ON GENUINE CHARACTERS OF THE METAPLECTIC GROUP OF $\mathrm{SL}_2(\mathfrak{o})$ AND THETA FUNCTIONS

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Abstract. This is a write up based on the author's talk given at the RIMS conference "Automorphic forms, Automorphic representations, Galois representations, and its related topics".

Let F be a totally real number field and $\mathfrak{o}$ the ring of integers of F. We study theta functions which are Hilbert modular forms of half-integral weight for the Hilbert modular group $\mathrm{SL}_2(\mathfrak{o})$. We obtain an equivalent condition that there exists a multiplier system of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$. We determine the condition of F that there exists a theta function which is a Hilbert modular form of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$. The theta function is defined by a sum on a fractional ideal $\mathfrak{a}$ of F.

1. Introduction

Put $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. It is known that the modular forms of $\mathrm{SL}_2(\mathbb{Z})$ of weight $1/2$ and $3/2$ are the Dedekind eta function $\eta(z)$ and its cubic power $\eta^3(z)$ up to constant, respectively. Here, $\eta(z)$ is given by

$$\eta(z) = e(z/24) \prod_{m\geq 1} (1 - e(mz)) \qquad (z \in \mathfrak{h}),$$

where $\mathfrak{h}$ is the upper half plane. It is known that

$$\eta(z) = \frac{1}{2} \sum_{m\in\mathbb{Z}} \chi_{12}(m) e(mz/24), \qquad \eta^3(z) = \frac{1}{2} \sum_{m\in\mathbb{Z}} m\, \chi_4(m) e(mz/8).$$

Here, χ_{12} and χ_4 are the primitive character mod 12 and mod 4, respectively. Note that $\eta(z)$ and $\eta^3(z)$ are theta functions defined by a sum on $\mathbb{Z}$.

The function $\eta(z)$ has the transformation formula with respect to modular transformations (see [11, 12, 16]). Let $\left(\frac{\cdot}{\cdot}\right)$ be the Jacobi symbol. We define $\left(\frac{\cdot}{\cdot}\right)^*$ and $\left(\frac{\cdot}{\cdot}\right)_*$ by

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right), \qquad \left(\frac{c}{d}\right)_* = t(c,d) \left(\frac{c}{d}\right)^*, \qquad t(c,d) = \begin{cases} -1 & c, d < 0 \\ 1 & \text{otherwise}, \end{cases}$$

for $c \in \mathbb{Z}\backslash\{0\}$ and $d \in 2\mathbb{Z}+1$ such that $(c,d) = 1$. We understand

$$\left(\frac{0}{\pm 1}\right)^* = \left(\frac{0}{1}\right)_* = 1, \qquad \left(\frac{0}{-1}\right)_* = -1$$

(see [8, Chapter 4 §1]).

For $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathfrak{h}$, put

$$(1) \qquad J(g,z) = \begin{cases} \sqrt{d} & \text{if } c=0, d>0 \\ -\sqrt{d} & \text{if } c=0, d<0 \\ (cz+d)^{1/2} & \text{if } c \neq 0, \end{cases} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here, we choose $\arg(cz+d)$ such that $-\pi < \arg(cz+d) \leq \pi$. Then we have

$$(2) \qquad \eta(\gamma(z)) = \mathbf{v}_\eta(\gamma) J(\gamma, z)\eta(z), \qquad \gamma(z) = \frac{az+b}{cz+d} \in \mathfrak{h}$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, where the multiplier system $\mathbf{v}_\eta(\gamma)$ is given by

$$(3) \quad \mathbf{v}_\eta(\gamma) = \begin{cases} \left(\dfrac{d}{c}\right)^* e\left(\dfrac{(a+d)c - bd(c^2-1) - 3c}{24}\right) & c: \text{odd} \\ \left(\dfrac{c}{d}\right)_* e\left(\dfrac{(a+d)c - bd(c^2-1) + 3d - 3 - 3cd}{24}\right) & c: \text{even}. \end{cases}$$

It is natural to ask the following problem. When does a Hilbert modular theta series of weight 1/2 with respect to $\mathrm{SL}_2(\mathfrak{o})$ exist? Here, $\mathfrak{o}$ is the ring of integers of a totally real number field F.

In 1983, Feng [1] studied this problem. She gave a sufficient condition for the existence of a Hilbert modular theta series of weight 1/2 with respect to $\mathrm{SL}_2(\mathfrak{o})$ and constructed certain Hilbert modular theta series. These series are defined by a sum on $\mathfrak{o}$. In 1984, Naganuma [10] obtained a Hilbert modular form of level 1 for a real quadratic $\mathbb{Q}(\sqrt{D})$, $D \equiv 1 \bmod 8$ with class number one, using modular imbeddings, from the theta constant with the characteristic (1/2, 1/2, 1/2, 1/2) of degree 2.

In this paper, we solve the problem above completely. We consider theta functions defined by a sum on a fractional ideal $\mathfrak{a}$ of F.

2. Multiplier systems for $\mathrm{SL}_2(\mathfrak{o})$

From now on, let F be a totally real number field such that $[F:\mathbb{Q}] = n$. Let v be a place of F and $\mathbb{A}$ the adele ring of F. We denote the completion of F at v by F_v. If v is an infinite place, we write $v \mid \infty$. Otherwise, we write $v < \infty$. For $v < \infty$, let $\mathfrak{o}_v$, $\mathfrak{p}_v$ and q_v be the ring of integers of F_v, the maximal ideal of $\mathfrak{o}_v$ and the order of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$, respectively.

For any v, let $\iota_v : F \to F_v$ be the embedding. The entrywise embeddings of $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(F_v)$ are also denoted by ι_v. Let $\{\infty_1, \cdots, \infty_n\}$ be the set of infinite places of F. Put $\iota_i = \iota_{\infty_i}$ for $1 \leq i \leq n$. We embed $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{R})^n$ by $r \mapsto (\iota_1(r), \cdots, \iota_n(r))$.

