# Beilinson–Bloch conjecture for unitary Shimura varieties

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## 1 The BSD conjecture

- $E : y^2 = x^3 + Ax + B$ an elliptic curve over $\mathbb{Q}$.
- *Algebraic rank*: the rank of the finitely generated abelian group $E(\mathbb{Q})$

$$r_{\mathrm{alg}}(E) := \operatorname{rank} E(\mathbb{Q}).$$

- *Analytic rank*: the order of vanishing of $L(E, s)$ at the central point $s = 1$

$$r_{\mathrm{an}}(E) := \operatorname{ord}_{s=1} L(E, s).$$

Conjecture (Birch–Swinnerton-Dyer, 1960s)

(1) (Rank)
$$r_{\mathrm{an}}(E) \overset{?}{=} r_{\mathrm{alg}}(E),$$

(2) (Leading coefficient) For $r = r_{\mathrm{an}}(E)$,
$$\frac{L^{(r)}(E,1)}{r!} \overset{?}{=} \frac{\Omega(E)R(E)\prod_p c_p(E) \cdot |Ш(E)|}{|E(\mathbb{Q})_{\mathrm{tor}}|^2}$$
where $R(E) = \det(\langle P_i, P_j \rangle_{\mathrm{NT}})_{r \times r}$ is the regulator for the Néron–Tate height pairing
$$\langle\ ,\ \rangle_{\mathrm{NT}} : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$$
and $Ш(E)$ is the Tate–Shafarevich group.

*Remark.* (Tate, *The Arithmetic of Elliptic Curves*, 1974)

> This remarkable conjecture relates the behavior of a function $L$ at a point where it is not at present known to be defined to the order of a group *Ш* which is not known to be finite!

## 2 What is known about BSD?

The BSD conjecture is still widely open in general, but much progress has been made in the rank 0 or 1 case.

Theorem (Gross–Zagier, Kolyvagin, 1980s)

$$r_{\mathrm{an}}(E) = 0 \Rightarrow r_{\mathrm{alg}}(E) = 0, \quad r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) = 1,$$

*Remark.* When $r = r_{\mathrm{an}}(E) \in \{0,1\}$, many cases of the formula for $L^{(r)}(E,1)$ are known. The proof combines two inequalities:

(1) (Gross–Zagier formula)

$$r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) \geq 1.$$

(2) (Kolyvagin's Euler system)

$$r_{\mathrm{an}}(E) \in \{0,1\} \Rightarrow r_{\mathrm{alg}}(E) \leq r_{\mathrm{an}}(E).$$

Both steps rely on *Heegner points* on modular curves.

## 3 The Beilinson–Bloch conjecture

- $X$: smooth projective variety over a number field $K$.
- $\mathrm{CH}^m(X)$: the Chow group of algebraic cycles of codimension $m$ on $X$.
- $\mathrm{CH}^m(X)^0 \subseteq \mathrm{CH}^m(X)$: the subgroup of geometrically cohomologically trivial cycles.
- Beilinson–Bloch height pairing

$$\langle\ ,\ \rangle_{\mathrm{BB}} : \mathrm{CH}^m(X)^0 \times \mathrm{CH}^{\dim X+1-m}(X)^0 \to \mathbb{R}.$$

- $L(H^{2m-1}(X),s)$: the motivic $L$-function for $H^{2m-1}(X_{\bar{K}}, \mathbb{Q}_\ell)$.

Conjecture (Beilinson–Bloch, 1980s)

(1) (Rank) $\qquad \operatorname{ord}_{s=m} L(H^{2m-1}(X),s) \overset{?}{=} \operatorname{rank} \mathrm{CH}^m(X)^0.$

(2) (Leading coefficient) $\qquad L^{(r)}(H^{2m-1}(X),m) \overset{?}{\sim} \det(\langle Z_i, Z_j' \rangle_{\mathrm{BB}})_{r\times r}$

Example ($X/K = E/\mathbb{Q}$ and $m = 1$)
BB recovers the BSD conjecture as

$$\mathrm{CH}^1(E)^0 \simeq E(\mathbb{Q}), \quad L(H^1(E),s) = L(E,s), \quad \langle\ ,\ \rangle_{\mathrm{BB}} = -\langle\ ,\ \rangle_{\mathrm{NT}}.$$

*Remark.* In general both sides are only conditionally defined!

(1) $L(H^{2m-1}(X),s)$ is not known to be analytically continued to the central point $s = m$.

(2) $\mathrm{CH}^m(X)^0$ is not known to be finitely generated.

