Moduli stacks of (φ, Γ)-modules

Toby Gee (joint with Matthew Emerton)

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Galois representations

$K/\mathbf{Q}_p$ finite extension.

$G_K := \mathrm{Gal}(\overline{K}/K)$.

R a topological ring.

Is there a moduli space of representations $\rho : G_K \to \mathrm{GL}_d(R)$?

Moduli spaces of Galois representations

There is a good theory over $\mathbf{Z}_l$, $l \neq p$, or even $\mathbf{Z}[1/p]$.

Basic point: the action of wild inertia doesn't deform, and tame inertia is "easy". Can work with Weil–Deligne representations and get finite type lci moduli spaces (Helm,...)

Over $\mathbf{Z}_p$: doesn't work.

Mazur: fix $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$, $\mathbf{F}/\mathbf{F}_p$ finite, and consider lifts of $\overline{\rho}$ to $\rho : G_K \to \mathrm{GL}_d(R)$, where R is Artin local with residue field $\mathbf{F}$. Moduli space: $\mathrm{Spf}\, R_{\overline{\rho}}$.

How can we let $\overline{\rho}$ vary?

Existence of the stack

$K/\mathbf{Q}_p$ finite extension, $G_K := \mathrm{Gal}(\overline{K}/K)$.

Theorem

There is a Noetherian formal algebraic stack $\mathcal{X}_d$ over $\mathrm{Spf}\, \mathbf{Z}_p$, such that $\mathcal{X}_d(\mathrm{Spf}\, \overline{\mathbf{Z}_p})$ is naturally equivalent to the groupoid of continuous representations $G_K \to \mathrm{GL}_d(\overline{\mathbf{Z}_p})$.

The underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack of finite type over $\mathbf{F}_p$, and is equidimensional of dimension $[K : \mathbf{Q}_p]d(d-1)/2$.

Similarly for $\mathcal{X}_d(\overline{\mathbf{F}}_p)$. *Not* true for general p-adically complete $\mathbf{Z}_p$-algebras, e.g. $\mathbf{F}_p[x]$.

n.b. $\mathcal{X}_d$ is not a p-adic formal algebraic stack. It is probably $[K : \mathbf{Q}_p]d^2$-dimensional.

The 1-dimensional case

For simplicity from now, unless otherwise stated: $K = \mathbf{Q}_p$.

Characters $G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$ are of the form $\lambda_a \overline{\varepsilon}^i$, $0 \le i < p-1$, where λ_a is the unramified character taking $\mathrm{Frob} \mapsto a$.

$\mathcal{X}_1$ has $(p-1)$ irreducible components, indexed by i.

$\mathcal{X}_{1,\mathrm{red}}$ is 0-dimensional: one dimension for a, but automorphisms are $\mathbf{G}_m$.

Definition of the stack

A a finite type $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \ge 1$.

Definition

$\mathcal{X}_d(A) :=$ *rank d projective étale (φ, Γ)-modules with A-coefficients.*

Extend to general p-adically complete A by taking limits.

Identification of $\mathcal{X}_d(\overline{\mathbf{Z}}_p)$ with $G_K \to \mathrm{GL}_d(\overline{\mathbf{Z}}_p)$ is due to Fontaine.

étale (φ, Γ)-modules

A a finite type $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$.

A *rank d projective (φ, Γ)-module with A-coefficients* is: a rank d projective $A((T))$-module M with commuting semilinear actions of φ and Γ.

$\varphi : A((T)) \to A((T))$ is A-linear, $\varphi(1+T) = (1+T)^p$.

$\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$, $\varepsilon : \Gamma \to \mathbf{Z}_p^\times$ cyclotomic character.

$\gamma : A((T)) \to A((T))$ is A-linear, $\gamma(1+T) = (1+T)^{\varepsilon(\gamma)}$,

étale: $M = A((T)) \cdot \varphi(M)$.

Closed points and specializations

Irreducible representations $G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ are again discrete up to unramified twist, and give 0-dimensional substacks of the 1-dimensional algebraic stack $\mathcal{X}_{2,\mathrm{red}}$.

Reducible indecomposable representations $\begin{pmatrix} \lambda_a \overline{\varepsilon}^i & * \\ 0 & \lambda_b \overline{\varepsilon}^j \end{pmatrix}$ with $* \neq 0$ can specialize to $* = 0$, so these are not closed points.