The metaplectic group of $\mathrm{SL}_2(F_v)$ is denoted by $\widetilde{\mathrm{SL}_2(F_v)}$, which is a nontrivial double covering group of $\mathrm{SL}_2(F_v)$. Set-theoretically, it is

$$\{[g,\tau] \mid g \in \mathrm{SL}_2(F_v), \tau \in \{\pm 1\}\}.$$

Its multiplication law is given by $[g,\tau][h,\sigma] = [gh, \tau\sigma c(g,h)]$ for $[g,\tau], [h,\sigma] \in \widetilde{\mathrm{SL}_2(F_v)}$, where $c(g,h)$ is the Kubota 2-cycle on $\mathrm{SL}_2(F_v)$. Put $[g] = [g,1]$.

Let $\tilde{H}$ be the inverse image of a subgroup H of $\mathrm{SL}_2(F_v)$ in $\widetilde{\mathrm{SL}_2(F_v)}$. For $v < \infty$, a function $\epsilon_v : \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} \to \mathbb{C}$ is genuine if $\epsilon_v([1_2, -1]\gamma) = -\epsilon_v(\gamma)$ for any $\gamma \in \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$.

We denote the embedding of $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{A})$ by ι. The finite part of $\mathrm{SL}_2(\mathbb{A})$ is denoted by $\mathrm{SL}_2(\mathbb{A}_f)$. Let $\iota_f : \mathrm{SL}_2(F) \to \mathrm{SL}_2(\mathbb{A}_f)$ be the projection of the finite part and $\iota_\infty : \mathrm{SL}_2(F) \to \mathrm{SL}_2(F_\infty) = \mathrm{SL}_2(\mathbb{R})^n$ that of the infinite part. Then we have $\iota(g) = \iota_f(g)\iota_\infty(g)$ for any $g \in \mathrm{SL}_2(F)$. The embedding of F into $\mathbb{A}_f$ is also denoted by ι_f.

The adelic metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is a double covering of $\mathrm{SL}_2(\mathbb{A})$ and there exists a canonical embedding $\widetilde{\mathrm{SL}_2(F_v)} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$ for each v. Let $\tilde{H}$ be the inverse image of a subgroup H of $\mathrm{SL}_2(\mathbb{A})$ in $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. It is known that $\mathrm{SL}_2(F)$ can be canonically embedded into $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. The embedding $\tilde{\iota}$ is given by $g \mapsto ([\iota_v(g)])_v$ for each $g \in \mathrm{SL}_2(F)$. We define the maps $\tilde{\iota}_f : \mathrm{SL}_2(F) \to \widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ and $\tilde{\iota}_\infty : \mathrm{SL}_2(F) \to \widetilde{\mathrm{SL}_2(F_\infty)}$ by

$$\tilde{\iota}_f(g) = ([\iota_v(g)])_{v<\infty} \times ([1_2])_{v|\infty}, \qquad \tilde{\iota}_\infty(g) = ([1_2])_{v<\infty} \times ([\iota_i(g)])_{v|\infty}.$$

Then we have $\tilde{\iota}(g) = \tilde{\iota}_f(g)\tilde{\iota}_\infty(g)$ for any $g \in \mathrm{SL}_2(F)$.

For $\gamma = [g, \tau] \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathfrak{h}$, $\tilde{j} : \widetilde{\mathrm{SL}_2(\mathbb{R})} \times \mathfrak{h} \to \mathbb{C}$ is an automorphy factor given by

$$\tilde{j}(\gamma, z) = \begin{cases} \tau\sqrt{d} & \text{if } c = 0, d > 0, \\ -\tau\sqrt{d} & \text{if } c = 0, d < 0, \\ \tau(cz+d)^{1/2} & \text{if } c \neq 0. \end{cases} \tag{4}$$

Here, we choose $\arg(cz + d)$ such that $-\pi < \arg(cz + d) \leq \pi$. Note that $\tilde{j}([g, \tau], z)$ is the unique automorphy factor such that $\tilde{j}([g, \tau], z)^2 = j(g, z)$, where $j(g, z)$ is the usual automorphy factor on $\mathrm{SL}_2(\mathbb{R}) \times \mathfrak{h}$ (see [6, §7]). Note that $\tilde{j}([g], z) = J(g, z)$, where $J(g, z)$ is defined in (1).

Definition 1. Let $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ be a congruence subgroup. the map $\mathbf{v} = \mathbf{v}(\gamma) : \Gamma \to \mathbb{C}^\times$ is said to be a multiplier system of half-integral weight if $\mathbf{v}(\gamma)\prod_{i=1}^n \tilde{j}([\iota_i(\gamma)], z_i)$ is an automorphy factor for $\Gamma \times \mathfrak{h}^n$, where $\tilde{j}$ is the automorphy factor in (4).

Lemma 1. A function $\mathbf{v} : \Gamma \to \mathbb{C}^\times$ is a multiplier system of half-integral weight if and only if we have

$$\mathbf{v}(\gamma_1)\mathbf{v}(\gamma_2) = c_\infty(\gamma_1, \gamma_2)\mathbf{v}(\gamma_1\gamma_2) \qquad \gamma_1, \gamma_2 \in \Gamma,$$

where $c_\infty(\gamma_1, \gamma_2) = \prod_{i=1}^n c_\mathbb{R}(\iota_i(\gamma_1), \iota_i(\gamma_2))$. Here, $c_\mathbb{R}(\cdot, \cdot)$ is the Kubota 2-cocycle at infinite places.

Let $K_\Gamma \subset \mathrm{SL}_2(\mathbb{A}_f)$ be the closure of $\iota_f(\Gamma)$ in $\mathrm{SL}_2(\mathbb{A}_f)$. Then K_Γ is a compact open subgroup and we have $\iota_f^{-1}(K_\Gamma) = \Gamma$. Let $\tilde{K}_\Gamma$ be the inverse image of K_Γ in $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$.

Lemma 2. Let $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ be a genuine character. Put $\mathbf{v}_\lambda(\gamma) = \lambda(\tilde{\iota}_f(\gamma))$ for $\gamma \in \Gamma$. Then $\mathbf{v}_\lambda$ is a multiplier system of half-integral weight for Γ.