## 4 Testable BB conjecture: $X$ = Shimura varieties

- Langlands–Kottwitz/Langlands–Rapoport: express the motivic $L$-functions of Shimura varieties $X = \mathrm{Sh}_G$, as a product of automorphic $L$-functions $L(s, \pi)$ on $G$,
$$L(H^{2m-1}(X), s+m) = \prod_{\pi} L(s+1/2, \pi).$$
- Assume from now (the most interesting case):
  (1) $2m-1 = \dim X$ (middle cohomology).
  (2) $\pi$ is tempered cuspidal.
- Analytic properties of $L(s, \pi)$ can be established.
- $\mathrm{CH}^m(X)^0$ is not known to be finitely generated, but we can test if it is nontrivial.

Unconditional prediction of BB conjecture, in the same spirit of Gross–Zagier:

Conjecture (Beilinson–Bloch for Shimura varieties)

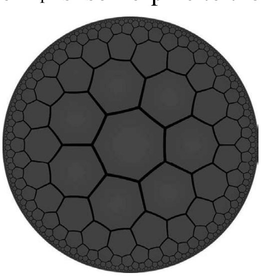
$$\mathrm{ord}_{s=1/2} L(s, \pi) = 1 \overset{?}{\Longrightarrow} \operatorname{rank} \mathrm{CH}^m(X)^0_\pi \geq 1.$$

*Remark.* Conjecture was only known for:
(1) $X$ = modular curves (Gross–Zagier)
(2) $X$ = Shimura curves (S. Zhang, Kudla–Rapoport–Yang, Yuan–Zhang–Zhang, Liu).
(3) $X = \mathrm{U}(1,1) \times \mathrm{U}(2,1)$ Shimura threefolds and $\pi$ = endoscopic (Xue).

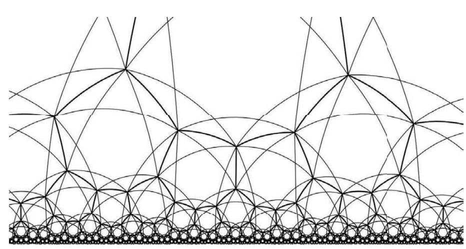
Theorem A (L.-Liu [1], impressionist version)
Conjecture holds for $\mathrm{U}(2m-1, 1)$-Shimura varieties and $\pi$ satisfying local assumptions.

## 5 The Hermitian symmetric space for $\mathrm{U}(n-1, 1)$

- Hermitian symmetric space for $\mathrm{U}(n-1,1)$,
$$\mathbb{D}_{n-1} := \{z \in \mathbb{C}^{n-1} : |z| < 1\} \cong \frac{\mathrm{U}(n-1,1)}{\mathrm{U}(n-1) \times \mathrm{U}(1)}.$$
- We have an action
$$\mathrm{U}(n-1,1) \curvearrowright \mathbb{D}_{n-1}.$$
- Notice $\mathbb{D}_1$ is isomorphic to the upper half plane $\mathbb{H}$.



$\longleftrightarrow$



## 6 Unitary Shimura varieties $X$

- $E/F$: CM extension of a totally real number field.
- $\mathbb{V}$: totally definite *incoherent* $\mathbb{A}_E/\mathbb{A}_F$*-hermitian space* of rank $n$.
- Incoherent: $\mathbb{V}$ is not the base change of a global $E/F$-hermitian space, or equivalently $\prod_v \varepsilon(\mathbb{V}_v) = -1$, where $\mathbb{V}_v := \mathbb{V} \otimes_{\mathbb{A}_F} F_v$.
- Any place $w|\infty$ of $F$ gives a nearby *coherent* $E/F$*-hermitian space* $V$ such that
$$V_v \cong \mathbb{V}_v, v \neq w, \quad \text{but } V_w \text{ has signature } (n-1, 1).$$
- $G = \mathsf{U}(\mathbb{V})$.
- $K \subseteq G(\mathbb{A}_F^\infty) \cong \mathsf{U}(V)(\mathbb{A}_F^\infty)$: open compact subgroup.
- $X$: *unitary Shimura variety* of dimension $n-1$ over $E$ such that for any place $w|\infty$ inducing $\iota_w : E \hookrightarrow \mathbb{C}$,
$$X(\mathbb{C}) = \mathsf{U}(V)(F)\backslash[\mathbb{D}_{n-1} \times \mathsf{U}(V)(\mathbb{A}_F)^\infty/K].$$
- $X$ is a Shimura variety *of abelian type*.
- Its étale cohomology and $L$-function are computed in the forthcoming work of Kisin–Shin–Zhu, under the help of the endoscopic classification for unitary groups (Mok, Kaletha–Minguez–Shin–White).