Fact: closed points of $\mathcal{X}_d$ = semisimple $G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$.

Closed points and specializations

Are there other specializations?

Consider the family of étale φ-modules over $\overline{\mathbf{F}}_p$ given by

$$\varphi = \begin{pmatrix} a_p & -1 \\ T^i & 0 \end{pmatrix}$$

with $1 \leq i \leq p-2$ and $a_p \in \overline{\mathbf{F}}_p$. (This can be equipped with an action of Γ).

If $a_p \neq 0$, corresponding $G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is reducible, but if $a_p = 0$ it is irreducible.

This cannot happen for a literal family of representations!

Irreducible components

Example from the previous slide is a feature, not a bug.

For each $0 \leq i, j < p-1$ consider $\begin{pmatrix} \lambda_a \overline{\varepsilon}^i & * \\ 0 & \lambda_b \overline{\varepsilon}^j \end{pmatrix}$ with a, b varying.

The closure of this is a 1-dimensional substack.

In fact if $i \equiv j+1 \pmod{p-1}$ we have an extra irreducible component with $a = b$.

Theorem

($K = \mathbf{Q}_p$) $\mathcal{X}_2$ has $p(p-1)$ irreducible components, indexed by i, j as above.

More generally, the irreducible components of $\mathcal{X}_d$ are indexed by $(k_1, \ldots, k_n)$ with $0 \leq k_i - k_{i+1} \leq p-1$, $0 \leq k_n < p-1$.

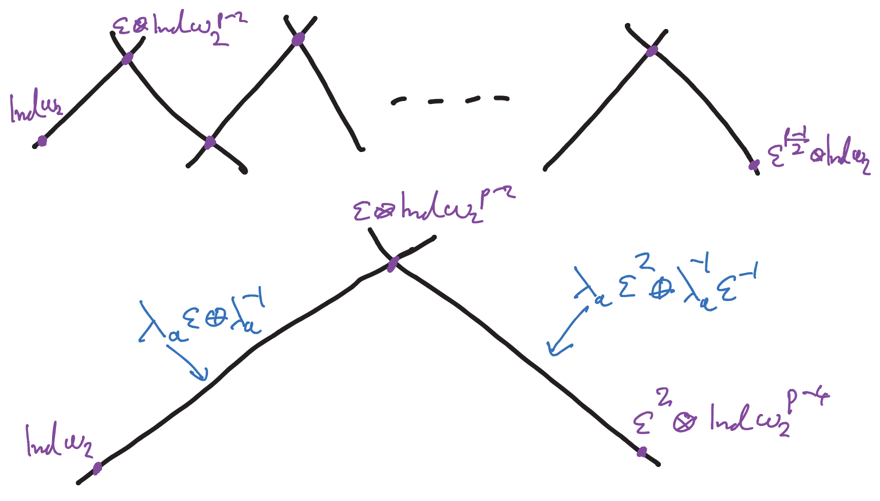
A coarse moduli space for $\mathrm{GL}_2(\mathbf{Q}_p)$

In general the specialisation relations are complicated and no obvious coarse moduli space.

For $\mathrm{GL}_2(\mathbf{Q}_p)$: the only interesting specialisations are those we wrote down before.

Fixing determinants, have a 1-dimensional scheme, in fact a chain of $\mathbf{P}^1$s.

A coarse moduli space for $\mathrm{GL}_2(\mathbf{Q}_p)$



The crystalline/potentially semistable substacks

Fix $\underline{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_d)$.

Theorem

There is a unique closed substack $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ of $\mathcal{X}_d$ which is flat over $\operatorname{Spf} \mathbf{Z}_p$, and such that if $A/\mathbf{Z}_p$ is finite flat, then

$$\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}} = \{\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_d(A) \mid \\ \rho \otimes \mathbf{Q}_p \textit{ is crystalline with Hodge–Tate weights } \underline{\lambda}\}.$$

$\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ is a p-adic formal algebraic stack.

If $\lambda_1 > \cdots > \lambda_d$ then $(\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}})_{\mathbf{F}_p}$ is equidimensional of dimension equal to $\dim \mathcal{X}_{d,\mathrm{red}}$.

Analogous result holds for (potentially) semistable representations, for any K.

The geometric Breuil–Mézard conjecture I

If $\lambda_1 > \cdots > \lambda_d$ then $(\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}})_{\mathbf{F}_p}$ is equidimensional of dimension $\dim \mathcal{X}_{d,\mathrm{red}}$.

