On supercuspidal representations of Sp_{2n} and Langlands parameters

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1 Introduction

Let $F/\mathbb{Q}_p$ be a finite extension with $p \neq 2$ whose integer ring O_F has unique maximal ideal $\mathfrak{p}_F$ wich is generated by ϖ_F. The residue class field $\mathbb{F} = O_F/\mathfrak{p}_F$ is a finite field of q-elements. The Weil group of F is denoted by W_F which is a subgroup of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ where $\overline{F}$ is a fixed algebraic closure of F in which we will take the algebraic extensions of F.

Let G be a connected semi-simple linear algebraic group defined over F. For the sake of simplicity, we will assume that G splits over F. Then the L-group ${}^LG$ of G is equal to the dual group $G^\frown$ of G. An admissible representation

$$\varphi : W_F \times SL_2(\mathbb{C}) \to {}^LG$$

of the Weil-Deligne group of F is called a discrete parameter of G over F if the centralizer $\mathcal{A}_\varphi = Z_{{}^LG}(\mathrm{Im}\varphi)$ of the image of φ in ${}^LG$ is a finite group. Let us denote by $\mathcal{D}_F(G)$ the $G^\frown$-conjugacy classes of the discrete parameters of G over F. The conjectural parametrization of $\mathrm{Irr}_2(G)$ (resp. $\mathrm{Irr}_s(G)$), the set of the equivalence classes of the irreducible admissible square-integrable (resp. supercuspidal) representations of G, by $\mathcal{D}_F(G)$ is (see [7, p.483, Conj.7.1] for the details)

Conjecture 1.0.1 *For every $\varphi \in \mathcal{D}_F(G)$, there exists a finite subset Π_φ of $\mathrm{Irr}_2(G)$ such that*

1) $\mathrm{Irr}_2(G) = \bigsqcup_{\varphi \in \mathcal{D}_F(G)} \Pi_\varphi,$

2) *there exists a bijection of Π_φ onto the equivalence classes $\mathcal{A}_\varphi^\frown$ of the irreducible complex linear representations of $\mathcal{A}_\varphi$,*

3) $\Pi_\varphi \subset \mathrm{Irr}_s(G)$ *if* $\varphi|_{SL_2(\mathbb{C})} = 1$.

The finite set Π_φ is called a L-packet of φ.

According to this conjecture, any $\pi \in \mathrm{Irr}_2(G)$ is determined by $\varphi \in \mathcal{D}_F(G)$ and $\sigma \in \mathcal{A}_\varphi^\frown$. So the formal degree of π should be determined by these data. The formal degree conjecture due to Hiraga-Ichino-Ikeda [8] is (with the formulation of [7])

Conjecture 1.0.2 *The formal degree* d_π *of* π *with respect to the absolute value of the Euler-Poincaré measure (see* [12, §3] *for the details) on* $G(F)$ *is equal to*

$$\frac{\dim \sigma}{|\mathcal{A}_\varphi|} \cdot \left| \frac{\gamma(\varphi, \mathrm{Ad}, \psi, d(x), 0)}{\gamma(\varphi_0, \mathrm{Ad}, \psi, d(x), 0)} \right| .$$

Here

$$\gamma(\varphi, \mathrm{Ad}, \psi, d(x), s) = \varepsilon(\varphi, \mathrm{Ad}, d(x), s) \cdot \frac{L(\varphi^\vee, \mathrm{Ad}, 1-s)}{L(\varphi, \mathrm{Ad}, s)}$$

is the gamma-factor associated with the φ combined with the adjoint representation Ad of $G^\frown$ on its Lie algebra $\mathfrak{g}^\frown$, and a continuous additive character ψ of F such that $\{x \in F \mid \psi(xO_F) = 1\} = O_F$ and the Haar measure $d(x)$ on the additive group F such that $\int_{O_F} d(x) = 1$. See [7, pp.440-441] for the details.

$$\varphi_0 : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} SL_2(\mathbb{C}) \to G^\frown$$

is the principal parameter (see [7, p.447] for the definition).

The formal degree conjecture concerns with the absolute value of the epsilon-factor

$$\varepsilon(\varphi, \mathrm{Ad}, d(x), s) = w(\varphi, \mathrm{Ad}) \cdot q^{a(\varphi, \mathrm{Ad})(\frac{1}{2}-s)}$$

where $a(\varphi, \mathrm{Ad})$ is the Artin-conductor and $w(\varphi, \mathrm{Ad})$ is the root number. The root number conjecture says that

Conjecture 1.0.3 [7, p.493, Conj.8.3]

$$\frac{w(\varphi, \mathrm{Ad})}{w(\varphi_0, \mathrm{Ad})} = \pi(\epsilon)$$

where ϵ *is a special central element of* G *(see* [7, p.492, (65)] *for the definition).*

We assume that G splits over F so that $w(\varphi_0, \mathrm{Ad}) = 1$ (see [7, p.448]).

In this report we will construct explicitly supercuspidal representations $\pi_{\beta,\theta}$ of $G = Sp_{2n}(F)$ by means of a tamely ramified extension K/F of degree $2n$ (Theorem 3.1.2) and a group homomorphism (see subsection 3.3)

$$\varphi : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} W_F \to G^\frown = SO_{2n+1}(\mathbb{G})$$

with which the formal degree conjecture and the root umber conjecture are valid (Theorem 3.3.1 and Corollary 3.4.4 respectively).

After my talk at the conference, several participants suggested certain relations to works of T.Kaletha and J.K.Yu especially [10] and [19]. We will discuss these relations in the section 4.

The details are presented in [18]. Parallel problems with the special linear group are discussed in [17].

2 Regular characters of hyperspecial compact groups

2.1 General theory

Let $G \subset GL_n$ be a closed smooth O_F-group subscheme, and $\mathfrak{g}$ the Lie algebra scheme of G which is a closed affine O_F-subscheme of $\mathfrak{gl}_n$ the Lie algebra scheme of GL_n. We assume that the fibers $G \otimes_{O_F} K$ ($K = F$ or $K = \mathbb{F}$) are non-commutative algebraic K-groups (that is smooth K-group schemes). Let

$$B : \mathfrak{gl}_n \times_{O_F} \mathfrak{gl}_n \to \mathbb{A}^1_{O_F}$$

be the trace form on $\mathfrak{gl}_n$, that is $B(X, Y) = \mathrm{tr}(XY)$ for all $X, Y \in \mathfrak{gl}_n(R)$ with any O_F-algebra R. The smoothness of G implies that the canonical group homomorphism $G(O_F) \to G(O_F/\mathfrak{p}^r)$ is surjective due to the formal smoothness [2, p.111, Cor. 4.6]. For any $0 < l < r$, let us denote by $K_l(O_F/\mathfrak{p}^r)$ the kernel of the canonical surjection $G(O_F/\mathfrak{p}^r) \to G(O_F/\mathfrak{p}^l)$. Let us assume that the following three conditions on G are satisfied:

I) $B : \mathfrak{g}(\mathbb{F}) \times \mathfrak{g}(\mathbb{F}) \to \mathbb{F}$ is non-degenerate,

II) for any integers $r = l + l'$ with $0 < l' \leq l$, we have a group isomorphism

$$\mathfrak{g}(O_F/\mathfrak{p}^{l'}) \tilde{\to} K_l(O_F/\mathfrak{p}^r)$$

defined by $X \,(\mathrm{mod}\ \mathfrak{p}^{l'}) \mapsto 1 + \varpi^l X \,(\mathrm{mod}\ \mathfrak{p}^r)$,

III) if $r = 2l - 1 \geq 3$ is odd, then we have a mapping

$$\mathfrak{g}(O_F) \to K_{l-1}(O_F/\mathfrak{p}^r)$$

defined by $X \mapsto (1 + \varpi^{l-1} X + 2^{-1} \varpi^{2l-2} X^2) \,(\mathrm{mod}\ \mathfrak{p}^r)$.