## 7 Automorphic representations $\pi$

- $W = E^{2m}$: the standard $E/F$-skew-hermitian space with matrix $\left(\begin{smallmatrix} 0 & 1_m \\ -1_m & 0 \end{smallmatrix}\right)$.
- $\mathsf{U}(W)$: quasi-split unitary group of rank $n = 2m$.
- $\pi$: cuspidal automorphic representation of $\mathsf{U}(W)(\mathbb{A}_F)$.

*Assumptions.*

(1) $E/F$ is unramified at all finite places (so $F \neq \mathbb{Q}$), and split at all 2-adic places. Assume that $E/\mathbb{Q}$ is Galois or contains an imaginary quadratic field (for simplicity).
(2) For $v|\infty$, $\pi_v$ is the holomorphic discrete series with Harish-Chandra parameter $\{\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-n+3}{2}, \frac{-n+1}{2}\}$.
(3) For $v \nmid \infty$, $\pi_v$ is tempered.
(4) For $v \nmid \infty$ split in $E$, $\pi_v$ is a principal series.
(5) For $v \nmid \infty$ inert in $E$, $\pi_v$ is unramified or almost unramified. If $\pi_v$ is almost unramified, then $v$ is unramified over $\mathbb{Q}$.

*Remark.*

(1) $\pi_v$ is *almost unramified*: $\pi_v$ has a nonzero Iwahori-fixed vector and its Satake parameter contains $\{q_v, q_v^{-1}\}$ and $2m-2$ complex numbers of norm 1. Equivalently, the theta lift of $\pi_v$ to the non-split local hermitian space has nontrivial invariants under the stabilizer of an almost self-dual lattice.
(2) Let $S_\pi = \{v \text{ inert} : \pi_v \text{ almost unramified}\}$. Then under *Assumptions*, the global root number for the (complete) standard $L$-function $L(s,\pi)$ equals
$$\varepsilon(\pi) = (-1)^{|S_\pi|}$$
by the epsilon dichotomy (Harris–Kudla–Sweet, Gan–Ichino).

## 8 Main result A: BB conjecture

When $\mathrm{ord}_{s=1/2} L(s,\pi) = 1$:

- $\varepsilon(\pi) = -1$ and so $|S_\pi|$ is odd.
- $\mathbb{V} = \mathbb{V}_\pi$: totally definite incoherent space of rank $n = 2m$ such that
$$\varepsilon(\mathbb{V}_v) = -1 \text{ exactly for } v \in S_\pi.$$
- Associated unitary Shimura variety $X$ of dimension $n-1 = 2m-1$ over $E$.
- $\mathrm{CH}^m(X)^0_\pi$ the localization of $\mathrm{CH}^m(X)^0_\mathbb{C}$ at the maximal ideal $\mathfrak{m}_\pi$ of the Hecke algebra associated to $\pi$.

Theorem A (L.–Liu [1], 2020)
Let $\pi$ be a cuspidal automorphic representation of $\mathsf{U}(W)(\mathbb{A}_F)$ satisfying *Assumptions*. Then the implication
$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \Longrightarrow \mathrm{rank}\, \mathrm{CH}^m(X)^0_\pi \geq 1$$
holds when the level $K \subseteq G(\mathbb{A}_F^\infty)$ is sufficiently small.

Nontrivial cycles constructed via the method of *arithmetic theta lifting* (Kudla, Liu).
Next: a baby example of Heegner points.

## 9 The Gross–Zagier formula

- Modular curve
$$X_0(N) = \Gamma_0(N)\backslash\mathbb{H} \cup \{\text{cusps}\} = \{(E_1 \xrightarrow[N\text{-isogeny}]{\text{cyclic}} E_2)\}$$
- For certain imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$, get a *Heegner divisor*
$$Z(d) := \{(E_1 \to E_2) \text{ with endomorphisms by } O_K\} \in \mathrm{CH}^1(X_0(N)).$$
- The theory of complex multiplication: $Z(d)$ is defined over $K$.
- $E/\mathbb{Q}$ elliptic curve of conductor $N$ has a modular parametrization
$$\varphi_E : X_0(N) \to E.$$
- Get a *Heegner point*
$$P_K \in \varphi_E(Z(d) - \deg Z(d) \cdot \infty) \in E(K).$$

Theorem (Gross–Zagier, 1980s)
Up to simpler nonzero factors,
$$L'(E_K, 1) \sim \langle P_K, P_K \rangle_{\mathrm{NT}}.$$
*Remark.*

(1) Choosing $K$ suitably gives the implications $r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) \geq 1$.

