}}{3y_1y_2^2} & 0 & 0 & 0 \\ -\frac{1}{24y_1^2y_2^2} & 0 & -\frac{1}{8y_1^2y_2^2} & 0 & -\frac{1}{8y_1^2y_2^2} & 0 & -\frac{1}{24y_1^2y_2^2} \\ \frac{\sqrt{-1}x_2-y_2}{24y_1^2y_2^2} & 0 & \frac{\sqrt{-1}x_2+y_2}{8y_1^2y_2^2} & 0 & -\frac{\sqrt{-1}x_2-y_2}{8y_1^2y_2^2} & 0 & -\frac{\sqrt{-1}x_2+y_2}{24y_1^2y_2^2} \\ 0 & 0 & 0 & \frac{\sqrt{-1}}{6y_1^3y_2} & 0 & 0 & 0 \\ \frac{\sqrt{-1}}{24y_1^2y_2^2} & 0 & \frac{\sqrt{-1}}{8y_1^2y_2^2} & 0 & -\frac{\sqrt{-1}}{8y_1^2y_2^2} & 0 & -\frac{\sqrt{-1}}{24y_1^2y_2^2} \\ \frac{\sqrt{-1}x_2-y_2}{48y_1y_2^3} & 0 & -\frac{\sqrt{-1}x_2-3y_2}{16y_1y_2^3} & 0 & \frac{\sqrt{-1}x_2+3y_2}{16y_1y_2^3} & 0 & -\frac{\sqrt{-1}x_2+y_2}{48y_1y_2^3} \\ 0 & \frac{\sqrt{-1}}{8y_1^2y_2^2} & 0 & \frac{\sqrt{-1}}{12y_1^2y_2^2} & 0 & \frac{\sqrt{-1}}{8y_1^2y_2^2} & 0 \\ \frac{\sqrt{-1}}{48y_1y_2^3} & 0 & -\frac{\sqrt{-1}}{16y_1y_2^3} & 0 & \frac{\sqrt{-1}}{16y_1y_2^3} & 0 & -\frac{\sqrt{-1}}{48y_1y_2^3} \\ 0 & -\frac{1}{8y_1^2y_2^2} & 0 & 0 & 0 & \frac{1}{8y_1^2y_2^2} & 0 \end{pmatrix}.$$

The proofs of Lemma 2.1 and Lemma 2.2 are done by direct calculation using Mathematica and Maxima.

2.4. **Space of cusp forms.** We introduce the space of cusp forms $\mathcal{S}_{\boldsymbol{\lambda}}^{(3)}(\mathcal{K}_3)$ associated with $\pi^{(3)}$, which is necessary to define Eichler-Shimura maps. Since $\mathrm{Hom}_{\mathrm{O}(n)}(\tau_{\boldsymbol{\lambda}}^{(3)}, \pi_\infty^{(n)})$ has dimension 1 by Schur's lemma and the theory of minimal K-types, we find a tuple

$$\boldsymbol{f} := \begin{pmatrix} f_{\lambda_3}^{\boldsymbol{\lambda}} & f_{\lambda_3-1}^{\boldsymbol{\lambda}} & \ldots & f_{-\lambda_3}^{\boldsymbol{\lambda}} \end{pmatrix}$$

consisting of cuspidal automorphic forms $f_\alpha^{\boldsymbol{\lambda}} \colon \mathrm{GL}_3(\mathbf{Q})\backslash\mathrm{GL}_3(\mathbf{Q}_\mathbf{A}) \to \mathbf{C}$ for $-\lambda_3 \leq \alpha \leq \lambda_3$ associated with $\pi^{(3)}$ satisfying

$$\begin{pmatrix} f_{\lambda_3}^{\boldsymbol{\lambda}}(gtu) & f_{\lambda_3-1}^{\boldsymbol{\lambda}}(gtu) & \ldots & f_{-\lambda_3}^{\boldsymbol{\lambda}}(gtu) \end{pmatrix} = \begin{pmatrix} f_{\lambda_3}^{\boldsymbol{\lambda}}(g) & f_{\lambda_3-1}^{\boldsymbol{\lambda}}(g) & \ldots & f_{-\lambda_3}^{\boldsymbol{\lambda}}(g) \end{pmatrix} t^{3\nu_3} M_{\boldsymbol{\lambda}}(u)$$

for each $g \in \mathrm{GL}_3(\mathbf{Q}_\mathbf{A})$, $t \in \mathbf{R}_{>0}^\times$ and $u \in \mathrm{O}(3)$. We call tuples $\boldsymbol{f}$ satisfying the above identity as cusp forms associated with $\pi^{(3)}$. For each $\boldsymbol{\lambda} \in \Lambda_3^{\mathrm{coh}}$, define $\mathcal{S}_{\boldsymbol{\lambda}}^{(3)}(\mathcal{K}_3)$ to be the $\mathbf{C}$-vector space spanned by cusp forms associated with certain cohomological irreducible cuspidal automorphic representations of $\mathrm{GL}_3(\mathbf{Q}_\mathbf{A})$, which are invariant under the right translation by $\mathcal{K}_3$ and whose archimedean parts have the minimal $\mathrm{O}(3)$-type $\tau_{\boldsymbol{\lambda}}^{(3)}$.

