A BRIEF REPORT ON p-ADIC SPIN L-FUNCTIONS FOR GSp_6

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Abstract. This document summarizes the content of the author's presentation at the remote RIMS conference "Automorphic forms, Automorphic representations, Galois representations, and its related topics." In particular, we report on a paper-in-preparation (joint with S. Shah and G. Rosso) on p-adic Spin L-functions for GSp_6.

1. Introduction

The main goal of the talk at RIMS was to introduce a construction of p-adic L-functions for GSp_6 (a joint paper-in-preparation with S. Shah and G. Rosso). As a first step, we introduced some key aspects of p-adic L-functions. Then we elaborated on related results, especially for symplectic groups. Finally, we gave an overview of the key ingredients and steps of proofs of main results.

The theory of p-adic L-functions could not exist without results about rationality of special values of L-functions. As a first example, consider the Riemann zeta function $\zeta(s)$. In the 1700s, Euler proved that for each positive integer k, $\zeta(2k) = \frac{-(2\pi i)^{2k}}{2\cdot(2k-1)!}\cdot\frac{B_{2k}}{2k}$, where B_{2k} denotes the $2k$-th Bernoulli number. Since we are particularly interested in the rational part of this value, we often consider instead the values $\zeta(1-2k) = -\frac{B_{2k}}{2k} \in \mathbb{Q}$. In the 1800s, as part of his study of ideals and class numbers for cyclotomic fields, Kummer proved that for a prime number p and k, k' positive integers such that $(p-1) \nmid 2k, 2k'$, if $2k \equiv 2k' \mod \phi(p^d)$, then $\zeta^{(p)}(1-2k) \equiv \zeta^{(p)}(1-2k') \mod p^d$, where $\zeta^{(p)}(1-2k) := (1-p^{2k-1})\zeta(1-2k)$. Kummer's death came about a decade before Hensel's discovery of the p-adic numbers. In the 1960s, however, Kubota and Leopoldt constructed the first p-adic zeta function, a p-adic analytic function that p-adically interpolates values of $\zeta^{(p)}$.

Since then, there has been much interest in the possibility of constructing other p-adic L-functions, i..e p-adic analytic functions whose values at certain points interpolate particular values of (appropriately normalized) classical L-functions. This began with Iwasawa's work in the middle of the twentieth century. It continues today, thanks to work building on an approach introduced by Serre in the 1970s.

2. Connections with modular forms

In the 1970s, Serre introduced an approach to constructing p-adic zeta functions, which used modular forms. Consider the Eisenstein series of weight $2k \geq 2$:

$$\tilde{G}_{2k}(z) := \sum_{(0,0)\neq(m,n)\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{(mz+n)^{2k}}.$$

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The Fourier expansion of $G_{2k}(z) := \frac{(2k-1)!}{(2\pi i)^{2k}} \tilde{G}_{2k}(z)$ is

$$G_{2k}(z) = \zeta(1-2k) + 2\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where $q = e^{2\pi i z}$ and $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$. This is a special case of the Eisenstein series

$$G_{k,\chi}(q) = L(1-k,\chi) + 2\sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n,$$

where χ is Dirichlet character and $\sigma_{2k-1,\chi}(n) = \sum_{d|n} \chi(d) d^{2k-1}$.

In the 1970s, Serre introduced a new approach to constructing p-adic L-functions: Consider congruences between Fourier coefficients of modular forms (and construct *p-adic modular forms*). In the 1970s, Serre proved that if we have modular forms

$$f(q) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$$
$$g(q) = b_0 + b_1 q + b_2 q^2 + b_3 q^3 + \cdots$$

such that $a_n \equiv b_n \mod p^d$ for $n \geq 1$, then $a_0 \equiv b_0 \mod p^d$. Applying this to the *p-stabilization* of $G_{k,\chi}$,

$$G_{k,\chi}^{(p)}(q) := L^{(p)}(1-k,\chi) + 2\sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{(p)}(n) q^n,$$

where $L^{(p)}(1-k,\chi) := (1-\chi(p)p^{k-1})L(1-k,\chi)$ and $\sigma_{k-1,\chi}^{(p)}(n) := \sum_{d|n, p\nmid d} \chi(d) d^{k-1}$, we recover the results of Kummer and Kubota–Leopoldt.

