ON ICHINO-IKEDA TYPE FORMULA OF BESSEL PERIOD FOR (SO(5), SO(2)).

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Abstract. In this paper, we announce the results given in a joint work with Masaaki Furusawa (Osaka City University) on refined Gan-Gross-Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$, namely Ichino-Ikeda type formula of Bessel periods for $(\mathrm{SO}(5), \mathrm{SO}(2))$

1. Notation

Let F be a number field, and we denote its ring of adeles by $\mathbb{A}$. Let ψ be a non-trivial character of $\mathbb{A}/F$. For a place v of F, we denote by F_v the completion of F at v. Let E be a quadratic extension field of F and $\mathbb{A}_E$ be its ring of adeles. We write $E_v = E \otimes F_v$. Let σ denote the non-trivial element of $\mathrm{Gal}(E/F)$ and let us denote by $N_{E/F}$ the norm map from E to F. Let us take $d \in F^\times$ such that $\eta = \sqrt{-d}$ satisfies $E = F(\eta)$ and $\sigma(\eta) = -\eta$. We define a character ψ_E of $\mathbb{A}_E/E$ by $\psi_E(x) = \psi\left(\frac{x+\sigma(x)}{2}\right)$.

1.1. Groups.

Let D be a quaternion algebra over F containing E, which is possibly split. Let us denote its reduced norm by N_D and reduced trace by Tr_D. We denote the canonical involution of D by $x \mapsto \bar{x}$ for $x \in D$. Then we define a similitude quaternionic unitary group H_D over F by

$$H_D(F) = \left\{ g \in \mathrm{GL}_2(D) : {}^t\bar{g} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} g = \lambda_D(g) \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},\ \lambda_D(g) \in F^\times \right\}.$$

Here, $\bar{g} = \left(\begin{smallmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{smallmatrix}\right)$ for $g = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{GL}_2(D)$. When $D(F) \simeq \mathrm{Mat}_{2\times 2}(F)$, we know that H_D is isomorphic to a similitude symplectic group H of degree 2 defined by

$$H(F) = \left\{ g \in \mathrm{GL}_4(F) : {}^t gJg = \lambda(g)J,\ \lambda(g) \in F^\times \right\}$$

where

$$J = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}.$$

We note that for any orthogonal space V over F such that $\dim V = 5$ and its Witt index is at least 1, there is a quaternion algebra D over F such that

$$PH_D \simeq \mathrm{SO}(V).$$

Let us define symmetric matrices $S_i \in \mathrm{Mat}_{i+2,i+2}(F)$ by

$$S_0 = \begin{pmatrix} 2 & \\ & -2d \end{pmatrix} \quad \text{and} \quad S_i = \begin{pmatrix} & & 1 \\ & S_{i-1} & \\ 1 & & \end{pmatrix}.$$

Then we define a similitude orthogonal group $\mathrm{GO}_{n+2,n}$ over F by

$$\mathrm{GO}_{n+2,n}(F) = \left\{ g \in \mathrm{GL}_{2n+2}(F) : {}^t g S_n g = \mu(g) S_n, \mu(g) \in F^\times \right\}.$$

We denote its identity component by $\mathrm{GSO}_{n+2,n}$, indeed it is given by

$$\mathrm{GSO}_{n+2,n} = \{ g \in \mathrm{GO}_{n+2,n} : \det(g) = \mu(g)^{n+1} \}.$$

Also, we define a similitude quaternionic unitary group $\mathrm{GU}_{3,D}^{\eta} = \mathrm{GU}_{3,D}$ of degree 3 over F by

$$\mathrm{GU}_{3,D}(F) = \left\{ g \in \mathrm{GL}_3(D) : {}^t \bar{g} J_\eta g = \mu_D(g) J_\eta \right\}$$

where

$$J_\eta = \begin{pmatrix} & & \eta \\ & \eta & \\ \eta & & \end{pmatrix}.$$

We denote its identity component by $\mathrm{GSU}_{3,D}$.

2. Bessel periods

Let us define Bessel periods in several cases.