The condition I) implies that $B : \mathfrak{g}(O_F/\mathfrak{p}^l) \times \mathfrak{g}(O_F/\mathfrak{p}^l) \to O_F/\mathfrak{p}^l$ is non-degenerate for all $l > 0$. Then, by the condition II), the characters of the commutative group $K_l(O_F/\mathfrak{p}^r)$ of the form

$$\chi_\beta(1 + \varpi^l X \pmod{\mathfrak{p}^r}) = \psi\left(\varpi^{-l'} B(X, \beta)\right) \quad (X \pmod{\mathfrak{p}^{l'}} \in \mathfrak{g}(O_F/\mathfrak{p}^{l'}))$$

with $\beta \,(\mathrm{mod}\ \mathfrak{p}^{l'}) \in \mathfrak{g}(O_F/\mathfrak{p}^{l'})$.

Since any finite dimensional complex continuous representation of the compact group $G(O_F)$ factors through the canonical surjection $G(O_F) \to G(O_F/\mathfrak{p}^r)$ for some $0 < r \in \mathbb{Z}$, it is enough to know the irreducible complex representations of the finite group $G(O_F/\mathfrak{p}^r)$. Let us assume that $r > 1$ and put $r = l + l'$ with the minimal integer l such that $0 < l' < l$.

Let δ be an irreducible complex representation of $G(O_F/\mathfrak{p}^r)$. The Clifford's theorem says that the restriction $\delta|_{K_l(O_F/\mathfrak{p}^r)}$ is a sum of the $G(O_F/\mathfrak{p}^r)$-conjugates of characters of $K_l(O_F/\mathfrak{p}^r)$:

$$\delta|_{K_l(O_F/\mathfrak{p}^r)} = \left(\bigoplus_{\dot{\beta} \in \Omega} \chi_\beta\right)^m$$

with an adjoint $G(O_F/\mathfrak{p}^{l'})$-orbit $\Omega \subset \mathfrak{g}(O_F/\mathfrak{p}^{l'})$. In this way the irreducible complex representations of $G(O_F/\mathfrak{p}^r)$ correspond to adjoint $G(O_F/\mathfrak{p}^{l'})$-orbits in $\mathfrak{g}(O_F/\mathfrak{p}^{l'})$.

Fix an adjoint $G(O_F/\mathfrak{p}^{l'})$-orbit $\Omega \subset \mathfrak{g}(O_F/\mathfrak{p}^{l'})$ and let us denote by $\Omega^\frown$ the set of the equivalence classes of the irreducible complex representations of $G(O_F/\mathfrak{p}^{l'})$ correspond to Ω. Then [16] gives a parametrization of $\Omega^\frown$ as follows:

Theorem 2.1.1 *Take a representative $\beta \,(\mathrm{mod}\ \mathfrak{p}^{l'}) \in \Omega$ ($\beta \in \mathfrak{g}O_F$)) and assume that*

1) *the centralizer $G_\beta = Z_G(\beta)$ of $\beta \in \mathfrak{g}(O_F)$ in G is smooth over O_F,*
2) *the characteristic polynomial $\chi_{\overline{\beta}}(t) = \det(t \cdot 1_n - \overline{\beta})$ of $\overline{\beta} = \beta \ (\mathrm{mod}\ \mathfrak{p}) \in \mathfrak{g}(\mathbb{F}) \subset \mathfrak{gl}_n(\mathbb{F})$ is the minimal polynomial of $\overline{\beta} \in M_n(\mathbb{F})$.*

Then we have a bijection $\theta \mapsto \delta_{\beta,\theta}$ of the set

$$\{\theta \in G_\beta(O_F/\mathfrak{p}^r)^\frown \ \text{ s.t. } \ \theta = \chi_\beta \text{ on } G_\beta(O_F/\mathfrak{p}^r) \cap K_l(O_F/\mathfrak{p}^r)\}$$

onto $\Omega^\frown$.

2.2 Regular Lie elements

Let us assume that the connected O_F-group scheme G is reductive, that is, the fibers $G \otimes_{O_F} K$ ($K = F, \mathbb{F}$) are reductive K-algebraic groups. In this case the dimension of a maximal torus in $G \otimes_{O_F} K$ is independent of K which is denoted by $\mathrm{rank}(G)$. For any $\beta \in \mathfrak{g}(O_F)$ we have

$$\dim_K \mathfrak{g}_\beta(K) = \dim \mathfrak{g}_\beta \otimes_{O_F} K \geq \dim G_\beta \otimes_{O_F} K \geq \mathrm{rank}(G). \tag{2.2.1}$$

We say β is *smoothly regular* over K if $\dim_K \mathfrak{g}_\beta(K) = \mathrm{rank}(G)$ (see [14, (5.7)]). In this case $G_\beta \otimes_{O_F} K$ is smooth over K. Then the general theory (see Th. 3.10 and Cor. 4.4 of [3]) implies

Proposition 2.2.1 *The centralizer $G_\beta = Z_G(\beta)$ of β in G is smooth over O_F if the following two conditions are fulfilled:*

1) *$\beta \in \mathfrak{g}(O_F)$ is smoothly regular over F and $\mathbb{F}$, and*
2) *$G_\beta \otimes_{O_F} F$ and $G_\beta \otimes_{O_F} \mathbb{F}$ are connected.*

If $G = Sp_{2n}$ over O_F, then, for $\beta \in \mathfrak{g}(O_F)$, the following two statements are equivalent:

1) $\overline{\beta} \in \mathfrak{g}(K)$ is smoothly regular over K,
2) the characteristic polynomial of $\overline{\beta} \in \mathfrak{g}(K) \subset \mathfrak{gl}_{2n}(K)$ is equal to its minimal polynomial

where $\overline{\beta} \in \mathfrak{g}(K)$ is the image of $\beta \in \mathfrak{g}(O_F)$ by the canonical morphism $\mathfrak{g}(O_F) \to \mathfrak{g}(K)$ with $K = F$ or $\mathbb{F}$. If further $\overline{\beta} \in \mathfrak{g}(K) \subset \mathfrak{gl}_{2n}(K)$ is nonsingular, then $G_\beta \otimes_{O_F} K$ is connected.

3 Supercuspidal representations of $Sp_{2n}(F)$

3.1 Construction

Let K_+/F be a tamely ramified extension such that $(K_+ : F) = n$ and K/K_+ a quadratic extension with $\mathrm{Gal}(K/K_+) = \langle\tau\rangle$. Then we have

Proposition 3.1.1 *There exists a $\beta \in O_K$ such that $O_K = O_F[\beta]$ and $\beta^\tau + \beta = 0$ if and only if K/K_+ is unramified or K/F is totally ramified.*

Now we will assume this and take a $\beta \in O_K$ such that $O_K = O_F[\beta]$ and $\beta^\tau + \beta = 0$. Then a F-symplectif form on K is defined by

$$D(x, y) = \frac{1}{2} T_{K/F}\left(\omega^{-1}\varpi_{K_+}^{1-e_+} x^\tau y\right)$$

for $x, y \in K$, where $e_+ = e(K_+/F)$ is the ramification index and $O_K = O_{K_+}[\omega]$ with $\omega^\tau + \omega = 0$. Then $\beta \in \mathfrak{sp}_F(K, D)$ by the regular representation of β on K, and $\beta \in \mathfrak{sp}_{2n}(O_F)$ with respect to a suitable symplectic basis. It is shown by [13] that the characteristic polynomial of $\beta\,(\mathrm{mod}\ \mathfrak{p}) \in \mathfrak{sp}_{2n}(\mathbb{F}) \subset \mathfrak{gl}_{2n}(\mathbb{F})$ is equal to the minimal polynomial of it.

In order to guarantee the smoothness as O_F-group scheme of the centralizer $G_\beta = Z_G(\beta)$, we should exclude the case of K/F being totally ramified.