For $v < \infty$, we define a map $s_v : \mathrm{SL}_2(\mathfrak{o}_v) \to \{\pm 1\}$ by

$$s_v(g) = \begin{cases} 1 & c \in \mathfrak{o}_v^\times \\ \langle c, d\rangle_v & c \in \mathfrak{p}_v \backslash \{0\} \\ \langle -1, d\rangle_v & c = 0 \end{cases} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}_v).$$

Here, $\langle \cdot, \cdot \rangle_v$ is the quadratic Hilbert symbol for F_v. A map $\mathbf{s}_v : \mathrm{SL}_2(\mathfrak{o}_v) \to \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$ is given by $\mathbf{s}_v(g) = [g, s_v(g)]$ for $g \in \mathrm{SL}_2(\mathfrak{o}_v)$. This map is the splitting on $K_1(4)_v$, where

$$K_1(4)_v = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}_v) \mid c \equiv 0, d \equiv 1 \bmod 4\}.$$

If $K_\Gamma \subset K_1(4)_f = \prod_{v<\infty} K_1(4)_v$, we may define a splitting $\mathbf{s} : K_\Gamma \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$ by

$$\mathbf{s}(\gamma) = (\mathbf{s}_v(\iota_v(\gamma)))_{v<\infty} \times ([1_2])_{v|\infty}.$$

We consider it as a homomorphism. Then we have $\tilde{K}_\Gamma = \mathbf{s}(K_\Gamma) \cdot \{[1_2, \pm 1]\}$. Note that $\mathbf{s}(K_\Gamma) \subset \widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ is a compact open subgroup.

For any congruence subgroup Γ, a map $\mathbf{v}_0 : \Gamma \to \mathbb{C}^\times$ is defined by $\mathbf{v}_0(\gamma) = \prod_{v<\infty} s_v(\iota_v(\gamma))$, which is not always a multiplier system of half-integral weight for Γ.

Corollary 1. If $\Gamma \subset \Gamma_1(4) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}) \mid c \equiv 0, d \equiv 1 \bmod 4\right\}$, then $\mathbf{v}_0$ is a multiplier system of half-integral weight for Γ.

Proof. Since $\Gamma \subset \Gamma_1(4)$, we have $K_\Gamma \subset K_1(4)_f$. We define a genuine character $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ by

$$\lambda(\mathbf{s}(k)[1_2, \tau]) = \tau, \qquad k \in K_\Gamma, \tau \in \{\pm 1\}.$$

Put $\mathbf{v}_\lambda(\gamma) = \lambda(\tilde{\iota}_f(\gamma))$ for $\gamma \in \Gamma$. Since $\mathbf{s}(\gamma) = ([\iota_v(\gamma), s_v(\iota_v(\gamma))])_{v<\infty}$, we have

$$\mathbf{v}_\lambda(\gamma) = \lambda(\mathbf{s}(\gamma)[1_2, \mathbf{v}_0(\gamma)]) = \mathbf{v}_0(\gamma).$$

Therefore Lemma 2 proves the corollary. □

Now suppose that $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ is a congruence subgroup and that $\mathbf{v} : \Gamma \to \mathbb{C}^\times$ is a multiplier system of half-integral weight.

Lemma 3. There exists a genuine character $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$ if and only if there exists a congruence subgroup $\Gamma' \subset \Gamma \cap \Gamma_1(4)$ such that $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for any $\gamma \in \Gamma'$.

Proposition 1. If $F \neq \mathbb{Q}$, then any multiplier system $\mathbf{v}$ of half-integral weight of any congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ is obtained from a genuine character of $\tilde{K}_\Gamma$.

Proof. By Lemma 3, it suffices to show that there exists a congruence subgroup $\Gamma' \subset \Gamma \cap \Gamma_1(4)$ such that $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for any $\gamma \in \Gamma'$. We assume that a congruence subgroup Γ satisfies $\Gamma \subset \Gamma_1(4)$ by replacing Γ with $\Gamma \cap \Gamma_1(4)$. Since $\mathbf{v}_0(\gamma)/\mathbf{v}(\gamma)$ is a character of Γ, we have $\mathbf{v}_0(\gamma)/\mathbf{v}(\gamma) = 1$ for any $\gamma \in D(\Gamma)$. By the congruence subgroup property, $D(\Gamma)$ contains a congruence subgroup Γ' (see [14, Corollary 3 of Theorem 2] or [7, §3]). Thus we have $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for any $\gamma \in \Gamma'$, which proves this proposition. □

By Lemma 3 and Proposition 1, the multiplier system of half-integral weight of a congruence subgroup Γ associated to an automorphy factor in the sense of Shimura [15] is obtained from a genuine character of $\tilde{K}_\Gamma$.

Put

$$K_f = \prod_{v<\infty} \mathrm{SL}_2(\mathfrak{o}_v).$$

Then K_f is a compact open group of $\mathrm{SL}_2(\mathbb{A}_f)$. The inverse image of K_f in $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ is denoted by $\tilde{K}_f$. We have $\mathrm{SL}_2(\mathfrak{o}) = \mathrm{SL}_2(F) \cap K_f \cdot \mathrm{SL}_2(F_\infty)$.

Proposition 2. Let $\mathbf{v}$ be a multiplier system of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$. Then there exists a genuine character $\lambda : \tilde{K}_f \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$.

Proof. If $F \neq \mathbb{Q}$, the assertion is proved by Proposition 1. If $F = \mathbb{Q}$, then we have

$$\mathbf{v}_0(g) = \begin{cases} \left(\dfrac{d}{c}\right)^* & c: \text{ odd} \\ \left(\dfrac{c}{d}\right)_* & c: \text{ even}, \end{cases} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Put

$$\Gamma(12) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \bmod 12 \right\}$$

and let $\mathbf{v}_\eta$ be the multiplier system of $\eta(z)$ in (3). Then we have $\mathbf{v}_\eta(\gamma) = \mathbf{v}_0(\gamma)$ for $\gamma \in \Gamma(12)$. Since $\mathbf{v}_\eta(\gamma)/\mathbf{v}(\gamma) = 1$ for any $\gamma \in D(\mathrm{SL}_2(\mathbb{Z}))$, we have $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for any $\gamma \in D(\mathrm{SL}_2(\mathbb{Z})) \cap \Gamma(12)$, which is a congruence subgroup. By Lemma 3, there exists a genuine character $\lambda : \tilde{K}_f \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$. □

Corollary 2. There exists a multiplier system $\mathbf{v}$ of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$ if and only if 2 splits completely in $F/\mathbb{Q}$. There exists a genuine character of $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$ for any $v < \infty$, provided that this condition holds.