2.1. $(\mathrm{U}(2n), \mathrm{U}(1))$**-case.** Let $(V, (\ ,\ )_V)$ be a $2n$-dimensional hermitian spaces over E with a non-degenerate hermitian pairing $(\ ,\ )_V$ such that its Witt index of V is at least $n-1$. Then we have Witt decomposition

$$V = X^+ \oplus L \oplus X^-$$

where L is a 2-dimensional hermitian space over E, and $X^\pm$ are totally isotropic $(n-1)$-dimensional subspaces of V which are dual to each other and orthogonal to L. We take a basis $\{e_1, \ldots, e_{n-1}\}$ of X^+ and a basis $\{e_{-1}, \ldots, e_{-n+1}\}$ of X^-, respectively so that

$$(e_i, e_{-j})_V = \delta_{i,j} \tag{2.1.1}$$

for $1 \le i, j \le n-1$, where $\delta_{i,j}$ denotes Kronecker's delta.

Suppose $G = \mathrm{U}(V)$ and let us fix an anisotropic vector $e \in L$. Let P' be the maximal parabolic subgroup of G preserving the isotropic subspace X^-. Let M' and S' denote the Levi part and the unipotent part of P' respectively. We define a character χ_e of $S'(\mathbb{A})$ by

$$\chi_e \begin{pmatrix} 1_{n-1} & A & B \\ & 1_2 & A' \\ & & 1_{n-1} \end{pmatrix} = \psi_E((Ae, e_{n-1})).$$

Let U_{n-1} denote the group of upper unipotent matrices in $\mathrm{Res}_{E/F}\mathrm{GL}_{n-1}$. For $u \in U_{n-1}$, we define $\check{u} \in P'$ by

$$\check{u} = \begin{pmatrix} u & & \\ & 1_2 & \\ & & u^* \end{pmatrix}.$$

Then we define an unipotent subgroup S of P' by

$$S = S'S'' \quad \text{where} \quad S'' = \{ \check{u} : u \in U_{n-1} \}$$

and we extend χ_e to a character of $S(\mathbb{A})$ by putting

$$\chi_e(\check{u}) = \psi_E(u_{1,2} + \cdots + u_{n-2,n-1}) \quad \text{for} \quad u \in U_{n-1}(\mathbb{A}).$$

We define a subgroup D_e of G by

$$D_e = \left\{ \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix} : h \in \mathrm{U}(L),\, he = e \right\}$$

and let $R_e = D_e S$. Then the elements of $D_e(\mathbb{A})$ stabilize a character χ_e by conjugation. We note that

$$D_e(F) \simeq \mathrm{U}_1(F) := \{a \in E^\times : \bar{a}a = 1\}.$$

Let Λ be a character of $\mathbb{A}_E^\times / E^\times$ such that $\Lambda|_{\mathbb{A}^\times}$ is trivial. Then we may regard Λ as a character of $D_e(\mathbb{A})$ by $d \mapsto \Lambda(\det d)$. Then we define a character $\chi_{e,\Lambda}$ of $R_e(\mathbb{A})$ by

$$\chi_{e,\Lambda}(ts) = \Lambda(t)\chi_e(s) \quad \text{for} \quad t \in D_e(\mathbb{A}), s \in S(\mathbb{A}).$$

Let Σ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character and we denote its space by V_Σ. For a cusp form $\varphi \in V_\Sigma$, we define (e, ψ, Λ)-Bessel period of φ by

$$B_{e,\psi,\Lambda}(\varphi) = \int_{D_e(F)\backslash D_e(\mathbb{A})} \int_{S(F)\backslash S(\mathbb{A})} \chi_{e,\Lambda}^{-1}(ts)\varphi(ts)\, ds\, dt.$$

We say that (Σ, V_Σ) has (e, ψ, Λ)-Bessel period if $B_{e,\psi,\Lambda}(-) \not\equiv 0$ on V_Σ.

Also, when φ is a cusp form on the similitude unitary group $\mathrm{GU}(V)$, we define (e, ψ, Λ)-Bessel period of φ by $B_{e,\psi,\Lambda}(\varphi) := B_{e,\psi,\Lambda}(\varphi|_{\mathrm{U}(V,\mathbb{A})})$.