(2) BSD formula for $E_K$ reduces to a precise relation between $P_K$ and $\text{Ш}(E_K)$.

## 10 Generating series of Heegner points

Take $P_d = \mathrm{tr}_{K/\mathbb{Q}} P_K \in E(\mathbb{Q})$. It may depend on the choice of $d$, even when $E(\mathbb{Q}) \cong \mathbb{Z}$.

Example ($E = 37a1 = X_0^+(37) : y^2 + y = x^3 - x$)

- $E(\mathbb{Q}) \cong \mathbb{Z}$ with a generator $P = (0,0)$.
- $E$ corresponds to the modular form $f \in S_2(37)$,

$$f = \sum_{n\geq 1} a_n q^n = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} - 6q^{12} - 2q^{13} + \cdots$$

- Table of Heegner points $P_d$:

| $d$ | 3 | 4 | 7 | 11 | 12 | 16 | 27 | $\cdots$ | 67 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $P_d$ | $(0,-1)$ | $(0,-1)$ | $(0,0)$ | $(0,-1)$ | $(0,0)$ | $(1,0)$ | $(-1,-1)$ | $\cdots$ | $(6,-15)$ | $\cdots$ |
| $c_d$ | $-1$ | $-1$ | 1 | $-1$ | 1 | 2 | 3 | $\cdots$ | $-6$ | $\cdots$ |

  where $P_d = c_d \cdot P$.

Miracle. The coefficients $c_d$ appear as the Fourier coefficients of $\phi \in S^+_{3/2}(4 \cdot 37)$,

$$\phi = \sum_{d\geq 1} c_d q^d = -q^3 - q^4 + q^7 - q^{11} + q^{12} + 2q^{16} + 3q^{27} + \cdots - 6q^{67} + \cdots,$$

which maps to $f$ under the Shimura–Waldspurger–Kohnen correspondence

$$\theta : S^+_{3/2}(4N) \to S_2(N), \quad \theta(\phi) = f.$$

## 11 Arithmetic theta lifting

- The generating series of Heegner points

$$\sum_{d\geq 1} P_d \cdot q^d = \sum_{d\geq 1} c_d P \cdot q^d = \phi \cdot P \in S^+_{3/2}(4 \cdot 37) \otimes E(\mathbb{Q})_{\mathbb{C}}$$

  is a modular form valued in $E(\mathbb{Q})_{\mathbb{C}}$.
- More generally, we may define a generating series of Heegner divisors on $X_0(N)$,

$$Z := \sum_d Z(d) q^d \in M_{3/2}(4N) \otimes \mathrm{CH}^1(X_0(N))_{\mathbb{C}},$$

  which may be viewed as an *arithmetic theta series*.
- Use $Z$ as the kernel to define *arithmetic theta lifting*

$$\Theta(\phi) := (Z, \phi)_{\mathrm{Pet}} \in \mathrm{CH}^1(X_0(N))^0_{f,\mathbb{C}} = E(\mathbb{Q})_{\mathbb{C}}.$$

- Now $\Theta(\phi)$ does not depend on any particular choice of $d$ or $K$.

Theorem (Gross–Kohnen–Zagier, 1980s)
Up to simpler nonzero factors,

$$L'(E,1) \sim \langle \Theta(\phi), \Theta(\phi) \rangle_{\mathrm{NT}}.$$

## 12 Special cycles on $X$

- For any $y \in V$ with $(y, y) > 0$. Its orthogonal complement $V_y \subseteq V$ has rank $n-1$. The embedding $\mathsf{U}(V_y) \hookrightarrow \mathsf{U}(V)$ defines a Shimura subvariety of codimension 1

$$\mathrm{Sh}_{\mathsf{U}(V_y)} \to X = \mathrm{Sh}_{\mathsf{U}(V)} .$$

- For any $x \in V(\mathbb{A}_F^\infty)$ with $(x, x) \in F_{>0}$, there exists $y \in V$ and $g \in \mathsf{U}(V)(\mathbb{A}_F^\infty)$ such that $y = gx$. Define the the *special divisor*

$$Z(x) \to X$$

to be $g$-translate of $\mathrm{Sh}_{\mathsf{U}(V_y)}$.