Proposition 3. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $\mathbf{v}_\lambda$ be a multiplier system of half-integral weight of $\mathrm{SL}_2(\mathfrak{o})$, where $\lambda = \prod_{v<\infty} \lambda_v$ is a genuine character of $\tilde{K}_f$. Put $S_2 = \{v < \infty \mid F = \mathbb{Q}_2\}$ and $T_3 = \{v < \infty \mid q_v = 3\}$. Then there exist continuous functions $\kappa_v(\iota_v(\gamma))$ for $v \in S_2 \cup T_3$ such that

$$\mathbf{v}_\lambda(\gamma) = \mathbf{v}_0(\gamma) \prod_{v \in S_2 \cup T_3} \kappa_v(\iota_v(\gamma)) \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}).$$

We omit the proof of this proposition and give one example instead. For $F=\mathbb{Q}$, we have $\mathbf{v}_\eta(g)=\mathbf{v}_0(g)\kappa_2(g)\kappa_3(g)$, where

$$\mathbf{v}_0(g)=\begin{cases}\left(\dfrac{d}{c}\right)^* & c:\ \text{odd}\\ \left(\dfrac{c}{d}\right)_* & c:\ \text{even},\end{cases}\qquad g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}),$$

$$\kappa_2(g)=\begin{cases}e(\frac{3}{8}[(a+d)c-3c]) & c:\ \text{odd}\\ e(\frac{3}{8}[(b-c)d+3(d-1)]) & c:\ \text{even},\end{cases}$$

$$\kappa_3(g)=e(\tfrac{-1}{3}[(a+d)c-bd(c^2-1)]).$$

3. The condition of the existence of a theta function

Suppose that 2 splits completely in $F/\mathbb{Q}$. In this case, there exists a genuine character $\lambda_v:\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}\to\mathbb{C}^\times$ for any $v<\infty$. If $v<\infty$, put $K_v=\mathrm{SL}_2(\mathfrak{o}_v)$. If $v\mid\infty$, put $K_v=\mathrm{SO}(2)$. Then K_v is a maximal compact subgroup of $\mathrm{SL}_2(F_v)$ for any v. Let $\psi:\mathbb{A}/F\to\mathbb{C}^\times$ be an additive character such that its v-component $\psi_v(x)$ equals $e(x)$ for any $v\mid\infty$. Put $\psi_\beta(x)=\psi(\beta x)$ and $\psi_{\beta,v}(x)=\psi_v(\beta x)$ for $\beta\in F^\times$.

For any v, let $S(F_v)$ be the Schwartz space of F_v. We denote the Weil representation of $\widetilde{\mathrm{SL}_2(F_v)}$ by $\omega_{\psi_\beta,v}$. For a genuine character $\lambda_v:\tilde{K}_v\to\mathbb{C}^\times$, we define the set $(\omega_{\psi_\beta,v},S(F_v))^{\lambda_v}$ by

$$(\omega_{\psi_\beta,v},S(F_v))^{\lambda_v}=\{f\in S(F_v)\mid \omega_{\psi_\beta,v}(\gamma)f=\lambda_v(\gamma)f \text{ for any } \gamma\in\tilde{K}_v\}.$$

We have an irreducible decomposition

$$\omega_{\psi_\beta,v}=\omega^+_{\psi_\beta,v}\oplus\omega^-_{\psi_\beta,v},$$

where $\omega^+_{\psi_\beta,v}$ (resp. $\omega^-_{\psi_\beta,v}$) is an irreducible representation of the set of even (resp. odd) functions in $S(\mathbb{R})$ (see [9, Lemma 2.4.4]).

The group $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ has a maximal compact subgroup $\widetilde{\mathrm{SO}(2)}$, which is the inverse image of $\mathrm{SO}(2)$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. It is known that if $\lambda_v:\widetilde{\mathrm{SO}(2)}\to\mathbb{C}^\times$ is a genuine character, $\dim_{\mathbb{C}}(\omega_{\psi_\beta,v},S(\mathbb{R}))^{\lambda_v}$ is at most 1. Let $\lambda_{\infty,1/2}$ be a genuine character of lowest weight 1/2 with respect to $(\omega^+_{\psi_\beta,v},S(\mathbb{R}))$ and $\lambda_{\infty,3/2}$ of lowest weight 3/2 with respect to $(\omega^-_{\psi_\beta,v},S(\mathbb{R}))$. For $\beta>0$, $(\omega^+_{\psi_\beta,v},S(\mathbb{R}))^{\lambda_{\infty,1/2}}=\mathbb{C}\,e(i\iota_v(\beta)x^2)$ and $(\omega^-_{\psi_\beta,v},S(\mathbb{R}))^{\lambda_{\infty,3/2}}=\mathbb{C}\,xe(i\iota_v(\beta)x^2)$ are spaces of lowest weight vectors. If $\beta<0$, there exist no lowest weight vectors with respect to $(\omega^+_{\psi_\beta,v},S(\mathbb{R}))$ or $(\omega^-_{\psi_\beta,v},S(\mathbb{R}))$.