2.2. $(\mathrm{SO}(5), \mathrm{SO}(2))$**-case.** Let P_D be Siegel parabolic subgroup of H_D with the Levi decomposition $P_D = M_{H_D} N_{H_D}$ where

$$M_{H_D}(F) = \left\{ \begin{pmatrix} g & \\ & \lambda \cdot \bar{g}^{-1} \end{pmatrix} : g \in D^\times, \lambda \in F^\times \right\}$$

and

$$N_{H_D}(F) = \left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} : a \in D^1(F), \right\}.$$

Here, we define $D^1 = \{a \in D : \mathrm{Tr}_D(a) = 0\}$. For $S_D \in D^1(F) \cap D^\times(F)$, let us define a character $\psi_{S_D,D}$ of $N_{H_D}(\mathbb{A})$ by

$$\psi_{S_D,D}\left(\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \right) = \psi(\mathrm{Tr}_D(S_D a)).$$

Then the identity component of the stabilizer T_{D,S_D} of $\psi_{S_D,D}$ in M_{H_D} is

$$\left\{ \begin{pmatrix} h & \\ & N_D(h) \cdot \bar{h}^{-1} \end{pmatrix} : h \in T_{S_D} \right\}.$$

where

$$T_{S_D} = \left\{ t \in D^\times : \bar{t} S_D t = N_D(t) S_D \right\}.$$

We write this torus of H_D by the same symbol T_{S_D}. Let Λ be a character of $T_{S_D}(\mathbb{A}) / T_{S_D}(F)$ such that $\Lambda|_{\mathbb{A}^\times}$ is trivial. Let π_D be an irreducible cuspidal automorphic representation of $H_D(\mathbb{A})$ with trivial central character and we denote its space by V_{π_D}. Then we define (S_D, ψ, Λ)-Bessel period of $f \in V_{\pi_D}$ by

$$B_{S_D,\psi,\Lambda}(f) = \int_{\mathbb{A}^\times T_{D,S_D}(F)\backslash T_{D,S_D}(\mathbb{A})} \int_{N_{H_D}(F)\backslash N_{H_D}(\mathbb{A})} f(tu)\Lambda^{-1}(t)\psi_{S_D,D}^{-1}(u)\, dt\, du.$$

We say that (π_D, V_{π_D}) has (S_D, ψ, Λ)-Bessel period when $B_{S_D,\psi,\Lambda}(-) \not\equiv 0$ on V_{π_D}.

2.3. PGSO(6)**-case.** Let us recall the following accidental isomorphisms for PGU(4)

$$\mathrm{PGSO}_{4,2} \simeq \mathrm{PGU}(2,2) \qquad \text{and} \qquad \mathrm{PGSU}_{3,D} \simeq \mathrm{PGU}(3,1).$$

In [4], we give these isomorphisms explicitly. Using these isomorphisms, we can define Bessel periods for $\mathrm{PGSO}_{4,2}$ and $\mathrm{PGSU}_{3,D}$. We shall write these Bessel periods corresponding to $B_{e,\psi,\Lambda}$ by the same symbol.

3. Main results

Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $H_D(\mathbb{A})$ with trivial central character. For $\phi_1, \phi_2 \in V_\pi$, we define the Petersson inner product $(-,-)_\pi$ of π by

$$(\phi_1, \phi_2)_\pi = \int_{\mathbb{A}^\times H_D(F)\backslash H_D(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg$$

where we take the Tamagawa measure dg. Let us take hermitian $H_D(F_v)$-invariant local pairing $(-,-)_{\pi_v}$ of π_v so that $(-,-)_\pi = \prod_v (-,-)_{\pi_v}$. Hereafter, we fix S_D such that $T_{S_D} \simeq E^\times$ and fix a character Λ of $\mathbb{A}_E^\times/E^\times$ such that $\Lambda|_{\mathbb{A}^\times}$ is trivial. For each place v of F, we can define the local (S_D, ψ_v, Λ_v)-Bessel period $\alpha^{H_D}_{S_D,\Lambda_v}(-,-)$ as certain stable integral at non-archimedean places and certain Fourier tranform at archimedean places (for precise definition, see [6]).

We define Bessel period $B_{S_D,\psi,\Lambda}$ using Tamagawa measures dt and du on $T_{S_D}(\mathbb{A})$ and $N_{H_D}(\mathbb{A})$, respectively. Further, we fix a decomposition of these measures into local measures by

$$dt = C_{S_D} \prod dt_v$$

and $du = \prod du_v$. Our main result is the following Ichino-Ikeda type formula of Bessel periods for (SO(5), SO(2)).