Then the general theory presented in the section 2.1 gives a parametrization of $\Omega^\frown$ where $\Omega \subset \mathfrak{sp}_{2n}(O_F/\mathfrak{p}^{l'})$ is the $G(O_F/\mathfrak{p}^{l'})$-orbit of $\beta\,(\mathrm{mod}\ \mathfrak{p}^{l'}) \in \mathfrak{sp}_{2n}(O_F/\mathfrak{p}^{l'})$. Here

$$\begin{aligned} G_\beta(O_F) =& G(O_F) \cap O_F[\beta] = G_\beta(O_F) \cap O_K \\ =& U_{K/K_+} = \left\{\varepsilon \in O_K^\times \mid N_{K/K_+}(\varepsilon) = 1\right\} \end{aligned}$$

and a bijection $\theta \mapsto \delta_{\beta,\theta}$ onto $\Omega^\frown$ of the continuous unitary character θ of U_{K/K_+} such that

1) θ factors through the canonical morphism $U_{K/K_+} \to (O_K/\mathfrak{p}_K^{er})^\times$,

2) $\theta(\varepsilon) = \chi\left(\varpi_F^{-r} T_{K/F}(\beta(\varepsilon - 1))\right)$ for all $\varepsilon \in U_{K/K_+} \cap (1 + \mathfrak{p}_K^{el})$.

Fix a continuous unitary character θ of U_{K/K_+} which satisfies the two conditions given above and the corresponding irreducible unitary representation $\delta_{\beta,\theta}$ of $G(O_F/\mathfrak{p}^r)$ which is considered as an irreducible unitary representation of $G(O_F)$ via the canonical morphism $G(O_F) \to G(O_F/\mathfrak{p}^r)$. Then we have

Theorem 3.1.2 *If $l' = \left[\frac{r}{2}\right] \geq \mathrm{Max}\{2, 2(e-1)\}$, then the compactly induced representation $\pi_{\beta,\theta} = \text{cmp-ind}_{G(O_F)}^{G(F)} \delta_{\beta,\theta}$ is an irreducible supercuspidal representation of $G(F) = Sp_{2n}(F)$ such that*

1) *the multiplicity of $\delta_{\beta,\theta}$ in $\pi_{\beta,\theta}|_{G(O_F)}$ is one, and $\delta_{\beta,\theta}$ is the unique irreducible unitary constituent of $\pi_{\beta,\theta}|_{G(O_F)}$ which factors through the canonical morphism $G(O_F) \to G(O_F/\mathfrak{p}^r)$,*

2) *with respect to the Haar measure on* $G(F)$ *such that the volume of* $G(O_F)$ *is one, the formal degree of* $\pi_{\beta,\theta}$ *is equal to*

$$\dim \delta_{\beta,\theta} = \frac{q^{n^2 r}}{1+q^{-f_+}} \cdot \prod_{k=1}^{n} (1-q^{-2k})$$

where $f_+ = f(K_+/F)$ *is the inertial degree of* K_+/F,

3) $\pi_{\beta,\theta}$ *is generic if* K/F *is unramified.*

3.2 Structures of Galois group of tamely ramified extension

From now on we will assume that the tamely ramified extension K/F is normal. The structure of the Galois group $\mathrm{Gal}(K/F)$ of the tamely ramified extension K/F is well understood:

$$\mathrm{Gal}(K/F) = \langle \delta, \rho \rangle$$

where $\mathrm{Gal}(K/K_0) = \langle \delta \rangle$ with the maximal unramified subextension K_0/F of K/F and $\rho|_{K_0} \in \mathrm{Gal}(K_0/F)$ is the inverse of the Frobenius automorphism. We have a relation $\rho^{-1}\delta\rho = \delta^q$ due to Iwasawa [9] and put

$$\rho^f = \delta^m \ \text{ with } \ 0 \le m < e = e(K/F), f = f(K/F).$$

The structure of the elements of order two in $\mathrm{Gal}(K/F)$ plays an important role in our arguments, and we have

Proposition 3.2.1 $H = \{\gamma \in \mathrm{Gal}(K/F) \mid \gamma^2 = 1\} \subset Z(\mathrm{Gal}(K/F))$ *and*

$$H = \begin{cases} \{1, \delta^{\frac{e}{2}}\} & : f = \text{odd or } \begin{cases} e = \text{even}, \\ m = \text{odd} \end{cases} \\ \{1, \rho^{\frac{f}{2}}\delta^{-\frac{m}{2}}\} & : e = \text{odd}\,, m = \text{even} \\ \{1, \rho^{\frac{f}{2}}\delta^{\frac{e-m}{2}}\} & : e = \text{odd}\,, m = \text{odd} \\ \{1, \delta^{\frac{e}{2}}, \rho^{\frac{f}{2}}\delta^{-\frac{m}{2}}, \rho^{\frac{f}{2}}\delta^{\frac{e-m}{2}}\} & : f = \text{even}\,, e = \text{even}\,, m = \text{even}. \end{cases}$$

Remark 3.2.2 *If* $\tau = \delta^{\frac{e}{2}}$, *then* K/F *is totally ramified. We have excluded this case.*

3.3 Candidate of the Langlands parameter

We have fixed a character $\theta : U_{K/K_+} \to \mathbb{C}^\times$. Chose a character $c : U_{K/K_+} \to \mathbb{C}^\times$ such that

1) $c(x) = 1$ for all $x \in U_{K/K_+} \cap (1+\mathfrak{p}_K^{el})$,

2) $c(-1) = \begin{cases} 1 & : |\{\sigma \in \mathrm{Gal}(K/F) \mid \sigma^2 = 1\}| = 2, \\ -(-1)^{\frac{q-1}{2}f_+} & : |\{\sigma \in \mathrm{Gal}(K/F) \mid \sigma^2 = 1\}| = 4 \end{cases}.$

Put $\vartheta = \theta \cdot c$ and define a character $\widetilde{\vartheta} : K^\times \to \mathbb{C}^\times$ by $\widetilde{\vartheta}(x) = \vartheta(x^{1-\tau})$ $(x \in K^\times)$.

Let $W_{K/F}$ be the relative Weil group of K/F;

$$W_{K/F} = W_F/\overline{[W_K, W_K]} = \mathrm{Gal}(K/F) \ltimes_{\alpha_{K/F}} K^\times$$

where $\alpha_{K/F} \in Z^1(\mathrm{Gal}(K/F), K^\times)$ is the fundamental class of K/F. The representation space V_Θ of the induced representation $\Theta = \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}$ has a $W_{K/F}$-invariant quadratic form

$$S(\varphi, \psi) = \sum_{\gamma \in \mathrm{Gal}(K/F)} \widetilde{\vartheta}\left(\alpha_{K/F}(\gamma, \tau)\right)^{-1} \cdot \varphi(\gamma)\psi(\gamma\tau)$$

for $\varphi, \psi \in V_\Theta$. So we have a group homomorphism

$$\Theta : W_{K/F} \to O(V_\Theta, S) = O_{2n}(\mathbb{C}).$$

and then

$$\varphi : W_F \xrightarrow{\text{can.}} W_{K/F} \xrightarrow{\Theta \oplus \det \Theta} SO_{2n+1}(\mathbb{C}) \tag{3.3.1}$$

with $|\mathcal{A}_\varphi| = 2$. We have $a(\varphi, \mathrm{Ad}) = 2n^2 r$ and $L(\varphi, \mathrm{Ad}, s) = \left(1 + q^{-f_+ s}\right)^{-1}$, hence

$$\gamma(\varphi, \mathrm{Ad}, \psi, d(x), 0) = w(\varphi, \mathrm{Ad}) \cdot q^{n^2 r} \cdot \frac{2}{1 + q^{-f_+}}.$$

On the other hand

$$\varphi_0 : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} SL_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}_{2n}} SO_{2n+1}(\mathbb{C})$$

is the principal parameter, and we have

$$a(\varphi_0, \mathrm{Ad}) = 2n^2, \quad w(\varphi_0, \mathrm{Ad}) = 1, \quad L(\varphi_0, \mathrm{Ad}, s) = \prod_{k=1}^{n} \left(1 - q^{-(2k-1)} q^{-s}\right)^{-1}.$$

So the γ-factor is $\gamma(\varphi_0, \mathrm{Ad}, \psi, d(x), 0) = q^{n^2} \prod_{k=1}^{n} \frac{1 - q^{-(2k-1)}}{1 - q^{-2k}}$.