Furthermore, Serre proved that modular forms can be put into p-adic families, and you can take p-adic limits of them. That is, if $\{f_{k_i}(q) := \sum_{n\geq 0} a_n(k_i) q^n\}_i$ is a sequence of modular forms of weight k_i with k_i converging in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ and $\lim_{i\to\infty} a_n(k_i) = a_n$ for $n \geq 1$, then $a_0(k_i)$ converge p-adically to some a_0. This builds on work of Hecke, Klingen, and Siegel on algebraicity and of Atkin and Swinnerton-Dyer on congruences.

This is the first instance of properties of modular forms driving the study of p-adic (or algebraic) properties of values of L-functions. The behavior of families of Galois representations is also tied to properties of p-adic modular forms, as developed in work of Hida in the 1980s via systems of Hecke eigenvalues. This approach turns out to be fruitful, because it generalizes naturally.

In analogue with how Klingen and Siegel's work on algebraicity inspired Serre's approach to congruences and p-adic L-functions, we have:

Theorem 2.1 (Shimura, 1970s)**.** *Let $f(q) = \sum_{n>0} a_n q^n$ be a cusp form of weight k and $g(q) = \sum_{n\geq 0} b_n q^n$ be a modular form of weight ℓ. Consider the Dirichlet series*

$$D(s,f,g) := \sum_{n=1}^{\infty} a_n b_n n^{-s}.$$

Then for $\ell < k$ and $\frac{k+\ell-2}{2} < m < k$

$$\frac{D(m,f,g)}{\pi^k \langle f,f\rangle_{\mathrm{Pet}}} \in \overline{\mathbb{Q}}$$

(in fact, lies in $\mathbb{Q}(\{a_n, b_n\}_n)$).

Here $\langle , \rangle_{\mathrm{Pet}}$ denotes the usual Petersson pairing on modular forms.

In the 1980s, Hida constructed p-adic Rankin–Selberg L-functions (as p-adic measures) by reinterpreting the pairing in Shimura's result p-adically. This is an instance of a broader phenomenon, in which there are connections between approaches to proving algebraicity and to constructing p-adic L-functions.

3. Overview of some Conjectures and Results about p-adic L-functions

In view of the discussion above, it is reasonable to ask:

Question 1. *For which ($\mathbb{C}$-valued) L-functions $L(s,M)$ and data M does there exist a p-adic function $\mathfrak{L}_{p-\mathrm{adic}}$ such that*

$$(*)\mathfrak{L}_{p-\mathrm{adic}}(n,M) = (*')L^{(p)}(n,M)$$

for all (n,M) meeting appropriate conditions (and such that the construction exploits congruences between Fourier expansions of modular forms)?

First, though, we must ask:

Question 2. *What can we say about algebraicity of values of $L(n,M)$? (We need to know that values of algebraic, and furthermore p-integral, before talking about p-adic properties.)*

There are precise conjectures addressing these questions. In particular, Coates and Perrin-Riou conjectured the existence of a wide class of p-adic L-functions. The also predicted the form of the Euler factor at p. Building on the main conjecture of Iwasawa Theory (proved by Mazur and Wiles), Greenberg predicted the meaning of many p-adic L-functions through more extensive *Main Conjectures*, which concern p-adic variation of arithmetic data. As for the question about algebraicity, which are necessary to address before one can speak meaningfully of congruences or p-adic properties, Deligne made precise conjectures about algebraicity of special values.

Via applications of properties of p-adic families of modular forms, several mathematicians have constructed p-adic L-functions in various settings. Kubota and Leopoldt constructed p-adic Dirichlet L-functions. What about p-adic L-functions associated to Hecke characters on the ideles of a number field K? Coates and Sinnott did this for real quadratic fields, and Deligne–Ribet handled the more general case of K totally real without a condition on the degree of K. Katz then handled the case of K a CM field, under the condition that each prime above p split completely. Taking this further, Hida constructed p-adic L-functions attached to modular forms, including to ordinary families of modular forms. Later, Panchishkin extended this construction to the case of Coleman families (the finite slope case). Still more recently, Eischen–Harris–Li–Skinner constructed p-adic L-functions associated to families of ordinary cuspidal automorphic representations on unitary groups, under a similar splitting condition on p to the one Katz required.