Note that $\lambda_v(\mathbf{s}_v(\mathrm{SL}_2(\mathfrak{o}_v)))=1$ for any $v<\infty$ except for finitely many places. Then a genuine character $\lambda_f:\tilde{K_f}\to\mathbb{C}^\times$ is given by $\lambda_f(g)=\prod_{v<\infty}\lambda_v(g_v)$ for $g=(g_v)_v\in\tilde{K_f}$. Put $w=(w_1,\cdots,w_n)\in\{1/2,3/2\}^n$. We define an automorphy factor $j^{\lambda_f,w}(\gamma,z)$ for $\gamma\in\mathrm{SL}_2(\mathfrak{o})$ and $z=(z_1,\cdots,z_n)\in\mathfrak{h}^n$ by

$$j^{\lambda_f,w}(\gamma,z)=\prod_{v<\infty}\lambda_v([\iota_v(\gamma)])\prod_{i=1}^{n}\tilde{j}([\iota_i(\gamma)],z_i)^{2w_i}.$$

In particular, we have $j^{\lambda_f,w}(-1_2,z)=\prod_{v<\infty}\lambda_v([-1_2])\times(-1)^{\sum 2w_i}$.

Let $M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)$ be the space of Hilbert modular forms on $\mathfrak{h}^n$ with respect to $j^{\lambda_f,w}(\gamma,z)$. A holomorphic function $h(z)$ of $\mathfrak{h}^n$ belongs to the space $M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)$ if and only if

$$h(\gamma(z))=j^{\lambda_f,w}(\gamma,z)h(z),$$

where $\gamma(z)=(\iota_1(\gamma)(z_1),\cdots,\iota_n(\gamma)(z_n))$ for $\gamma\in\mathrm{SL}_2(\mathfrak{o})$ and $z\in\mathfrak{h}^n$. (When $F=\mathbb{Q}$, the usual cusp condition is also required.) If $j^{\lambda_f,w}(-1_2,z)$ does not equal 1, $M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)$ is $\{0\}$.

Put $K=K_f\times\prod_{v|\infty}\mathrm{SO}(2)$. There exists a genuine character $\lambda:\tilde{K}\to\mathbb{C}^\times$ such that its v-component equals λ_v, where λ_{∞_i} is $\lambda_{\infty,1/2}$ or $\lambda_{\infty,3/2}$ for $1\le i\le n$. Then we have an automorphy factor $j^{\lambda_f,w}(\gamma,z)$ corresponding to λ such that $\lambda_{\infty_i}=\lambda_{\infty,w_i}$.

For each $g\in\widetilde{\mathrm{SL}_2}(\mathbb{A})$, there exist $\gamma\in\mathrm{SL}_2(F)$, $g_\infty\in\widetilde{\mathrm{SL}_2(\mathbb{R})}^n$ and $g_f\in\tilde{K}_f$ such that $g=\gamma g_\infty g_f$ by the strong approximation theorem for $\mathrm{SL}_2(\mathbb{A})$. Put $\mathbf{i}=(\sqrt{-1},\cdots,\sqrt{-1})\in\mathfrak{h}^n$. For $h\in M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)$, put

$$\varphi_h(g)=h(g_\infty(\mathbf{i}))\lambda_f(g_f)^{-1}\prod_{i=1}^n\tilde{j}(g_{\infty_i},\sqrt{-1})^{-2w_i}.$$

Then φ_h is an automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$.

Let $\mathcal{A}_w(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})},\lambda_f)$ be the space of automorphic forms φ on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ satisfying the following conditions (1), (2), and (3).

(1) $\varphi(gk_\infty)=\varphi(g)\prod_{i=1}^n\tilde{j}(k_{\infty,i},\sqrt{-1})^{-2w_i}$ for any $g\in\widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $k_\infty=(k_{\infty,1},\ldots,k_{\infty,n})\in\widetilde{\mathrm{SO}(2)}^n$.

(2) φ is a lowest weight vector with respect to the right translation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}^n$.

(3) $\varphi(gk)=\lambda_f(k)^{-1}\varphi(g)$ for any $g\in\widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $k\in\tilde{K}_f$.

Then $\Phi:h\mapsto\varphi_h$ gives rise to an isomorphism

$$M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)\xrightarrow{\sim}\mathcal{A}_w(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})},\lambda_f).$$

For $\varphi\in\mathcal{A}_w(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})},\lambda_f)$, put $h=\Phi^{-1}(\varphi)$. Then we have

$$h(z)=\varphi(g_\infty)\prod_{i=1}^n\tilde{j}(g_{\infty_i},\sqrt{-1})^{2w_i},\qquad g_\infty\in\widetilde{\mathrm{SL}_2(\mathbb{R})}^n,\ g_\infty(\mathbf{i})=z.$$

When q_v is odd, there exists a genuine character $\epsilon_v:\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}\to\mathbb{C}^\times$ defined by $\epsilon_v([g,\tau])=\tau s_v(g)$. If $q_v\ge 5$, it is a unique genuine character of $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$.

Put $S_2=\{v<\infty\mid F_v=\mathbb{Q}_2\}$, $T_3=\{v<\infty\mid q_v=3\}$ and $S_3=\{v\in T_3\mid \lambda_v\ne\epsilon_v\}$. Since 2 splits completely in $F/\mathbb{Q}$, we have $|S_2|=n$. It is known that for $v<\infty$, $(\omega_{\psi_\beta,v},S(F_v))^{\lambda_v}$ is not 0 if and only if we have

$$\mathrm{ord}_v\psi_{\beta,v}\equiv\begin{cases}0 \bmod 2 & \text{if } \lambda_v=\epsilon_v\\ 1 \bmod 2 & \text{otherwise.}\end{cases}$$

Then, if $(\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v} \neq 0$ for any $v < \infty$, there exists a fractional ideal $\mathfrak{a}$ such that

(5) $$(8\beta)\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v = \mathfrak{a}^2,$$

where $\mathfrak{d}$ is the different of $F/\mathbb{Q}$. The set of totally positive elements of F is denoted by $F_+^\times$. Replacing β with $\beta\gamma^2$ and $\mathfrak{a}$ with $(\mathfrak{a}\gamma)^2$ in (5) for $\gamma \in F_+^\times$, we may assume $\mathrm{ord}_v \mathfrak{a} = 0$ for $v \in S_2 \cup S_3$. Then we have $\mathrm{ord}_v \psi_{\beta,v} = -1$ (resp. -3) for $v \in S_3$ (resp. S_2).