Let $d_{G(F)}$ be the Haar measure on $G(F)$ such that $\int_{G(O_F)} d_{G(F)}(x) = 1$. Then the Euler-Poincaré measure $\mu_{G(F)}$ on $G(F) = Sp_{2n}(F)$ is (see [12, p.150, Th.7])

$$d\mu_{G(F)}(x) = (-1)^n q^{n^2} \prod_{k=1}^{n} \left(1 - q^{-(2k-1)}\right) \cdot d_{G(F)}(x).$$

So Theorem 3.1.2 shows the following

Theorem 3.3.1 *The formal degree conjecture is valid for $\pi_{\beta,\theta}$ and φ of* (3.3.1).

3.4 Root number conjecture

The character of $\Theta = \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}$ is

$$\chi_\Theta(g) = \begin{cases} 0 & : \sigma \neq 1, \\ \sum_{\gamma \in \mathrm{Gal}(K/F)} \widetilde{\vartheta}\left(x^\gamma\right) & : \sigma = 1 \end{cases} \tag{3.4.1}$$

for $g = (\sigma, x) \in W_{K/F} = \mathrm{Gal}(K/F) \ltimes_{\alpha_{K/F}} K^\times$. Since $\mathrm{Ad} \circ \varphi(g) = \bigwedge^2 \varphi(g)$, we have

$$\chi_{\mathrm{Ad}\circ\Theta}(g) = \frac{1}{2}\left\{\chi_\Theta(g)^2 - \chi_\Theta(g^2)\right\}. \tag{3.4.2}$$

Now we can read the character formula (3.4.2), and we have

Theorem 3.4.1

$$\begin{aligned}&\mathrm{Ad} \circ (\mathbf{\Theta} \oplus \det \mathbf{\Theta}) \\ &= \bigoplus_{\substack{\pi \in \mathrm{Gal}(K/F)^\wedge \\ \pi(\tau)\neq 1}} \pi^{\dim \pi} \oplus \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta} \oplus \bigoplus_{\{\gamma \neq \gamma^{-1}\}} \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}_\gamma \oplus \bigoplus_{\substack{\gamma^2=1 \\ \gamma\neq 1,\tau}} \mathrm{Ind}_{W_{K/K_\gamma}}^{W_{K/F}} \chi_\gamma\end{aligned}$$

where, for $1 \neq \gamma \in \mathrm{Gal}(K/F)$*, we put* $\widetilde{\vartheta}_\gamma(x) = \widetilde{\vartheta}(x^{1+\gamma})$ *(*$x \in K^\times$*) and*

$$\begin{aligned}\chi_\gamma : W_{K/K_\gamma} = W_{K_\gamma}/\overline{[W_K, W_K]} &\xrightarrow[\text{can.}]{} W_{K_\gamma}/\overline{[W_{K_\gamma}, W_{K_\gamma}]} \\ &\xrightarrow[\text{l.c.f.t.}]{\sim} K_\gamma^\times \xrightarrow[(*,K/K_\gamma)\cdot\widetilde{\vartheta}]{} \mathbb{C}^\times\end{aligned}$$

with

$$\mathrm{Gal}(K/K_\gamma) = \langle\gamma\rangle, \quad (x, K/K_\gamma) = \begin{cases} 1 & : x \in N_{K/K_\gamma}(K^\times), \\ -1 & : x \notin N_{K/K_\gamma}(K^\times) \end{cases}$$

Then $\varepsilon(\mathrm{Ad} \circ \varphi, \psi, d_F(x), s)$ is a product of powers of q and Gauss sums:

$$G(\chi, \psi_K) = q^{-n/2} \sum_{t \in (O_K/\mathfrak{p}_K^n)^\times} \chi\left(\varpi_K^{-(n+d)} t\right) \cdot \psi_K\left(\varpi_K^{-(n+d)} t\right)$$

for $\chi : K^\times \to \mathbb{C}^\times$ s.t. $\chi|_{O_K^\times} \neq 1$ with

$$n = \mathrm{Min}\{0 < l \in \mathbb{Z} \mid \chi(1 + \mathfrak{p}_K^l) = 1\}, \qquad \mathcal{D}(K/\mathbb{Q}_p) = \mathfrak{p}_K^d$$

and

$$\psi_K : K \xrightarrow{T_{K/\mathbb{Q}_p}} \mathbb{Q}_p \xrightarrow{\text{can.}} \mathbb{Q}_p/\mathbb{Z}_p \tilde{\to} \mathbb{Q}/\mathbb{Z} \xrightarrow{\exp(2\pi\sqrt{-1}*)} \mathbb{C}^\times.$$

We have a trivial cancellation $G(\chi, \psi_K) \cdot G(\chi^{-1}, \psi_K) = \chi(-1)$. We have also (see [6, p.130, Th.3])

Theorem 3.4.2 *If* K/E *is a quadratic field extension such that* $\chi|_{E^\times} = 1$*, then*

$$G(\chi, \psi_K) = \chi(\varepsilon)$$

where $K = E(\varepsilon)$ *with* $\varepsilon^2 \in E^\times$*.*

After a lots of cancellation amog Gauss sums, we have

Theorem 3.4.3 $w(\varphi, \mathrm{Ad}) = \theta(-1)$.

On the other hand, the special central element ϵ of $Sp_{2n}(F)$ is $\epsilon = -1_{2n}$, and we have

$$\pi_{\beta,\theta}(\epsilon) = \delta_{\beta,\theta}(-1_{2n}) = \theta(-1).$$

So we have

Corollary 3.4.4 *The root number conjecture is valid for* $\pi_{\beta,\theta}$ *and* φ *of* (3.3.1).

4 Relations with the theory of Kaletha

4.1 Elliptic torus

Let K_+/F be a finite extension, K/K_+ a quadratic extension with a non-trivial element τ of $\mathrm{Gal}(K/K_+)$. Let us denote by L a Galois extension over F containing K in general for which let us denote by

$$\mathrm{Emb}_F(K, L) = \{\sigma|_K \mid \sigma \in \mathrm{Gal}(L/F)\}$$

the set of the embeddings over F of K into L.

Put $O_K = O_{K_+} \oplus \omega O_{K_+}$ with $\omega^\tau + \omega = 0$. Then $\mathrm{ord}_K(\omega) = e(K/K_+) - 1$. Let us denote by $\mathbb{V}$ the $\overline{F}$-algebra of the functions v on $\mathrm{Emb}_F(K, \overline{F})$ with values in $\overline{F}$ which is endowed with a symplectic $\overline{F}$-form

$$D(u, v) = \frac{1}{2} \sum_{\gamma \in \mathrm{Emb}_F(K, \overline{F})} \left(\omega^{-1} \varpi_{K_+}^{-d_+}\right)^\gamma u(\tau\gamma) \cdot v(\gamma)$$

($u, v \in \mathbb{V}$) where $\mathcal{D}(K_+/F) = \mathfrak{p}_{K_+}^{d_+}$ is the difference of K_+/F. The action of $\sigma \in \mathrm{Gal}(\overline{F}/F)$ on $v \in \mathbb{V}$ is defined by $v^\sigma(\gamma) = v(\gamma\sigma^{-1})^\sigma$. Then fixed point subspace $\mathbb{V}^{\mathrm{Gal}(\overline{F}/L)} = \mathbb{V}(L)$ is the set of the functions on $\mathrm{Emb}_F(K, L)$ with values in L, and $\mathbb{V}^{\mathrm{Gal}(\overline{F}/F)} = \mathbb{V}(F)$ is identified with K via $v \mapsto v(\mathbf{1}_K)$.