4. Summary of a strategy for constructing p-adic L-functions

This is an overview of the recipe employed by Hida, Panchishkin, E–Harris–Li–Skinner, and others, broken into three (nontrivial!) steps:

(1) Construct a p-adic family $\mathbb{E}$ of Eisenstein series (indexed by weights, like the example of G_{2k} at the beginning) on some group G
(2) Restrict ("pull back") to some (possibly smaller) subgroup H inside G, and pair against a cusp form on H
(3) Interpret (via an integral representation, such as the Rankin–Selberg method) as a (familiar) L-function

This builds on a strategy for proving algebraicity results (pioneered by Shimura in his study of Rankin–Selberg convolutions):

(1) Realize your L-function in terms of an integral representation (e.g. Rankin–Selberg convolution, doubling integral, ...), pairing an Eisenstein series against cusp form(s), so values of your L-function are realized as a pairing $\langle \phi, E \rangle$
(2) Find an Eisenstein series with algebraic Fourier coefficients
(3) Find an othonormal basis $\{\phi_i\}$ for your space of cusp forms over $\bar{\mathbb{Q}}$, and show that $\langle \phi_i, E \rangle / \langle \phi_i, \phi_i \rangle \in \bar{\mathbb{Q}}$. Use this prove similar statement for any cusp form ϕ in place of ϕ_i.

It is nontrivial to carry out this recipe, and it is even nontrivial to acquire some of the ingredients (in particular, the Fourier expansion of an Eisenstein series, an integral representation of an L-function, and furthermore an integral representation that is suitable for studying algebraicity).

5. Symplectic groups and Siegel modular forms

In analogue with the standard Langlands L-functions studied by E–Harris–Li–Skinner in the setting of unitary groups, p-adic standard Langlands L-functions associated to cuspidal automorphic forms on symplectic groups were recently constructed by Zheng Liu, further extending earlier work by Böcherer–Schmidt. This relies on the doubling method, due to Piatetski-Shapiro–Rallis and earlier work of Garrett. Shimura proved corresponding algebraicity results. In the setting of $\mathrm{GSp}_2 = \mathrm{GL}_2$, the standard L-function is $L(Sym^2 f, s)$.

In addition to the standard representation, we also have that ${}^L\mathrm{GSp}_{2n} = \mathrm{GSpin}_{2n+1}$, which has a natural 2^n-dimensional representation, the *spin* representation. What can we say about algebraicity and p-adic interpolation if we replace $L(s, \pi, \mathrm{std})$ (studied by Liu, Böcherer–Schmidt, $\ldots$) by $L(s, \pi, \mathrm{spin})$, with π a cuspidal automorphic representation of GSp_{2n}? When $n = 1$, this is $L(f, s)$ with f a modular form. Using modular symbols, Manin–Shimura proved algebraicity results, and Mazur–Tate–Teitelbaum constructed the corresponding p-adic L-function. When $n = 2$, using higher Hida theory, together with Harris's interpretation of L-values as cup products (useful for studying algebraicity), Loeffler–Pilloni–Skinner–Zerbes constructed p-adic L-function. This topic remains poorly understood for $n > 3$ (although Bump–Ginzburg constructed an integral representation for n=$3, 4, 5$ that appears not to be amenable to proving algebraicity results)

The case of $n = 3$ is the subject of the author/speaker's joint work with Rosso and Shah. Our work focuses on analytic p-adic L-functions, obtained via p-adic interpolation. On the algebraic side, there is recent related work of Cauchi–Jacinto and Cauchi–Lemma–Jacinto.

We provide a brief summary of the main results of the paper-in-preparation of the author, Rosso, and Shah:

Theorem 5.1 (E–Rosso–Shah)**.** *Let π be a cuspidal automorphic representation of* GSp_6 *associated to a Siegel modular form ϕ of weight $2r$. Then $\frac{L(\pi,spin,s)}{\pi^{4s-6r+6}\langle\phi|_{w_N},\phi\rangle}$ is algebraic, for $s \in \{3r-2,\dots,4r-5\}$. (Here, w_N is a certain involution for G.)*

Theorem 5.2 (E–Rosso–Shah)**.** *There is a two-variable p-adic L-function $L_p(f_{(2r)}, j)$ that varies p-adically in $2r$ and j, interpolating the values*

$$I_\infty I_p \frac{L^{(p)}(\pi_{(2r)}, spin, s)}{\langle f_{(2r)}|_{w_N}, f_{(2r)}\rangle},$$

where $3r-2 \le j \le 4r-5$ and $f_{(2r)}$ varies in a Hida family of ordinary Siegel modular forms. Here I_∞ denotes the archimedean Euler factor (a product of gamma factors) and I_p denotes the modified Euler factor at p.