- For any $\mathbf{x} = (x_1, \ldots, x_m) \in V(\mathbb{A}_F^\infty)^m$ with $T(\mathbf{x}) = ((x_i, x_j)) \in \mathsf{Herm}_m(F)_{>0}$, define the *special cycle* (of codimension $m$)

$$Z(\mathbf{x}) = Z(x_1) \cap \cdots \cap Z(x_m) \to X.$$

- More generally, for a Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_F^\infty)^m)^K$ and $T \in \mathsf{Herm}_m(F)_{>0}$, define the *weighted special cycle*

$$Z_\varphi(T) = \sum_{\substack{\mathbf{x} \in K \backslash V(\mathbb{A}_F^\infty)^m \\ T(\mathbf{x}) = T}} \varphi(\mathbf{x}) Z(\mathbf{x}) \in \mathrm{CH}^m(X)_{\mathbb{C}}.$$

- With extra care, we can also define $Z_\varphi(T) \in \mathrm{CH}^m(X)_{\mathbb{C}}$ for any $T \in \mathsf{Herm}_m(F)_{\geq 0}$.

## 13 Arithmetic theta lifting

Define *Kudla's generating series of special cycles*

$$Z_\varphi(\tau) = \sum_{T \in \mathsf{Herm}_m(E)_{\geq 0}} Z_\varphi(T) q^T.$$

Conjecture (Kudla's modularity, 1990s)
The formal generating series $Z_\varphi(\tau)$ converges absolutely and defines a modular form on $\mathsf{U}(W)$ valued in $\mathrm{CH}^m(X)_{\mathbb{C}}$.

*Remark.*

(1) The analogous modularity in Betti cohomology is known (Kudla–Millson, 1980s).
(2) Conjecture is known for $m = 1$. For general $m$, the modularity follows from the absolute convergence (Liu, 2011).
(3) The analogous conjecture for orthogonal Shimura varieties over $\mathbb{Q}$ is known (Bruinier–Raum, 2014).
(4) Conjecture is known when $E/F$ is a norm-Euclidean imaginary quadratic field (Xia, 2021).

Assuming Kudla's modularity conjecture, for $\phi \in \pi$, define *arithmetic theta lifting*

$$\Theta_\varphi(\phi) = (Z_\varphi(\tau), \phi)_{\mathrm{Pet}} \in \mathrm{CH}^m(X)^0_\pi.$$

## 14 Main result B: Arithmetic inner product formula

Theorem B (L.–Liu [1], 2020)
Let $\pi$ be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying *Assumptions*. Assume $\varepsilon(\pi) = -1$. Assume *Kudla's modularity*. Then for any $\phi \in \pi$ and $\varphi \in \mathcal{S}(\mathbb{V}(\mathbb{A}_F^\infty)^m)$, up to simpler factors depending on $\phi$ and $\varphi$,

$$L'(1/2, \pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}}.$$

*Remark.* The simpler factors can be further made explicit. For example, if

- $\pi$: unramified or almost unramified at all finite places,
- $\phi \in \pi$: holomorphic newform such that $(\phi, \overline{\phi})_\pi = 1$,
- $\varphi$: characteristic function of self-dual or almost self-dual lattices at all finite places.

Then $$\frac{L'(1/2, \pi)}{\prod_{i=1}^{2m} L(i, \eta_{E/F}^i)} C_m^{[F:\mathbb{Q}]} \prod_{v \in S_\pi} \frac{(-1)^m q_v^{m-1}(q_v + 1)}{(q_v^{2m-1} + 1)(q_v^{2m} - 1)} = \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}},$$

where $C_m = 2^{m(m-1)} \pi^{m^2} \frac{\Gamma(1) \cdots \Gamma(m)}{\Gamma(m+1) \cdots \Gamma(2m)}$.

*Remark.*

- Riemann hypothesis predicts $L'(1/2, \pi) \geq 0$.
- Beilinson's Hodge index conjecture predicts $(-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}} \geq 0$.

Compatible with our formula!