The action of $\sigma \in \mathrm{Gal}(\overline{F}/F)$ on $g \in Sp(\mathbb{V}, D)$ is defined by $v \cdot g^\sigma = (v^{\sigma^{-1}} \cdot g)^\sigma$. Then the fixed point subgroup $Sp(\mathbb{V}, D)^{\mathrm{Gal}(\overline{F}/F)}$ is identified with $Sp(K, D)$ via $g \mapsto g|_K$.

Put $S = \mathrm{Res}_{K/F}\mathbb{G}_m$ which is identified with the multiplicative group $\mathbb{V}^\times$. Then $S(F)$ is identified with the multiplicative group $K^\times$.

Let T be a subtorus of S wich is identified with the multiplicative subgroup of $\mathbb{V}^\times$ consisting of the functions s on $\mathrm{Emb}_F(K, \overline{F})$ to $\overline{F}^\times$ such that $s(\tau\gamma) = s(\gamma)^{-1}$ for all $\gamma \in \mathrm{Emb}_F(K, \overline{F})$. In other words T is a maximal torus of $Sp(\mathbb{V}, D)$ by identifying $s \in T$ with $[v \mapsto v \cdot s] \in Sp(\mathbb{V}, D)$. The fixed point subgroup $T^{\mathrm{Gal}(\overline{F}/F)} = T(F)$ is identified with

$$U_{K/K_+} = \{\varepsilon \in O_K^\times \mid N_{K/K_+}(\varepsilon) = 1\} \quad \text{by } s \mapsto s(\mathbf{1}_K).$$

4.2 Building

Fix a finite Galois extension L/F such that $K \subset L$. A function $\alpha : \mathbb{V}(L) \to \mathbb{R} \cup \{\infty\}$ is called a norm if

1) $\alpha(u + v) \geq \mathrm{Max}\{\alpha(u), \alpha(v)\}$ for all $u, v \in \mathbb{V}(L)$,

2) $\alpha(\xi v) = \mathrm{ord}_L(\xi) + \alpha(v)$ for all $\lambda \in L, v \in \mathbb{V}(L)$,

3) $\alpha(v) = \infty$ if and only if $v = 0$.

A norm α on $\mathbb{V}(L)$ is split over a L-basis $\{v_i, \cdots, v_{2n}\}$ of $\mathbb{V}(L)$ if

$$\alpha(v) = \mathop{\mathrm{Min}}_{1 \leq i \leq 2n} \{\mathrm{ord}_L(\xi_i) + \alpha(v_i)\}$$

for all $v = \sum_{i=1}^{2n} \xi_i v_i \in \mathbb{V}(L)$. An ordering $\alpha \leq \beta$ among norms α, β on $\mathbb{V}(L)$ is defined by $\alpha(v) \leq \beta(v)$ for all $v \in \mathbb{V}(L)$. A norm α on $\mathbb{V}(L)$ is *maximinorante* with respect to the symplectic form D if α is a maximal norm among the norms on $\mathbb{V}(L)$ such that

$$\alpha(u) + \alpha(v) \leq \mathrm{ord}_L D(u, v)$$

for all $u, v \in \mathbb{V}(L)$. The set $\mathcal{N}_D(\mathbb{V}(L))$ of the maximinorantes norms on $\mathbb{V}(L)$ with respect to D is identified with the Bruhat-Tits building of $Sp(\mathbb{V}(L), D)$ ([1, Th. 2.12]). The action of $g \in Sp(\mathbb{V}(L), D)$ on $\alpha \in \mathcal{N}_D(\mathbb{V}(L))$is defined by $(g \cdot \alpha)(v) = \alpha(v \cdot g)$.

Let $T' \subset Sp(\mathbb{V}(L), D)$ be a maximal L-split torus defined by a symplectic L-basis $\{v_{\pm i}\}_{1 \leq i \leq n}$, that is

$$D(v_i, v_j) = D(v_{-i}, v_{-j}) = 0, \quad D(v_i, v_{-j}) = \delta_{ij} \quad (1 \leq i, j \leq n)$$

and

$$T' = \{g \in Sp(\mathbb{V}(L), D) \mid \langle v_i \rangle_L g = \langle v_i \rangle_L, \ -n \leq i \leq n\}$$

Then the set $\mathcal{N}_D(\mathbb{V}(L); T')$ of the norm $\alpha \in \mathcal{N}_D(\mathbb{V}(L))$ splitting over $\{v_i\}_{-n \leq i \leq n}$ is identified with the apartment $X(T') \otimes_{\mathbb{Z}} \mathbb{R}$ of $Sp(\mathbb{V}(L), D)$ associated with T'.

The action of $\sigma \in \mathrm{Gal}(L/F)$ on $\alpha \in \mathcal{N}_D(\mathbb{V}(L))$ is defined by $\alpha^\sigma(v) = \alpha(v^{\sigma^{-1}})$. Then the fixed point subset $\mathcal{N}_D(\mathbb{V}(L); T)^{\mathrm{Gal}(L/F)}$, with the elliptic torus T defined in subsection 4.1, consists of the single element α such that

$$\alpha(v) = \inf_{\gamma \in \mathrm{Emb}_F(K,L)} \mathrm{ord}_L(v(\gamma)) - \frac{1}{2} e(L/K) \cdot \{e(K/K_+)(d_+ + 1) - 1\} \quad (v \in \mathbb{V}(L)),$$

and the corresponding element in $\mathcal{N}_D(K)$ is

$$\begin{aligned} \alpha_K &= e(L/F)^{-1} \alpha|_K \\ &= e(K/F)^{-1} \mathrm{ord}_K - \frac{1}{2} e(K/F)^{-1} \cdot \{e(K/K_+)(d_+ + 1) - 1\}. \end{aligned}$$

If K/F is a tamely ramified extension, then the fixed point subgroup

$$Sp(K, D)_{\alpha_K} = \{g \in Sp(K, D) \mid g \cdot \alpha_K = \alpha_K\}$$

is the intersection of $Sp(K, D)$ with an hereditary order

$$\left\{ \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1e} \\ \varpi_F A_{21} & A_{22} & \cdots & A_{2e} \\ \vdots & \vdots & \ddots & \vdots \\ \varpi_F A_{e1} & \varpi_F A_{e2} & \cdots & A_{ee} \end{bmatrix} \in M_{2n}(O_F) \,\middle|\, A_{ij} \in M_f(O_F) \right\}$$

of $M_{2n}(F)$ with respect to a suitably ordered symplectic F-basis of K which is an O_F-basis of O_K. Here we put $e = e(K/F)$ and $f = f(K/F)$. Then the open compact subgroup $T(F) \cdot Sp(K, D)_{\alpha_K}$ is a subgroup of $Sp_{2n}(O_F)$. Kaletha-Yu theory gives supercuspidal representations as compact inductions to $Sp(K, D) = Sp_{2n}(F)$ of irreducible representations of open compact subgroups of $T(F) \cdot Sp(K, D)_{\alpha_K}$. This means that the compact induction passes through the open compact subgroup $Sp_{2n}(O_F)$ and that the supercuspidal representation is the compact induction to $Sp_{2n}(F)$ of an irreducible representation δ of $Sp_{2n}(O_F)$. It seems provable that this δ is the representation $\delta_{\beta,\theta}$ given in subsection 3.1. But the exact matching of the two representations needs rather delicate arguments because, first of all, the correspondence of Theorem 2.1.1 is not canonical, and also because the construction of Kaletha-Yu contains certain uncertainties, for example the Howe factorization of (T,θ), not saying about recent modification by Fintzen-Kaletha-Spice [5]. For that reason, although it is provable that the supercuspidal representation given in Theorem 3.1.2 is the one which is parametrized by (T,θ) in the theory of Kaletha [10], I have not yet proved it at the moment of writing this report.