We work with the Rankin–Selberg-style interval studied by Aaron Pollack in this setting. Let ϕ be a cusp form associated to a holomorphic Siegel modular form on GSp_6 of positive, even scalar weight $2r$. Consider the pairing

$$\langle\phi, E_{2r}(g,s)\rangle = I_{2r}^*(\phi,s) = \int_{\mathrm{GSp}_6(\mathbb{Q})Z(\mathbb{A})\backslash\mathrm{GSp}_6(\mathbb{A})} \phi(g)E_{2r}(g,s)dg,$$

for $E_{2r}(g,s)$ certain Eisenstein series on a group G (to be defined in a few moments) restricted to $\mathrm{GSp}_6 \subset G$.

If you're familiar with the doubling method for Siegel modular forms or the Rankin–Selberg integral for modular forms, it might *feel* like you can use what you know about those settings to proceed here. There are some key differences, though:

(1) G is closely related to the exceptional group GE_7 (and although G is related to a unitary group, that's a *quaternionic* unitary group).
(2) There's currently no known moduli problem for G, no q-expansion principle, etc.
(3) The Rankin–Selberg style pairing here is a non-unique model.

Fix:

- a field F of characteristic 0
- a quaternion algebra B over F

Define an F vector space

$$W := F \oplus H_3(B) \oplus H_3(B) \oplus F,$$

where $H_3(B)$ denotes 3×3 hermitian matrices over B, i.e. matrices of the form

$$\begin{pmatrix} c_1 & a_3 & a_2^* \\ a_3^* & c_2 & a_1 \\ a_2 & a_1^* & c_3 \end{pmatrix},$$

with $c_1, c_2, c_3 \in F$ and $a_1, a_2, a_3 \in B$. Here, a^* denotes the conjugate of a in the quaternion algebra B.

The group G will be defined as a certain subgroup of $\mathrm{GL}(W) \times \mathrm{GL}_1(F)$. First, though, we establish a few conventions for operations on $H_3(B)$, following the conventions employed by Pollack. Given $h = \begin{pmatrix} c_1 & a_3 & a_2^* \\ a_3^* & c_2 & a_1 \\ a_2 & a_1^* & c_3 \end{pmatrix} \in H_3(B)$, we put

$$N(h) := c_1c_2c_3 - c_1n(a_1) - c_2n(a_2) - c_3n(a_3) + \mathrm{tr}(a_1a_2a_3),$$

where $n(a) := aa^*$ for all $a \in B$. We also define

$$h^{\#} := \begin{pmatrix} c_2c_3 - n(a_1) & a_2^*a_1^* - c_3a_3 & a_3a_1 - c_2a_2^* \\ a_1a_2 - c_3a_3^* & c_1c_2 - n(a_2) & a_3^*a_2^* - c_1a_1 \\ a_1^*a_3^* - c_2a_2 & a_2a_3 - c_1a_1^* & c_1c_2 - n(a_3) \end{pmatrix}$$

Given an element $h' \in H_3(B)$, we define

$$\mathrm{tr}(h, h') := \mathrm{trace}(hh' + h'h).$$

Define

- A symplectic form $\langle , \rangle : W \times W \to F$ by
$$\langle (a, b, c, d), (a', b', c', d') \rangle := ad' - \mathrm{tr}(b, c') + \mathrm{tr}(c, b') - da',$$
where $\mathrm{tr}(b, c) := \frac{1}{2}(bc + cb)$ for all $b, c \in H_3(B)$.
- A quartic form $Q : W \to F$ by
$$Q((a, b, c, d)) := (ad - \mathrm{tr}(b, c))^2 + 4aN(c) + 4dN(b) - 4\mathrm{tr}(b^{\#}, c^{\#}).$$

We define G to be the similitude algebraic group consisting of all $(g, \nu(g)) \in \mathrm{GL}(W) \times \mathrm{GL}_1(F)$ such that for all $u, v, w \in W$:

$$\langle ug, vg \rangle = \nu(g)\langle u, v \rangle$$
$$Q(wg) = \nu(g)^2 Q(w)$$

We note a few properties of the group G:

- There's a group that is a double cover of G and $GU_6(B)$ (quaternionic unitary group)
- G is closely related to the exceptional group GE_7
- G is a half-spin group of type D_6

The Hermitian symmetric space for $G(\mathbb{R})$ is

$$\mathcal{H} := \{Z \in H_3(B \otimes_{\mathbb{R}} \mathbb{C}) | Z = X + iY, withX, Y \in H_3(B_{\mathbb{R}}), Y \text{ positive definite } \}$$

Continuing to follow Pollack's conventions, we have a map $r : \mathcal{H} \to W$, $Z \mapsto (1, -Z, Z^{\#}, -N(Z))$. The action of G on $\mathcal{H}$ and the factor of automorphy $j_{G(\mathbb{R})}(g, Z)$ are defined simultaneously by

$$r(Z)g^{-1} = j_{G(\mathbb{R})}(g, Z)r(gZ).$$

There is an embedding $\mathrm{GSp}_6 \hookrightarrow G$ and corresponding embedding of Siegel upper half space into $\mathcal{H}$.

A key input to the Rankin–Selberg integral is (a family of) Eisenstein series. We have an Eisenstein series $E_{2r}(g,s)$ that is holomorphic at $s=r$ and whose Fourier expansion when $s=r$ has coefficients of the form

$$N(h)^{r-5}\left(\prod_{\ell\,|\,N(h)} P_\ell(\ell^{2r})\right),$$

with P_ℓ a polynomial with rational coefficients and h is a 3×3 hermitian quaternionic matrix. We easily see that these Fourier coefficients satisfy congruences, which would be nice for constructing an Eisenstein measure; but we don't have a q-expansion principle here. Note, though, that it pulls back nicely to the space for GSp_6, where we *do* have a q-expansion principle and can construct a p-adic family of (pullbacks of) Eisenstein series. There is a differential operator $\mathcal{D}$ such that $\mathcal{D}E_{2r}(g,s)=E_{2r+2}(g,s)$. So it suffices to study $\mathcal{D}$ and $E_{2r}(g,r)$. This differential operator has a nice action on q-expansions.

We can show that the differential operator $\mathcal{D}$ acting on a weight k modular form $f(Z)$ on $\mathcal{H}$ has the form

$$N(Y)^{-k}\Delta\left(N(Y)^k f\right),$$

where Δ is the determinant of a matrix of partial derivatives. Note the similarity with the familiar Maass–Shimura operators in the setting of classical automorphic forms. Given the way the action of G and the automorphy factor are defined, how can we see that this operator raises the weight of a modular form on G by 2? The answer follows from two key equations (whose proofs are a nice exercise in exploring how to think about the action of G and the embedding $r:\mathcal{H}\hookrightarrow W$):For any $\alpha\in G^+(\mathbb{R})$ and $z,w\in\mathcal{H}$, we have:

$$N(\alpha z-\alpha w) = \nu(\alpha)^{-1}j(\alpha,z)^{-1}j(\alpha,w)^{-1}N(z-w) \tag{5.1}$$

$$N(\mathrm{Im}(\alpha z)) = |j(\alpha,Z)|^{-2}\nu(\alpha)^{-1}N(\mathrm{Im} z) \tag{5.2}$$

These rely on the observation that $\langle r(z),r(w)\rangle = N(z-w)$.

Similarly to Shimura's approach to proving algebraicity for Rankin–Selberg convolutions, the following key realization enables our algebraicity theorem: Write $\frac{L(\pi,spin,s)}{\pi^{4s-6r+6}\langle\phi|_{w_N},\phi\rangle} = \frac{\langle\phi|_{w_N},E_{2r}(z,s+5-3r)\rangle}{\langle\phi|_{w_N},\phi\rangle}$ with Eisenstein series E_{2r} and a Rankin–Selberg style pairing as above, and recall that $\mathcal{D}E_{2r}(z,s)=E_{2r+2}(z,s)$. Following Hida's approach to constructing p-adic L-functions associated to Rankin–Selberg convolutions of modular forms, we write $\ell_{f_{2r}}(E_{2r}(s)) = \frac{\langle f_{2r}|_{w_N},E_{2r}(z,s+5-3r)\rangle}{\langle\phi|_{w_N},f_{2r}\rangle}$, and we obtain a p-adic L-function associated to a Hida family of (ordinary) Siegel modular forms $f_{(2r)}$.

Full details will be in the paper, which will be available soon.

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