Theorem 3.1. *Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $H_D(\mathbb{A})$ with trivial central character. Then there is an integer $\ell(\pi)$ which depends only on (π, V_π) and for any non-zero decomposable cusp form $\phi = \otimes_v \phi_v \in V_\pi$, we have*

(3.0.1)

$$\frac{|B_{S_D,\psi,\Lambda}(\phi)|^2}{(\phi,\phi)_\pi} = 2^{-\ell(\pi)}\, C_{S_D} \cdot \left(\prod_{j=1}^{2} \zeta_F(2j)\right) \frac{L(1/2, \pi \times \theta(\Lambda))}{L(1,\pi,\mathrm{Ad})\, L(1,\chi_E)} \cdot \prod_v \frac{\alpha_v^\natural(\phi_v)}{(\phi_v,\phi_v)_{\pi_v}}$$

Here, $\theta(\Lambda)$ denotes the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ *associated to Λ, $\zeta_F(s)$ denotes the complete zeta function of F, and*

$$\frac{\alpha_v^\natural(\phi_v)}{(\phi_v,\phi_v)_{\pi_v}} = \frac{L(1,\pi_v,\mathrm{Ad})\, L(1,\chi_{E,v})}{L(1/2, \pi_v \times \theta(\Lambda)_v) \prod_{j=1}^2 \zeta_{F_v}(2j)} \frac{\alpha^{H_D}_{S_D,\chi_v}(\phi_v,\phi_v)}{(\phi_v,\phi_v)_{\pi_v}},$$

which is equal to one at almost all places by [6].

Remark 3.1. *Assume that the endoscopic classification of PH_D, i.e.* [1, Conjecture 9.4.2, Conjecture 9.5.4] *holds for PH_D. Then we can show that $|\mathcal{S}(\phi_\pi)| = 2^{\ell(\pi)}$ as conjectured in* [6, Conjecture 2.5 (3)] *where ϕ_π denotes the Arthur parameter of π. In particular, when $H_D \simeq H$, our theorem proves Liu's conjecture because of* [1].

As a corollary of this formula, thanks to an explicit computation of local Bessel periods by Dickson et al. [3], we can show a generalization of Böcherere's conjecture on an explicit formula of Bessel periods for Siegel modular forms of degree two.

Theorem 3.2. *Let Φ be a holomorphic Siegel cusp form of degree two and weight k with respect to $H^1(\mathbb{Z})$ which is a Hecke eigenform and $\pi(\Phi)$ the associated automorphic representation of $PH(\mathbb{A})$. Let*

$$\Phi(Z) = \sum_{T>0} a(\Phi, T) \exp\left[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)\right], \ Z \in \mathfrak{H}_2,$$

be the Fourier expansion where T runs over semi-integral positive definite two by two symmetric matrices and $\mathfrak{H}_2$ denotes the Siegel upper half space of degree two.

Let E be an imaginary quadratic extension of $\mathbb{Q}$ with discriminant $-D_E$ and the ideal class group Cl_E. Then we define $\mathcal{B}_\Lambda(\Phi, E)$ as follows. Since $a(\Phi, T)$ depends only on an $\mathrm{SL}_2(\mathbb{Z})$ equivalence class of T with the action $\gamma \cdot T = {}^t\gamma T \gamma$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the notation $a(\Phi, c)$ for $c \in \mathrm{Cl}_E$ makes sense. Suppose that Λ is a character of Cl_E. Then we define

$$\mathcal{B}_\Lambda(\Phi, E) = w(E)^{-1} \cdot \sum_{c \in \mathrm{Cl}_E} a(\Phi, c)\Lambda^{-1}(c)$$

where $w(E)$ denotes the number of distinct roots of unity inside E. Suppose that Φ is not a Saito-Kurokawa lift. Then we have

$$\frac{|\mathcal{B}_\Lambda(\Phi, E)|^2}{\langle \Phi, \Phi \rangle} = 2^{2k-4} \cdot D_E^{k-1} \cdot \frac{L(1/2, \pi(\Phi) \times \theta(\Lambda))}{L(1, \pi(\Phi), \mathrm{Ad})}.$$

Here

$$\langle \Phi, \Phi \rangle = \int_{\mathrm{Sp}_4(\mathbb{Z}) \backslash \mathfrak{H}_2} |\Phi(Z)|^2 \det(Y)^{k-3}\, dX\, dY \quad \text{where } Z = X + \sqrt{-1}\,Y.$$

4. Idea of proof of Theorem 3.1

In this section, we would like to explain an idea of our proof of Theorem 3.1. For simplicity, we only explain the case where π has (S_D, ψ, Λ)-Bessel period.