In the following subsections, we will see that the L-parameter of Kaletha corresponding to (T,θ) is exactly the candidate of Langlands parameter considered in subsection 3.3.

4.3 Local Langlands correspondence of torus

The group $X(S)$ of the characters over $\overline{F}$ of S is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis $\{b_\delta\}_{\delta\in\mathrm{Emb}_F(K,\overline{F})}$ where $b_\delta(s)=s(\delta)$ for $s\in S$. The dual torus $S^\frown = X(S)\otimes_{\mathbb{Z}}\mathbb{C}^\times$ is identified with the group of the functions s on $\mathrm{Emb}_F(K,\overline{F})$ with values in $\mathbb{C}^\times$. The action of $\sigma\in W_F\subset \mathrm{Gal}(\overline{F}/F)$ on S induces the action on $X(S)$ such that $b_\delta^\sigma = b_{\delta\sigma}$, and hence the action on $s\in S^\frown$ is defined by $s^\sigma(\gamma)=s(\gamma\sigma^{-1})$. Then the local Langlands correspondence for the torus S is the isomorphism

$$H^1(W_F,S^\frown)\tilde{\to}\mathrm{Hom}(W_K,\mathbb{C}^\times) \tag{4.3.1}$$

given by $[\alpha]\mapsto[\rho\mapsto\alpha(\rho)(\mathbf{1}_K)]$. Being restricted to continuous group homomorphisms, we have an isomorphism

$$H^1_{\mathrm{conti}}(W_F,S^\frown)\tilde{\to}\mathrm{Hom}_{\mathrm{conti}}(K^\times,\mathbb{C}^\times) \tag{4.3.2}$$

via (4.3.1) combined with the isomorphism of the local class filed theory

$$\delta_K : K^\times \tilde{\to} W_K/\overline{[W_K,W_K]}.$$

The restriction from S to T gives a surjection $X(S)\to X(T)$ whose kernel is the subgroup of $X(S)$ generated by $\{b_\delta+b_{\tau\delta}\mid\delta\in\mathrm{Emb}_F(K,L)\}$. Then the dual torus $T^\frown = X(T)\otimes_{\mathbb{Z}}\mathbb{C}^\times$ is identified with the group of the functions s on $\mathrm{Emb}_F(K,\overline{F})$ with values in $\mathbb{C}^\times$ such that $s(\tau\gamma)=s(\gamma)^{-1}$ for all $\gamma\in\mathrm{Emb}_F(K,\overline{F})$. The inclusion $T^\frown\subset S^\frown$ gives a canonical inclusion $H^1_{\mathrm{conti.}}(W_F,T^\frown)\subset H^1_{\mathrm{conti.}}(W_F,S^\frown)$. On the other hand, the surjection $x\mapsto x^{1-\tau}$ of $K^\times$ onto $U_{K/F}$ gives a canonical inclusion $\mathrm{Hom}_{\mathrm{conti.}}(U_{K/F},\mathbb{C}^\times)\subset\mathrm{Hom}_{\mathrm{conti.}}(K^\times,\mathbb{C}^\times)$. Now the restriction of the isomorphism (4.3.2) to these included subgroups gives the isomorphism

$$H^1_{\mathrm{conti}}(W_F,T^\frown)\tilde{\to}\mathrm{Hom}_{\mathrm{conti}}(U_{K/F},\mathbb{C}^\times). \tag{4.3.3}$$

See [20] for the details.

Put ${}^LT=W_F\ltimes T^\frown$. Then a cohomology class $[\alpha]\in H^1_{\mathrm{conti}}(W_F,T^\frown)$ defines a continuous group homomorphism

$$\widetilde{\alpha}:W_F\to {}^LT\quad(\sigma\mapsto(\sigma,\alpha(\sigma))) \tag{4.3.4}$$

and $\alpha\mapsto\widetilde{\alpha}$ induces a well-defined bijection

$$H^1_{\mathrm{conti}}(W_F,T^\frown)\tilde{\to}\mathrm{Hom}^*_{\mathrm{conti}}(W_F,{}^LT)/\text{“}T^\frown\text{-conjugate”}$$

where $\mathrm{Hom}^*_{\mathrm{conti}}(W_F, {}^L T)$ denotes the set of the continuous group homomorphisms ψ of W_F to ${}^L T$ such that $W_F \xrightarrow{\psi} {}^L T \xrightarrow{\text{proj.}} W_F$ is the identity map.

4.4 Howe factorization

From now on we will assume that K/F is a tamely ramified Galois extension and put $\Gamma = \mathrm{Gal}(K/F)$.

Let $\Phi(T)$ be the root system of $Sp(\mathbb{V}, D)$ with respect to T, that is

$$\Phi(T) = \{a_\gamma \cdot a_{\gamma'},\ a_\gamma^2 \mid \gamma, \gamma' \in \Gamma, \gamma \neq \gamma'\},$$

where $a_\gamma \in X(T)$ such that $a_\gamma(s) = s(\gamma)$. Note that $a_{\tau\gamma} = a_\gamma^{-1}$. Put

$$\Phi_k(T) = \left\{a \in \Phi(T) \mid \theta\left(N_{K/F}(a^\vee(x))\right) = 1\ \forall x \in 1 + \mathfrak{p}_K^{\lceil ek \rceil}\right\}$$

for any positive real number k ($1 + \mathfrak{p}_K^0 = O_K^\times$), where $a^\vee$ is the co-root of $a \in \Phi(T)$. If $1 < f < 2n$, then

$$\Phi_k(T) = \begin{cases} \Phi(T) & : k > r-1, \\ \left\{ a_\gamma \cdot a_{\gamma'}^{-1} \;\middle|\; \begin{array}{l} \dot\gamma, \dot\gamma' \in \Gamma/\langle\tau\rangle, \dot\gamma \neq \dot\gamma' \\ \gamma\gamma'^{-1} \in \mathrm{Gal}(K/K_0) \end{array} \right\} & : r - 1 - \dfrac{1}{e} < k \leq r-1, \\ \emptyset & : k \leq r - 1 - \dfrac{1}{e} \end{cases}.$$

Then the Howe factorization of (T, θ) is

$$G^0 = T \subsetneqq G^1 \subsetneqq G^2 = Sp_{2n}$$

with character

$$\begin{aligned} \phi_0 &: G^0 \to \mathbb{G}_m \text{ realized by } \mathrm{diag}((\beta - a_0)^\gamma)_{\gamma\in\Gamma}, \\ \phi_1 &: G^1 \to \mathbb{G}_m \text{ realized by } \mathrm{diag}(a_0^\gamma)_{\gamma\in\Gamma}, \\ \phi_2 = \mathrm{id} &: G^2 \to \mathbb{G}_m, \end{aligned}$$

where G^1 is a twisted Levi subgroup of Sp_{2n} such that

$$G^1 \otimes_F K = \underbrace{GL_e \times \cdots \times GL_e}_{f_+} \qquad (f_+ = f(K_+/F) = f/2)$$

and we put $\beta = \sum_{i=0}^{e-1} a_o \varpi_K^i$ with $a_i \in O_{K_0}$ and $\varpi_K \in K_+$ being a prime element of K (note that K/K_+ is unramified in this case). So, in this case, our supercuspidal representation $\pi_{\theta,\beta}$ may not be toric.

If $f = 2n$, then

$$\Phi_k(T) = \begin{cases} \Phi(T) & : k > r-1, \\ \emptyset & : k \leq r-1. \end{cases}$$

Then the Howe factorization of (T, θ) is

$$G^0 = T \subsetneqq G^1 = Sp_{2n}$$

with character $\phi_0 = \theta$, $\phi_1 = \mathrm{id}$.