Conversely, suppose that there exists a fractional ideal $\mathfrak{a}$ satisfying (5) for a subset $S_3 \subset T_3$. For $v < \infty$, put

$$\lambda_v = \begin{cases} \epsilon_v & \text{if } \mathrm{ord}_v \psi_{\beta,v} \equiv 0 \mod 2 \\ \mu_\beta & \text{if } \mathrm{ord}_v \psi_{\beta,v} \equiv 1 \mod 2, \end{cases}$$

where μ_β is a certain genuine character such that $(\omega_{\psi_\beta,v}, S(F_v))^{\mu_\beta} \neq 0$. Then we have $(\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v} \neq 0$ for any $v < \infty$. Let $\lambda : \tilde{K} \to \mathbb{C}^\times$ be a genuine character such that its v-component equals λ_v, where $\lambda_{\infty_i} = \lambda_{\infty,w_i}$ for $w_i \in \{1/2, 3/2\}$. Put $S_\infty = \{\infty_i \mid w_i = 3/2\}$.

From now on, suppose that $\beta \in F_+^\times$. Let $S(\mathbb{A})$ be the Schwartz space of $\mathbb{A}$ and $(\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ the set of functions $\phi = \prod_v \phi_v \in S(\mathbb{A})$ such that $\phi_v \in (\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v}$ for any v. For $\phi \in S(\mathbb{A})$, we define the theta function Θ_ϕ by

(6) $$\Theta_\phi(g) = \sum_{\xi \in F} \omega_{\psi_\beta}(g)\phi(\xi) \qquad g = (g_v) \in \widetilde{\mathrm{SL}_2(\mathbb{A})},$$

where $\omega_{\psi_\beta}(g)\phi(\xi) = \prod_v \omega_{\psi_\beta,v}(g_v)\phi_v(\iota_v(\xi))$ is essentially a finite product. We have $\Theta_\phi(gk) = \lambda(k)^{-1}\Theta_\phi(g)$ for any $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $k \in \tilde{K}_f$. If $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$, then Θ_ϕ is a Hilbert modular form of weight $w = (w_1, \cdots, w_n)$.

It is known that

$$\omega_{\psi_\beta} = \bigoplus_S \omega_{\psi_\beta,S}, \qquad \omega_{\psi_\beta,S} = \left(\bigotimes_{v \in S} \omega^-_{\psi_\beta,v}\right) \otimes \left(\bigotimes_{v \notin S} \omega^+_{\psi_\beta,v}\right),$$

where S ranges over all finite subsets of places of F (see [2, §3.4]). We define a map Θ from ω_{ψ_β} to the space of automorphic forms on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ by $\Theta(\phi)(g) = \Theta_\phi(g)$. Then it is known that

(7) $$\mathrm{Im}(\Theta) \simeq \bigoplus_{|S|:\text{even}} \omega_{\psi_\beta,S},$$

(see [2, Proposition 3.1]).

Let $\mathbf{G}$ be the set of triplets $(\beta, S_3, \mathfrak{a})$ of $\beta \in F_+^\times$, a subset $S_3 \subset T_3$ and a fractional ideal $\mathfrak{a}$ of F satisfying (5) and the condition (A),

(A) $$|S_2| + |S_3| + |S_\infty| \in 2\mathbb{Z}.$$

We define an equivalence relation $\sim$ on $\mathbf{G}$ by

$$(\beta, S_3, \mathfrak{a}) \sim (\beta', S_3', \mathfrak{a}') \Longleftrightarrow S_3 = S_3',\ \beta' = \gamma^2\beta,\ \mathfrak{a}' = \gamma\mathfrak{a} \text{ for some } \gamma \in F^\times.$$

Theorem 1. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $\beta \in F_+^\times$, $\lambda : \tilde{K} \to \mathbb{C}^\times$ and $w_1, \dots, w_n \in \{1/2, 3/2\}$ be as above. Then there exists $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ such that $\Theta_\phi \neq 0$ if and only if there exists a fractional ideal $\mathfrak{a}$ of F such that $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$.

Proof. Let $\lambda_v : \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} \to \mathbb{C}^\times$ be the v-component of λ for any $v < \infty$. We already proved that there exists $\prod_{v<\infty} \phi_v \neq 0$ such that $\phi_v \in (\omega_{\psi_\beta, v}, S(F_v))^{\lambda_v}$ for any $v < \infty$ if and only if there exists a fractional ideal $\mathfrak{a}$ of F satisfying (5). Suppose that the equivalent conditions hold. Since we have $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,1/2}} = \mathbb{C}\, e(i\iota_v(\beta)x^2)$ and $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,3/2}} = \mathbb{C}\, xe(i\iota_v(\beta)x^2)$ for any $v \mid \infty$, there exists a nonzero $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$. It is clear that if there exists a nonzero $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$, $\prod_{v<\infty} \phi_v \neq 0$ satisfies $\phi_v \in (\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v}$ for any $v < \infty$.

Suppose there exists a nonzero $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$. Note that $|S_2| + |S_3| + |S_\infty|$ is the number of v such that ϕ_v is an odd function. Then $|S|$ in (7) is $|S_2| + |S_3| + |S_\infty|$. By (7), it is clear that $\Theta_\phi \neq 0$ if and only if the condition (A) holds. □

Let H be a group of fractional ideals that consists of all elements of the form

$$\prod_{v \in T_3} \mathfrak{p}_v^{e_v}, \qquad \sum_v e_v \in 2\mathbb{Z}.$$

Let Cl^+ be the narrow ideal class group of F. Put $\mathrm{Cl}^{+2} = \{\mathfrak{c}^2 \mid \mathfrak{c} \in \mathrm{Cl}^+\}$. We denote the image of the group H (resp. $\mathfrak{b} \in \mathrm{Cl}^+$) in $\mathrm{Cl}^+/\mathrm{Cl}^{+2}$ by $\bar{H}$ (resp. $[\mathfrak{b}]$).

Theorem 2. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $w_1, \dots, w_n \in \{1/2, 3/2\}$ be as above.