2.5. **Definition of Eichler-Shimura map.** Suppose that $\mathcal{A}$ is a field of characteristic 0 and let $\mathcal{L}^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathcal{A})$ be the local system on $Y_{\mathcal{K}_3}^{(3)}$ associated with $\mathrm{GL}_3(\mathbf{Q})$-module $L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathcal{A})$. In this subsection, we introduce our description of Eichler-Shimura map:

$$\delta^{(3),i} : \mathcal{S}_{\boldsymbol{\lambda}}^{(3)}(\mathcal{K}_3) \longrightarrow H_{\mathrm{cusp}}^i(Y_{\mathcal{K}_3}^{(3)}, \mathcal{L}^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C})) \quad (i = 2, 3).$$

Although only $i = 2$ case is necessary for the proof of Theorem 1.1, we include $i = 3$ case here, since the explicit description of Eichler-Shimura maps is important itself and might be useful for further study.

We firstly recall the basic fact about $(\mathfrak{gl}_3(\mathbf{R}), K_3)$-cohomology, which is originally worked out by Clozel ([Clo90, Lemme 3.14]). Here we introduce slightly modified statement as follows:

Theorem 2.3. ([Mah05, Section 3.1.2] and [GR14, Section 5.5]) *Let $K_3 = \mathbf{R}_{>0}^\times \mathrm{SO}(3)$ and define $H_{\pi^{(3)}, K_3}$ to be the $(\mathfrak{gl}_3(\mathbf{R}), K_3)$-module associated with $\pi_\infty^{(3)}$. Then we have*

$$H^i(\mathfrak{gl}_3(\mathbf{R}), K_3; H_{\pi^{(3)}, K_3} \otimes L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C})) \cong \begin{cases} \mathbf{C} & \text{for } i = 2 \text{ and } 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let $i = 2, 3$. By using [BW80, II, Proposition 3.1], we obtain a natural isomorphism

$$H^i(\mathfrak{gl}_3(\mathbf{R}), K_3; H_{\pi^{(3)},K_3} \otimes L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C})) \cong \mathrm{Hom}_{\mathrm{SO}(3)}\left(\bigwedge^i \mathcal{P}_{3,\mathbf{C}}, H_{\pi^{(3)},K_3} \otimes L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C})\right).$$

Hence the $\pi^{(3)}$-isotypic component of $H^i_{\mathrm{cusp}}(Y^{(3)}_{\mathcal{K}_3}, \mathcal{L}^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C}))$ is given by

$$\left(H_{\pi^{(3)},K_3} \otimes L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C}) \otimes \bigwedge^i \mathcal{P}^*_{3,\mathbf{C}}\right)^{\mathrm{SO}(3)} \otimes \left(\pi^{(3)}_{\mathrm{fin}}\right)^{\mathcal{K}_3}, \tag{2.2}$$

where $\mathcal{P}^*_{3,\mathbf{C}}$ is the linear dual of $\mathcal{P}_{3,\mathbf{C}}$. For each cusp form $\boldsymbol{f} \in \mathcal{S}^{(3)}_{\boldsymbol{\lambda}}(\mathcal{K}_3)$, we construct an element $\delta^{(3),i}(\boldsymbol{f})$ in the above space.

Let us consider an element

$$P(X, Y, Z, A, B, C, z_1, z_2, z_3) \in (\mathbf{C}[X, Y, Z; A, B, C]_{w_{\boldsymbol{\lambda}}} \otimes \mathbf{C}[z_1, z_2, z_3]_{\lambda_3}/\mathcal{V}_{\boldsymbol{\lambda}})^{\oplus 7}$$

as follows:

$$\begin{aligned} &P(X, Y, Z, A, B, C, z_1, z_2, z_3) \\ &:= \left(\begin{pmatrix} X & Y & Z \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right)^{w_{\boldsymbol{\lambda}}} \otimes \left(\begin{pmatrix} A & B & C \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right)^{w_{\boldsymbol{\lambda}}} \begin{pmatrix} v_3^{(3,\delta)} & \dots & v_{-3}^{(3,\delta)} \end{pmatrix}. \end{aligned}$$

Then matrix $P(X, Y, Z, A, B, C, z_1, z_2, z_3)$ of polynomials satisfies the following distinguished properties:

Lemma 2.4. *Let $P(X, Y, Z, A, B, C, z_1, z_2, z_3)$ be as above.*

(i) *There exists a unique matrix*

$$\begin{aligned} &\mathcal{P}(X, Y, Z, A, B, C) \\ &= \begin{pmatrix} \mathcal{P}_{\lambda_3,3}(X, Y, Z, A, B, C) & \dots & \mathcal{P}_{\lambda_3,-3}(X, Y, Z, A, B, C) \\ \vdots & \ddots & \vdots \\ \mathcal{P}_{-\lambda_3,3}(X, Y, Z, A, B, C) & \dots & \mathcal{P}_{-\lambda_3,-3}(X, Y, Z, A, B, C) \end{pmatrix} \\ &\qquad \in \mathrm{M}_{2\lambda_3+1,7}(\mathbf{C}[X, Y, Z; A, B, C]_{w_{\boldsymbol{\lambda}},w_{\boldsymbol{\lambda}}}) \end{aligned}$$

such that $P(X, Y, Z, A, B, C, z_1, z_2, z_3) = \begin{pmatrix} v^{\boldsymbol{\lambda}}_{\lambda_3} & \dots & v^{\boldsymbol{\lambda}}_{-\lambda_3} \end{pmatrix} \mathcal{P}(X, Y, Z, A, B, C)$ holds. Moreover every entry of $\mathcal{P}(X, Y, Z, A, B, C)$ is an element of $L^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C})$ all of whose coefficients are contained in $\mathbf{Z}[2^{-1}, \sqrt{-1}]$.

(ii) *For each $u \in \mathrm{SO}(3)$, we have*

$$(\varrho^{(3)}_{\boldsymbol{\lambda}}(u)\mathcal{P})(X, Y, Z, A, B, C) = M^{-1}_{\boldsymbol{\lambda}}(u)\mathcal{P}(X, Y, Z, A, B, C)M_{(3,0)}(u).$$

For a cusp form $\boldsymbol{f} = \begin{pmatrix} f^{\boldsymbol{\lambda}}_{\lambda_3} & f^{\boldsymbol{\lambda}}_{\lambda_3-1} & \dots & f^{\boldsymbol{\lambda}}_{-\lambda_3} \end{pmatrix} \in \mathcal{S}^{(3)}_{\boldsymbol{\lambda}}(\mathcal{K}_3)$, we define $\delta^{(3),i}(\boldsymbol{f})$ to be

$$\delta^{(3),i}(\boldsymbol{f}) = \begin{pmatrix} f^{\boldsymbol{\lambda}}_{\lambda_3} & f^{\boldsymbol{\lambda}}_{\lambda_3-1} & \dots & f^{\boldsymbol{\lambda}}_{-\lambda_3} \end{pmatrix} \mathcal{P}(X, Y, Z, A, B, C) \begin{pmatrix} \boldsymbol{\omega}^i_3 \\ \vdots \\ \boldsymbol{\omega}^i_{-3} \end{pmatrix}.$$

Then Lemma 2.4 shows that $\delta^{(3),i}(\boldsymbol{f})$ is invariant under the $\mathrm{O}(3)$-action. This implies that $\delta^{(3),i}(\boldsymbol{f})$ gives an element in (2.2), and hence it gives a class of $H^i_{\mathrm{cusp}}(Y^{(3)}_{\mathcal{K}_3}, \mathcal{L}^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}}; \mathbf{C}))$.

3. Motivic interpretations

Assuming the existence of the conjectural motives attached to cohomological automorphic representations, Theorem 1.1 implies that the product $\Omega_{\pi^{(3)}}\Omega^{\pm}_{\pi^{(2)}}$ of Raghuram-Shahidi's Whittaker periods is related to Deligne's period of motives of Rankin-Selberg product. In this section, we clarify this relation by introducing the pure motives which is conjectured by Clozel ([Clo90]).

3.1. **Conjectural motives.** Notations are as in the previous sections. We briefly recall the conjectural description of motives attached to $\pi^{(n)}$.

Conjecture 3.1. ([Clo90, Conjecture 4.5]) *There exists a pure motive $\mathcal{M}[\pi^{(n)}]$ of rank n and of weight $-(2\nu_n - n + 1)$ which satisfies*

$$L(s, \mathcal{M}[\pi^{(n)}]) = L\left(s - \frac{n-1}{2}, \pi^{(n)}\right).$$

Since we normalized $\nu_n = -\frac{l_n}{2} + \frac{n-1}{2}$, $\mathcal{M}[\pi^{(n)}]$ has weight l_n. Let $\mathcal{M}(\pi^{(3)} \times \pi^{(2)})$ be the tensor product motive of $\mathcal{M}[\pi^{(3)}]$ and $\mathcal{M}[\pi^{(2)}]$. Then we find the following identity of L-functions:

$$L\left(s + \frac{1}{2}, \pi^{(3)} \times \pi^{(2)}\right) = L(s, \mathcal{M}(\pi^{(3)} \times \pi^{(2)})(2)),$$

where $\mathcal{M}(\pi^{(3)} \times \pi^{(2)})(2)$ denotes the 2-fold Tate twist of $\mathcal{M}(\pi^{(3)} \times \pi^{(2)})$.