If $f = 1$ so that K/F is totally ramified, then

$$\Phi_k(T) = \begin{cases} \Phi(T) & : k > r - 1 - \frac{1}{e}, \\ \emptyset & : k \leq r - 1 - \frac{1}{e}. \end{cases}$$

Then the Howe factorization of (T, θ) should be

$$G^0 = T \subsetneqq G^1 = Sp_{2n}$$

with character $\phi_0 = \theta$, $\phi_1 = \mathrm{id}$. But the genericity condition on the character $\phi_0 = \theta$ implies that K/F is unramified. So this case is excluded.

4.5 χ-data

Let us denote by $SO_{2n+1}(\mathbb{C})$ the complex special orthogonal group with respect to the symmetric matrix

$$S = \begin{bmatrix} S_0 & 0 \\ & -2 \end{bmatrix} \text{ with } S_0 = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}$$

and put

$$\mathbb{T}^\frown = \left\{ \begin{bmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{bmatrix} \middle| \, t = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}, \ t_i \in \mathbb{C}^\times \right\}$$

a maximal torus of $SO_{2n+1}(\mathbb{C})$. We have an isomorphism $T^\frown \tilde{\to} \mathbb{T}^\frown$ given by

$$s \mapsto \mathrm{diag}(s(\gamma_1), \cdots, s(\gamma_n), s(\gamma_{n+1}), \cdots, s(\gamma_{2n}), 1)$$

where $\mathrm{Emb}_F(K, \overline{F}) = \{\gamma_i\}_{1 \leq i \leq 2n}$ where $\gamma_1 = \mathbf{1}_K$ and $\gamma_{n+i} = \tau\gamma_i$ $(1 \leq i \leq n)$. The action of W_F on $T^\frown$ induces the action on $\mathbb{T}^\frown$ which factors through Γ.

The Weyl group $W(\mathbb{T}^\frown) = N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T}^\frown)/\mathbb{T}^\frown$ on $\mathbb{T}^\frown$ is identified with a subgroup of the permutation group S_{2n} generated by

$$\begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & 2n \\ \sigma(1) & \cdots & \sigma(n) & n+\sigma(1) & \cdots & n+\sigma(n) \end{pmatrix} \text{ with } \sigma \in S_n \text{ and}$$

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ n+1 & 2 & \cdots & n & 1 & n+2 & \cdots & 2n \end{pmatrix}.$$

Then any $w \in W(\mathbb{T}^\frown)$ is represented by

$$\widetilde{w} = \begin{bmatrix} [w] & 0 \\ 0 & \det[w] \end{bmatrix} \in N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T}^\frown),$$

where $[w] \in GL_{2n}(\mathbb{Z})$ is the permutation matrix corresponding to $w \in W(\mathbb{T}^\frown) \subset S_{2n}$.

For any $\delta \in \mathrm{Emb}_F(K, \overline{F}) = \Gamma$, let us denote by a_δ an element of $X(T^\frown)$ such that $a_\delta(s) = s(\delta)$ for all $s \in T^\frown$. Then

$$\Phi(T^\frown) = \{a_\delta \cdot a_{\delta'}, a_\delta \mid \delta, \delta' \in \Gamma, \ \delta \neq \delta'\}.$$

is the set of the roots of $SO_{2n+1}(\mathbb{C})$ with respect to $T^\frown = \mathbb{T}^\frown$ with the simple roots

$$\Delta = \{\alpha_i = a_{\gamma_i} \cdot a_{\tau\gamma_{i+1}}, \alpha_n = a_{\gamma_n} \mid 1 \le i < n\}.$$

Let $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ be the standard triple associate with a simple root $\alpha \in \Delta$. Then $s_\alpha \in W(\mathbb{T}^\frown)$ is represented by

$$n(s_\alpha) = \exp(X_\alpha) \cdot \exp(-X_{-\alpha}) \cdot \exp(X_\alpha) \in N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T}^\frown)$$

and $W(\mathbb{T}^\frown)$ is generated by $S = \{s_\alpha\}_{\alpha\in\Delta}$. For any $w \in W(\mathbb{T}^\frown)$, let $w = s_1 s_2 \cdots s_r$ $(s_i \in S)$ be a reduced presentation and put

$$n(w) = n(s_1)n(s_2)\cdots n(s_r) \in N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T}^\frown).$$

Then $r(w) = \widetilde{w}^{-1} n(w) \in \mathbb{T}^\frown$.

The action of $\sigma \in W_F$ on $X(T^\frown)$ induced from the action on $T^\frown$ is such that $a_\delta^\sigma = a_{\delta\sigma}$ for all $\delta \in \mathrm{Emb}_F(K, \overline{F})$, and it determines an element $w(\sigma) \in W(\mathbb{T}^\frown)$. Then [11] shows that the 2-cocycle $t \in Z^2(W_F, \mathbb{T}^\frown)$ defined by

$$t(\sigma, \sigma') = n(w(\sigma\sigma'))^{-1} n(w(\sigma)) \cdot n(w(\sigma')) \quad (\sigma, \sigma' \in W_F)$$

is split by $r_p : W_F \to \mathbb{T}^\frown$ defined by χ-data as follows.

For any $\lambda \in \Phi(T^\frown)$, put

$$\Gamma_\lambda = \{\sigma \in \Gamma \mid \lambda^\sigma = \lambda\}, \quad \Gamma_{\pm\lambda} = \{\sigma \in \Gamma \mid \lambda^\sigma = \pm\lambda\}$$

and put $F_\lambda = L^{\Gamma_\lambda}$, $F_{\pm\lambda} = L^{\Gamma_{\pm\lambda}}$. Then $(F_\lambda : F_{\pm\lambda}) = 1$ or 2 , and λ is called symmetric if $(F_\lambda : F_{\pm\lambda}) = 2$.

The Galois group Γ acts on $\Phi(T^\frown)$ and

$$\Phi(T^\frown)/\Gamma = \{a_{\mathbf{1}_K} a_\delta, a_{\mathbf{1}_K} \mid 1 \ne \delta \in \Gamma\}.$$

If $\lambda = a_{\mathbf{1}_K} a_\delta$, then λ is symmetric if and only if $\delta \ne \tau$. If further $\delta^2 \ne 1$, then $F_\lambda = K$ and $F_{\pm\lambda} = K_+$ and choose a continuous character $\chi_\lambda : F_\lambda^\times = K^\times \to \mathbb{C}^\times$ such that $\chi_\lambda|_{F_{\pm\lambda}^\times} : K_+^\times \to \{\pm 1\}$ is the character of the quadratic extension K/K_+. We may assume that $\chi_{a_{\mathbf{1}_K} a_{\delta^{-1}}} = \chi_{a_{\mathbf{1}_K} a_\delta}^{-1}$.

If $\delta^2 = 1$, then $F_\lambda = K_\delta$ and $F_{\pm\lambda} = E = K_\delta \cap K_+$ and choose a continuous character $\chi_\lambda : F_\lambda^\times = K_\delta^\times \to \mathbb{C}^\times$ such that $\chi_\lambda|_{F_{\pm\lambda}^\times} : E^\times \to \{\pm 1\}$ is the character of the quadratic extension K_δ/E.

If $\lambda = a_{\mathbf{1}_K}$ then $F_\lambda = K$ and $F_{\pm\lambda} = K_+$ and choose a continuous character $\chi_\lambda : F_\lambda^\times = K^\times \to \mathbb{C}^\times$ such that $\chi_\lambda|_{F_{\pm\lambda}^\times} : K_+^\times \to \{\pm 1\}$ is the character of the quadratic extension K/K_+.

These characters are parts of a system of χ-data $\chi_\lambda : F_\lambda \to \mathbb{C}^\times$ $(\lambda \in \Phi(\mathbb{T}^\frown))$ such that

1) $\chi_{-\lambda} = \chi_\lambda^{-1}$ and $\chi_{\lambda^\sigma} = \chi_\lambda(x^{\sigma^{-1}})$ for all $\sigma \in \Gamma$, and

2) $\chi_\lambda = 1$ if λ is not symmetric.