(1) Compute the Bessel period $B_{(e,\psi,\Lambda)}$ of the theta lift $\theta_1(\pi)$ of π to $\mathrm{GSU}_{3,D}$ for a suitable e. Then we see that it can be written as an integral involving $B_{(S_D,\psi,\Lambda)}$.

(2) When $H_D \simeq H$ and the theta lift of π to $\mathrm{GSO}_{3,1}$ is non-zero, $\theta_1(\pi)$ is not cuspidal. In this case, our formula has already been proved by Corbett [2]. Hence, we may assume that this case does not occur. Then we can check that $\theta_1(\pi) \neq 0$ and it is irreducible cuspidal tempered automorphic representation. Moreover, as in our previous paper [4], we can reduce Theorem 3.1 to an explicit formula of $B_{(e,\psi,\Lambda)}$.

(3) Regard $\theta_1(\pi)$ as an automorphic representation of $\mathrm{GU}(4)$. Compute suitable Whittaker period of the theta lift $\theta_\Lambda(\theta_1(\pi))$ of $\theta_1(\pi)$ to $\mathrm{GU}(2,2)$. Then we see that it can be written as an integral involving $B_{(e,\psi,\Lambda)}$. Here, we note that the theta lift $\theta_\Lambda(\theta_1(\pi))$ depends on a choice of Λ. Further, because of the temperedness, we can check that $\theta_\Lambda(\theta_1(\pi))$ is an irreducible

cuspidal automorphic representation of $\mathrm{GU}(2,2)$. Then as in the previous step, we can reduce an explicit formula of $B_{(e,\psi,\Lambda)}$ to an explicit formula of Whittaker periods on $\mathrm{GU}(2,2)$.

(4) Regard $\theta_\Lambda(\theta_1(\pi))$ as an automorphic representation of $\mathrm{GSO}_{4,2}$. Then because of the conservation relation by Sun–Zhu [9] and the results by Takeda [10] and Yamana [11] on the non-vanishing of global theta lifts, we see that the theta lift π' of $\theta_\Lambda(\theta_1(\pi))$ to H is non-zero. Further, by [7], the Whittaker periods of theta lift of π' to $\mathrm{GSO}_{4,2}$ is written as an integral involving Whittaker period on of $\theta_\Lambda(\theta_1(\pi))$. Then as in [6], we can reduce an explicit formula of Whittaker periods on $\mathrm{GU}(2,2)$ to an explicit formula of Whittaker periods of H.

(5) (a) Suppose that the theta lift σ of π' to $\mathrm{GSO}_{2,2}$ is non-zero. Then compute Whittaker periods of the theta lift $\pi' = \theta(\sigma)$ of σ to H, and it is well-known that it can be written as an integral involving Whittaker period on $\mathrm{GSO}_{2,2}$.

(b) Suppose that the theta lift π' to $\mathrm{GSO}_{2,2}$ is zero. Then compute the Whittaker periods the that lift of π' to $\mathrm{GSO}_{3,3}$, and we know that by Soudry [8], it can be written as an integral involvingWhittaker period on $\mathrm{GSO}_{3,3}$

(6) Recall that we have the accidental isomorphisms $\mathrm{PGSO}_{3,3} \simeq \mathrm{PGL}_4$ and $\mathrm{PGSO}_{2,2} \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$. From the above two steps, Theorem 3.1 is reduced to an explicit formula of Whittaker periods of PGL_n.

(7) Finally, an explicit formula of Whittaker periods of cusp forms on GL_n is proved by Lapid and Mao [5]. Then this completes our proof of Theorem 3.1.

Remark 4.1. *In this article, we only considered the case where* $\mathrm{SO}(2)$ *is not split, namely* $\mathrm{SO}(2) \simeq E^\times/F^\times$ *case. However, our proof of Theorem 3.1 can be applied to the split case with a slight modification. See* [4] *for details.*

Acknowledgements. This paper is based on the author's talk given at 2020 RIMS conference *Automorphic forms, Automorphic representations, Galois representations, and its related topics*. The author thanks the organizer of the conference, Takuya Yamauchi, for giving him the opportunity to talk.

The research of the author was supported in part by Grant-in-Aid for Young Scientists (B) 17K14166.

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