## 15 Summary

| BSD conjecture | BB conjecture |
|---|---|
| Modular curves $X_0(N)$ | Unitary Shimura varieties $X$ |
| Heegner points $Z(d)$ | Special cycles $Z_\varphi(T)$ |
| $Z = \sum_d Z(d) q^d \in \mathrm{CH}^1(X_0(N))_\mathbb{C}$ | $Z_\varphi = \sum_T Z_\varphi(T) q^T \in \mathrm{CH}^m(X)_\mathbb{C}$ |
| $\Theta(\phi) \in E(\mathbb{Q})_\mathbb{C}$ | $\Theta_\varphi(\phi) \in \mathrm{CH}^m(X)_\pi^0$ |
| Gross–Zagier formula<br>$L'(E, 1) \sim \langle \Theta(\phi), \Theta(\phi) \rangle_{\mathrm{NT}}$ | Arithmetic inner product formula<br>$L'(1/2, \pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}}$ |

## 16 Proof strategy: doubling method

- *Doubling method* (Piatetski-Shapiro–Rallis, Yamana)

$$L(s+1/2,\pi) \sim (\phi\otimes\overline{\phi}, \mathrm{Eis}(s,g))_{\mathsf{U}(W)^2},$$

where $\mathrm{Eis}(s,g)$ is a Siegel Eisenstein series on $\mathsf{U}(W\oplus W)$.

- By definition $\Theta_\varphi(\phi) = (Z_\varphi, \phi)_{\mathrm{Pet}}$ gives

$$\langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}} = (\phi\otimes\overline{\phi}, \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}})_{\mathsf{U}(W)^2}.$$

- To prove $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}$, it suffices to compare

$$\mathrm{Eis}'(0,g) \stackrel{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}}.$$

This can be viewed as an *arithmetic Siegel–Weil formula*.

- The Beilinson–Bloch height pairing is a sum of local indexes

$$\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}} = \sum_v \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},v}.$$

- The nonsingular Fourier coefficient decomposes as

$$\mathrm{Eis}'_T(0,g) = \sum_v \mathrm{Eis}'_{T,v}(0,g)$$

## 17 Proof strategy: comparison

- Nonsingular terms: it suffices to compare

$$\mathrm{Eis}'_{T,v}(0,g) \stackrel{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}.$$

- Gross–Zagier ($m=1$): compute both sides explicitly.
- Explicit computation infeasible for general $m$.
- $v\nmid\infty$
  (1) relate $\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}$ to arithmetic intersection numbers.
  (2) recent proof of *Kudla–Rapoport conjecture* (L.–Zhang).
- $v\mid\infty$:
  (1) archimedean arithmetic Siegel–Weil (Liu, Garcia–Sankaran).
  (2) avoidance of holomorphic projections.

To finish:

- Kill singular terms on both sides: Prove the existence of special $\varphi\in\mathcal{S}(V(\mathbb{A}_F^\infty)^m)$ with *regular support* at two split places with nonvanishing local zeta integrals.
- Theorem B for special $\varphi$: comparison of nonsingular terms.
- Theorem B for arbitrary $\varphi$: *multiplicity one* of doubling method (tempered case).
- Theorem A: same computation without Kudla's modularity (proof by contradiction).

## 18 Remarks on Assumptions

- When $v \nmid \infty$, the local index $\langle\ ,\ \rangle_{\mathrm{BB},v}$ is defined as a $\ell$-adic linking number. It is defined on a certain subspace $\mathrm{CH}^m(X)^{\langle\ell\rangle} \subseteq \mathrm{CH}^m(X)^0$ (conjecturally equal) and its independence on $\ell$ is not known in general.
- Find a Hecke operator $t \notin \mathfrak{m}_\pi$ such that $t^*Z \in \mathrm{CH}^m(X)^{\langle\ell\rangle}$, so BB height is defined.
- Find another Hecke operator $s \notin \mathfrak{m}_\pi$, so BB height of $s^*t^*Z$ can be computed in terms of the *arithmetic intersection number* of a nice extension $\mathcal{Z}$ on $\mathcal{X}$.
- Here $\mathcal{X}$ is a regular integral model of a related unitary Shimura variety of PEL type (Kudla–Rapoport, Rapoport–Smithling–Zhang).
- *Kudla–Rapoport conjecture*: arithmetic intersection number equals $\mathrm{Eis}'_{T,v}(0, g)$.
- The $\ell$-independence of $\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}$ then follows.
- Construction of Hecke operators and the proof of Kudla–Rapoport conjecture requires *Assumptions*.
- More recent work ([2]): remove some of *Assumptions*, and extend Kudla–Rapoport conjecture to include ramified places.

## References

[1]: Chao Li and Yifeng Liu. Chow groups and L-derivatives of automorphic motives for unitary groups. arXiv e-prints, page arXiv:2006.06139, June 2020.
[2]: Chao Li and Yifeng Liu. Chow groups and L-derivatives of automorphic motives for unitary groups, II. arXiv e-prints, page arXiv:2101.09485, January 2021.

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