With this χ-data and the gauge

$$p : \Phi(\mathbb{T}^\frown) \to \{\pm 1\} \text{ s.t. } p(\lambda) = \begin{cases} 1 & : \lambda > 0, \\ -1 & : \lambda < 0, \end{cases}$$

the mechanism of [11] gives a $r_p : W_F \to \mathbb{T}^\frown$ such that

$$t(\sigma,\sigma') = r_p(\sigma)^{\sigma'} r_p(\sigma\sigma')^{-1} r_p(\sigma') \text{ for all } \sigma,\sigma' \in W_F$$

and

$$r_p(\sigma) = \prod_{\delta\in\Gamma,\delta^2\neq 1} \prod_{0<\lambda\in\{a_{\mathbf{1}_K} a_\delta\}_\Gamma} \chi_\lambda(x)^{\check\lambda} \times \prod_{\substack{\delta\in\Gamma,\delta^2=1\\ \delta\neq 1,\tau}} \prod_{0<\lambda\in\{a_{\mathbf{1}_K} a_\delta\}_\Gamma} \chi_\lambda(N_{K/F_\lambda}(x))^{\check\lambda}$$
$$\times \prod_{0<\lambda\in\{a_{\mathbf{1}_K}\}_\Gamma} \chi_\lambda(x)^{\check\lambda}$$

if $\dot\sigma = (1,x) \in W_{K/F} = \Gamma \ltimes_{\alpha_{K/F}} K^\times$, where $\{\alpha\}_\Gamma$ is the Γ-orbit of $\alpha \in \Phi(\mathbb{T}^\frown)$ and $\check\lambda$ is the co-root of λ. Then we have a group homomorphism

$$^L T = W_F \ltimes \mathbb{T}^\frown \to SO_{2n+1}(\mathbb{C}) \quad ((\sigma,s) \mapsto n(w(\sigma)) r_p(\sigma)^{-1} s). \tag{4.5.1}$$

If we put $r(\sigma) = r(w(\sigma))$ for $\sigma \in W_F$, we have

$$t(\sigma,\sigma') = r(\sigma)^{\sigma'} r(\sigma\sigma')^{-1} r(\sigma') \qquad (\sigma,\sigma' \in W_F).$$

Now $\chi_p(\sigma) = r(\sigma)\cdot r_p(\sigma)^{-1}$ $(\sigma \in W_F)$ define an element of $Z^1(W_F, \mathbb{T}^\frown)$ and the group homomorphism (4.5.1) is

$$^L T = W_F \ltimes \mathbb{T}^\frown \to SO_{2n+1}(\mathbb{C}) \quad ((\sigma,s) \mapsto \widetilde{w}(\sigma)\chi_p(\sigma)\cdot s). \tag{4.5.2}$$

Let $c \in \mathrm{Hom}_{\mathrm{conti}}(U_{K/K_+}, \mathbb{C}^\times)$ be the character corresponding to the cohomology class $[\chi_p] \in H^1(W_F, \mathbb{T}^\frown)$ by the local Langlands correspondence of torus (4.3.3). Then we have

Proposition 4.5.1 1) *$c(x) = 1$ for all $x \in U_{K/K_+} \cap (1+\mathfrak{p}_K^2)$,*

$$2)\ c(-1) = \begin{cases} 1 & : |\{\sigma\in\Gamma \mid \sigma^2 = 1\}| = 2, \\ -(-1)^{\frac{q-1}{2}\cdot f_+} & : |\{\sigma\in\Gamma \mid \sigma^2 = 1\}| = 4 \end{cases}$$

Remark 4.5.2 *Note that we have excluded the case of K/F being totally ramified when we consider the Howe factorization of (T,θ).*

4.6 Langlands parameter

The continuous character θ of U_{K/K_+} which parametrizes the irreducible representation $\delta_{\beta,\theta}$ of $Sp_{2n}(O_F)$ determines the cohomology class $[\alpha] \in H^1_{\mathrm{conti}}(W_F, T^\frown)$. Then we have a group homomorphism

$$\varphi : W_F \xrightarrow{\widetilde{\alpha}} {}^L T \xrightarrow{(4.5.2)} SO_{2n+1}(\mathbb{C}). \tag{4.6.1}$$

The construction of φ shows that $\varphi(\sigma) \in SO_{2n+1}(\mathbb{C})$ is of the form

$$\varphi(\sigma) = \begin{bmatrix} \psi(\sigma) & 0 \\ 0 & \det\psi(\sigma) \end{bmatrix} \text{ with } \psi(\sigma) \in O(S_0, \mathbb{C}).$$

(4.6.1) shows that

$$\begin{aligned}\mathrm{tr}\psi(\sigma) &= \sum_{\gamma\in \mathrm{Emb}_F(K,\overline{F}),\gamma\sigma=\gamma} \chi_p(\sigma)(\gamma)\cdot\alpha(\sigma)(\gamma)\\ &= \sum_{\dot\gamma\in W_K\backslash W_F,\gamma\sigma\gamma^{-1}\in W_K} \chi_p(\sigma)(\gamma)\cdot\alpha(\sigma(\gamma))\\ &= \sum_{\dot\gamma\in W_K\backslash W_F,\gamma\sigma\gamma^{-1}\in W_K} \psi_c\cdot\psi_\theta(\gamma\sigma\gamma^{-1})\end{aligned}$$

where ψ_c (resp. ψ_θ) is the element of $\mathrm{Hom}_{\mathrm{conti}}(W_K,\mathbb{C})$ corresponding to c (resp. θ) of $\mathrm{Hom}_{\mathrm{conti}}(U_{K/K_+},\mathbb{C}^\times)\hookrightarrow \mathrm{Hom}_{\mathrm{conti}}(K^\times,\mathbb{C}^\times)$. More strictly we have

$$\mathrm{tr}\psi(\sigma,x) = \begin{cases} 0 & :\sigma\neq 1,\\ \sum_{\gamma\in\mathrm{Gal}(K/F)} \widetilde{c\cdot\theta}(x^\gamma) & :\sigma=1\end{cases}$$

for $(\sigma,x)\in W_F/\overline{[W_K,W_K]}=\mathrm{Gal}(K/F)\ltimes_{\alpha_{K/F}}K^\times$ where $\widetilde{c\cdot\theta}(x)=c\cdot\theta(x^{1-\tau})$ ($x\in K^\times$). Being compared with (3.4.1), this shows that parameter φ defined here is identical to the parameter (3.3.1) in the subsection 3.3.

5 Additional remarks

We have excluded the case of K/F being totally ramified because of

1) the assurance of the connectivity of the centralizer G_β, and
2) the genericity condition of the character appearing in the Howe factorization of (T,θ).

In fact, we can prove the surjectivity of the canonical homomorphisms

$$G_\beta(O_F)\to G_\beta(O_F/\mathfrak{p}^l),\quad \mathfrak{g}_\beta(O_F)\to\mathfrak{g}_\beta(O_F/\mathfrak{p}^l),$$

which is enough to make work the arguments for Theorem 2.1.1, and the arguments in section 3. In this case, the condition on the character c at the beginning of subsection 3.3 is

1) $c(x)=1$ for all $x\in U_{K/K_+}\cap(1+\mathfrak{p}_K^{el})$,
2) $c(-1)=\begin{cases}(-1)^{\frac{q-1}{4}} & :n=\text{even},\\ (-1)^{\frac{q-1}{2}\cdot\frac{n+1}{2}} & :n=\text{odd}\end{cases}$.

Note that if K/F is totally ramified, then $q-1$ is divisible by $e=2n$.

Recently [5] gives a remedy for a gap, which is pointed out by Fintzen [4], in the argument of [19]. On the other hand [4] shows that Yu's original method gives supercuspidal representations. So the situation is rather complicated, and we need further studies for putting properly our results of Section 3 in the framework of Kaletha-Yu theory.

Finally it should be noted that Schwein [15] shows that the formal degree conjecture is valid with respect to the regular supercuspidal representation and L-parameter constructed by Kalethat [10].

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