Not necessarily reduced: write $\mathcal{Z}(\underline{\lambda}) = \mathcal{Z}((\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}})_{\mathbf{F}_p})$, a formal sum of irreducible components of $\mathcal{X}_{d,\mathrm{red}}$.

Question (Breuil–Mézard): Which components? What are the multiplicities?

p-adic local Langlands

Expectation/hope: there is a sheaf $\mathcal{M}$ of $\mathrm{GL}_d(K)$-representations on $\mathcal{X}_d$ which:

- satisfies local-global compatibility for completed cohomology of locally symmetric spaces.
- encodes the weight part of Serre's conjecture.
- answers the question of Breuil–Mézard.
- ...

(p-adic analogue of Hellmann/Ben-Zvi–Chen–Helm–Nadler/Zhu.)

Dotto–Emerton–G. (in progress): holds for $\mathrm{GL}_2(\mathbf{Q}_p)$.

Idea: Colmez's construction of $(D \boxtimes \mathbf{P}^1)/(D^\natural \boxtimes \mathbf{P}^1)$ makes sense on $\mathcal{X}_2$.

The weight part of Serre's conjecture

$K = \mathbf{Q}_p$. Let σ be an irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_d(\mathbf{F}_p)$.

$\mathcal{M}(\sigma) := \mathrm{Hom}_{\mathrm{GL}_d(\mathbf{Z}_p)}(\mathcal{M}, \sigma^\vee)^\vee$.

Expectation: support of $\mathcal{M}(\sigma)$ is a union of irreducible components of $\mathcal{X}_{d,\mathrm{red}}$.

$\mathcal{Z}(\sigma) := \mathcal{Z}(\mathcal{M}(\sigma)))$.

If local-global compatibility holds, the support of $\mathcal{Z}(\sigma)$ exactly determines the weight part of Serre's conjecture.

True for modular curves.

The geometric Breuil–Mézard conjecture II

For $\underline{\lambda} = (\lambda_1 > \cdots > \lambda_d)$, set $\mathcal{M}(\underline{\lambda}) := \mathrm{Hom}_{\mathrm{GL}_d(\mathbf{Z}_p)}(\mathcal{M}, \pi_{\underline{\lambda}}^{\vee})^{\vee}$, where $\pi_{\underline{\lambda}} =$ irreducible algebraic $\mathrm{GL}_d(\mathbf{Z}_p)$-representation of highest weight $(\lambda_1 - (d-1), \ldots, \lambda_{d-1} - 1, \lambda_d)$.

Expectations imply: $\mathcal{Z}(\mathcal{M}(\underline{\lambda})) = \mathcal{Z}(\underline{\lambda}) = \mathcal{Z}((\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}})_{\mathbf{F}_p})$.

Then we have the *geometric Breuil–Mézard conjecture* $\mathcal{Z}(\underline{\lambda}) = \sum_\sigma n_\sigma(\underline{\lambda}) \mathcal{Z}(\sigma)$, where $n_\sigma(\underline{\lambda}) =$ multiplicity of σ in $\pi_{\underline{\lambda}} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$.

Known for $\mathrm{GL}_2(\mathbf{Q}_p)$ (Kisin, Paškūnas,. . .)

Extends to potentially crystalline/semistable case, using inertial local Langlands correspondence.

This version can sometimes be proved if $\underline{\lambda}$ is small, e.g. G.–Kisin, Le–Le Hung–Levin–Morra.

Patched modules and deformation rings

No construction is known of $\mathcal{M}$ other than for GL_1 or $\mathrm{GL}_2(\mathbf{Q}_p)$.

However there is a candidate after pulling back to the versal ring at a fixed $\overline{\rho} : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$: the *patched module* M_∞ of Caraiani–Emerton–G.–Geraghty–Paškūnas–Shin.

Construction via globalisation and Taylor–Wiles patching of cohomology of unitary Shimura varieties.

The pullback of the geometric Breuil–Mézard conjecture $\mathcal{Z}(\underline{\lambda}) = \sum_\sigma n_\sigma(\underline{\lambda}) \mathcal{Z}(\sigma)$ to the versal ring is equivalent to automorphy lifting theorems (Kisin).

e.g. can use solvable base change to reduce the potentially crystalline case to the crystalline case (G.–Kisin).