(1) Suppose that $|S_2| + |S_\infty|$ is even. Then there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ if and only if $[\mathfrak{d}] \in \bar{H}$.

(2) Suppose that $|S_2| + |S_\infty|$ is odd. Then there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ if and only if $T_3 \neq \emptyset$ and $[\mathfrak{d}\mathfrak{p}_{v_0}] \in \bar{H}$. Here, v_0 is any fixed element of T_3.

Proof. We prove the theorem in case (1). The proof for case (2) is similar.

If $[\mathfrak{d}] \in \bar{H}$, we have $(8\beta)\mathfrak{d} \prod_{v \in T_3} \mathfrak{p}_v^{e_v} = \mathfrak{a}'^2$ such that $\sum_v e_v$ is even for a fractional ideal $\mathfrak{a}'$ and $\beta \in F_+^\times$. Put $S_3 = \{v \in T_3 \mid e_v : \text{odd}\}$. Since $|S_2| + |S_3| + |S_\infty|$ is even, we have $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$, where

$$\mathfrak{a} = \prod_{v \in T_3 \setminus S_3} \mathfrak{p}_v^{-e_v/2} \mathfrak{a}'.$$

Conversely, if there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$, it satisfies (5) and $|S_3|$ is even. Then we have $[\mathfrak{d}] = \prod_{v \in S_3} [\mathfrak{p}_v] \in \bar{H}$. □

Let w_i be $1/2$ or $3/2$ for $1 \leq i \leq n$. Suppose that there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$. Replacing $(\beta, S_3, \mathfrak{a})$ with an equivalent element of $\mathbf{G}$, we may assume

$\mathrm{ord}_v\mathfrak{a} = 0$ for $v \in S_2 \cup S_3$. For $v \in S_2 \cup S_3$, put

$$f_v(x) = \begin{cases} 1 & \text{if } x \in 1 + 2\mathfrak{p}_v \\ -1 & \text{if } x \in -1 + 2\mathfrak{p}_v \\ 0 & \text{otherwise.} \end{cases}$$

Put

$$f = \prod_{v \in S_2 \cup S_3} f_v \times \prod_{v < \infty, v \notin S_2 \cup S_3} \mathrm{ch}\mathfrak{a}_v^{-1},$$

where $\mathfrak{a}_v = \mathfrak{a}\mathfrak{o}_v$. Here, $\mathrm{ch}A$ is the characteristic function of a set A. Put $\phi = f \times \prod_{i=1}^n f_{\infty,i}$, where $f_{\infty,i}(x) = x^{w_i - (1/2)} e(i\iota_i(\beta)x^2)$ for $x \in \mathbb{R}$. By Theorem 1, there exists $\Theta_\phi \neq 0$ of weight $w = (w_1, \cdots, w_n)$.

Put $z = (z_1, \cdots, z_n), \mathbf{i} = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathfrak{h}^n$. We define $x_i, y_i \in \mathbb{R}$ by $z_i = x_i + \sqrt{-1}y_i$ for $1 \leq i \leq n$. Then we have $z = g_\infty(\mathbf{i})$, where $g_\infty = (g_{\infty_1}, \cdots, g_{\infty_n}) \in \mathrm{SL}_2(\mathbb{R})^n$, $g_{\infty_i} = \begin{pmatrix} y_i^{1/2} & y_i^{1/2}x_i \\ 0 & y_i^{-1/2} \end{pmatrix}$. Since $\lambda_v([1_2]) = 1$ for $v < \infty$, we have

$$\Theta_\phi(g_\infty) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) \prod_{i=1}^n \omega_{\psi_\beta, \infty_i}([g_{\infty_i}]) f_{\infty,i}(\iota_i(\xi)).$$

Theorem 3. Let ϕ and Θ_ϕ be as above. We define a theta function $\theta_\phi : \mathfrak{h}^n \to \mathbb{C}$ by

$$\theta_\phi(z) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) \prod_{\infty_i \in S_\infty} \iota_i(\xi) \prod_{i=1}^n e(z_i \iota_i(\beta\xi^2)).$$

Then θ_ϕ is a nonzero Hilbert modular form of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with respect to a multiplier system.

Every theta function of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with a multiplier system may be obtained in this way.

Proof. Since

$$\omega_{\psi_\beta, \infty_i}([g_{\infty,i}]) f_{\infty,i}(\iota_i(\xi)) = y_i^{w_i/2} \iota_i(\xi)^{w_i - (1/2)} e(z_i \iota_i(\beta\xi^2)),$$

we have $\theta_\phi(z) = \Theta_\phi(g_\infty) \times \prod_{i=1}^n y_i^{-w_i/2}$. Then θ_ϕ is nonzero. Note that

$$\tilde{j}([g_{\infty_i}], \sqrt{-1})^{2w_i} = y_i^{-w_i/2}.$$

Since $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$, we have $\Theta_\phi \in \mathcal{A}_w(\mathrm{SL}_2(F) \backslash \widetilde{\mathrm{SL}_2(\mathbb{A})}, \lambda_f)$. Then we have $\theta_\phi = \Phi^{-1}(\Theta_\phi) \in M_w(\mathrm{SL}_2(\mathfrak{o}), \lambda_f)$. The multiplier system of θ_ϕ is $\mathbf{v}_\lambda$ given by

$$\mathbf{v}_\lambda(\gamma) = \mathbf{v}_0(\gamma) \prod_{v \in S_2 \cup T_3} \kappa_v(\iota_v(\gamma)) \qquad \gamma \in \mathrm{SL}_2(\mathfrak{o}),$$

where κ_v for $v \in S_2 \cup T_3$ is an continuous function in Proposition 3.