3.2. **Period relations.** We introduce a description of Raghuram-Shahidi's Whittaker periods $\Omega_{\pi^{(3)}}$ in terms of motives.

We write $a \sim_E b$ for $a, b \in \mathbf{C}$ and a number field E if $a = bc$ for some $c \in E^{\times}$. For a pure motive $\mathcal{M}$ over $\mathbf{Q}$, let $\delta(\mathcal{M})$ and $c^{\pm}(\mathcal{M})$ be the periods in [Del79].

By calculating the Γ-factor, we find the following relation of critical values:

$$\sqrt{-1}^{\frac{l_3}{2}-m-1}\frac{L(m+\frac{1}{2}, \pi^{(3)} \times \pi^{(2)})}{\Omega_{\pi^{(3)}}\Omega^{\pm}_{\pi^{(2)}}} \sim_{\mathbf{Q}} \sqrt{-1}^{-l_2+1}\frac{L_{\mathrm{fin}}(m+\frac{1}{2}, \pi^{(3)} \times \pi^{(2)})}{(2\pi\sqrt{-1})^{3m+6-\frac{l_3}{2}-l_2}\Omega_{\pi^{(3)}}\Omega^{\pm}_{\pi^{(2)}}}, \tag{3.1}$$

where $(-1)^{\delta+\frac{l_3}{2}+1} = \pm(-1)^m$. Note that Theorem 1.1 yields that the right-hand side of the above equality is algebraic.

On the other hand, [Del79, (5.1.8)] implies that

$$\frac{L_{\mathrm{fin}}(0, \mathcal{M}(\pi^{(3)} \times \pi^{(2)})(m+2))}{c^{+}(\mathcal{M}(\pi^{(3)} \times \pi^{(2)})(m+2))} = \frac{L_{\mathrm{fin}}(0, \mathcal{M}(\pi^{(3)} \times \pi^{(2)})(m+2))}{(2\pi\sqrt{-1})^{3(m+2)}c^{(-1)^{m+2}}(\mathcal{M}(\pi^{(3)} \times \pi^{(2)}))} \tag{3.2}$$

and [Del79, Conjecture 1.8] yields that the above quantity is an algebraic number.

Combining (3.1) with (3.2), we obtain the following

$$c^{\pm}(\mathcal{M}(\pi^{(3)} \times \pi^{(2)})) \sim_{\overline{\mathbf{Q}}} (2\pi\sqrt{-1})^{-\frac{l_3}{2}-l_2}\Omega_{\pi^{(3)}}\Omega^{\pm(-1)^{\delta+\frac{l_3}{2}+1}}_{\pi^{(2)}}.$$

By [Yos01, Proposition 12], $c^{\pm}(\mathcal{M}(\pi^{(3)} \times \pi^{(2)}))$ can be written a product of periods of $\mathcal{M}[\pi^{(2)}]$ and $\mathcal{M}[\pi^{(3)}]$. Hence we obtain the following corollary:

Corollary 3.2. *We have the following period relation:*

$$\Omega_{\pi^{(3)}} \sim_{\overline{\mathbf{Q}}} (2\pi\sqrt{-1})^{\frac{l_3}{2}}c^{+}(\mathcal{M}[\pi^{(3)}])c^{-}(\mathcal{M}[\pi^{(3)}]).$$

Remark 3.3. Independently of our work, S.-Y. Chen [Che] also obtains the same formula in Corollary 3.2.

Remark 3.4. Analogue of the period relation in Corollary 3.2 for general n is deduced from an explicit formula for the unspecified constant $p_\infty^{\epsilon,\eta}(\mu,\lambda)$ in [Rag16, Theorem1.1 and 2.50]. Assuming $p_\infty^{\epsilon,\eta}(\mu,\lambda)$ is given by the product of a Γ-factor and an algebraic number, we clarified a motivic background of Raghuram-Shahidi's Whittaker periods for general n and general base fields. Our assumption on the evaluation of $p_\infty^{\epsilon,\eta}(\mu,\lambda)$ is recently proved in [IM] if the base field is totally imaginary. Hence in this case, the motivic interpretation of $\Omega^{\pm}_{\pi(n)}$, which is the generalization of Corollary 3.2 for general n, is obtained in a similar manner. The detail will appear in our forthcoming paper [HN2].

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