By Proposition 2, if θ is a theta function of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with a multiplier system $\mathbf{v}$, we have a genuine character λ_f of $\tilde{K}_f$ such that $\mathbf{v} = \mathbf{v}_{\lambda_f}$. Let $\lambda = \lambda_f \times \prod_{i=1}^n \lambda_{\infty,w_i}$ be a genuine character of $\tilde{K}$. Then there exists nonzero $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ such that $\theta = \theta_\phi$ up to constant, which completes the proof. □

Proposition 4. Let Cl be the usual ideal class group of F. Let $Sq : \mathrm{Cl} \to \mathrm{Cl}^+$ be the homomorphism given by $[\mathfrak{a}] \mapsto [\mathfrak{a}^2]$ for a fractional ideal $\mathfrak{a}$ of F. The number of equivalence classes of $\mathbf{G}$ is equal to

$$[E^+ : E^2] \sum_{\substack{S_3 \subset T_3 \\ \text{(A)}}} |Sq^{-1}([\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v])|,$$

where S_3 ranges over all subset of T_3 satisfying (A). Here, E^+ is the group of totally positive units of F and E^2 is the subgroup of squares of units of F.

Proof. We follow the argument of Hammond [5] Theorem 2.9. For given S_3 satisfying (A), the number of ideal classes $[\mathfrak{a}]$ such that $\mathfrak{a}^2$ is narrowly equivalent to $\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v$ is equal to $|Sq^{-1}([\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v])|$. Then for a given fractional ideal $\mathfrak{a}$ such that $\mathfrak{a}^2$ is narrowly equivalent to $\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v$, the number of equivalence classes of triplets of the form $(\beta, S_3, \mathfrak{a})$ such that $\beta \in F_+^\times$ satisfying (5) is equal to $[E^+ : E^2]$. □

4. The case F is a real quadratic field

Now suppose that $F = \mathbb{Q}(\sqrt{D})$, where $D > 1$ is a square-free integer and $D \equiv 1 \bmod 8$. Then 2 splits in $F/\mathbb{Q}$ and we have $\mathfrak{d} = (\sqrt{D})$. When there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$, one of the followings holds.

(C1) $(8\beta)\mathfrak{d} = \mathfrak{a}^2$ and $S_3 = \emptyset$.
(C2) $(8\beta)\mathfrak{d}\mathfrak{p} = \mathfrak{a}^2$ such that $N_{F/\mathbb{Q}}(\mathfrak{p}) = 3$ and $S_3 = \{\mathfrak{p}\}$.
(C3) $(8\beta)\mathfrak{d}\mathfrak{p}\bar{\mathfrak{p}} = \mathfrak{a}^2$ such that $N_{F/\mathbb{Q}}(\mathfrak{p}) = N_{F/\mathbb{Q}}(\bar{\mathfrak{p}}) = 3$ and $S_3 = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$.

If $|S_\infty|$ is even, (C1) or (C3) holds. If $|S_\infty|$ is odd, (C2) holds.

Proposition 5. Suppose that $F = \mathbb{Q}(\sqrt{D})$, where $D > 1$ is a square-free integer such that $D \equiv 1 \bmod 8$.

(1) There exist $\beta \in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying (C1) if and only if $p \equiv 1 \bmod 4$ for any prime $p \mid D$.
(2) There exist $\beta \in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying (C2) if and only if $p \equiv 0$ or $1 \bmod 3$ for any prime $p \mid D$.
(3) There exists $\beta \in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying (C3) if and only if $D \equiv 1 \bmod 24$ and $p \equiv 1 \bmod 4$ for any prime $p \mid D$.

Proof. For a prime ideal $\mathfrak{p}$ such that $N_{F/\mathbb{Q}}(\mathfrak{p}) = 3$, the equation $(8\beta)\mathfrak{d}\mathfrak{p} = \mathfrak{a}^2$ implies that the narrow ideal class of $\mathfrak{d}\mathfrak{p}$ is a square. Note that a positive integer x is of the form $3u^2 + v^2$ for some $u, v \in \mathbb{N}$ if and only if any prime p which divides x satisfies $p \equiv 0$ or $1 \bmod 3$. Here, a necessary and sufficient condition that the narrow ideal class of $\mathfrak{d}\mathfrak{p}$ is a square for a prime ideal $\mathfrak{p}$ which has norm 3 is that D is of the form $3u^2 + v^2$ for some $u, v \in \mathbb{N}$, which proves the second assertion.

The equation $(8\beta)\mathfrak{d} = \mathfrak{a}^2$ implies that the narrow ideal class of $\mathfrak{d}$ is a square. Note that a positive integer x is of the form $u^2 + v^2$ for some

$u, v \in \mathbb{N}$ if and only if any prime p which divides x satisfies $p \equiv 1 \bmod 4$. Then [5] Proposition 3.1 proves the first assertion.

There exist two distinct prime ideal $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ such that such that $N_{F/\mathbb{Q}}(\mathfrak{p}) = N_{F/\mathbb{Q}}(\bar{\mathfrak{p}}) = 3$ if and only if 3 splits in $F/\mathbb{Q}$. This condition holds if and only if $D \equiv 1 \bmod 24$. In the case $D \equiv 1 \bmod 24$, we have $\mathfrak{p}\bar{\mathfrak{p}} = (3)$. Then the equation $(8\beta)\mathfrak{d}\mathfrak{p}\bar{\mathfrak{p}} = \mathfrak{a}^2$ implies that the narrow ideal class of $\mathfrak{d}$ is a square. Thus, similarly to the first assertion, [5] Proposition 3.1 proves the third assertion. □

Example: put $D = 793 = 13 \cdot 61$. Then there exist $\beta \in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying any condition of (C1), (C2) or (C3). Moreover, Cl^+ has order 8 and the fundamental unit ε of F has norm 1. For example, put $\rho = (5 + \sqrt{D})/2$. Since $N_{F/\mathbb{Q}}(\rho) = -3 \cdot 8^2$, we have $(\rho) = \mathfrak{q}_2^6\mathfrak{q}_3$, where $\mathfrak{q}_3 = (3, 1-\sqrt{D})$ and $\mathfrak{q}_2 = (2, (1+\sqrt{D})/2)$ are prime ideals. Put $\beta = \rho\sqrt{D}/8$ and $\mathfrak{a} = \mathfrak{d}\mathfrak{q}_2^3\mathfrak{q}_3$. Then we have $(8\beta)\mathfrak{d}\mathfrak{q}_3 = \mathfrak{a}^2$.

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