NONTEMPERED RESTRICTION PROBLEMS FOR CLASSICAL GROUPS

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1. Introduction

This short note is based on a talk given at the RIMS conference "Automorphic forms, Automorphic representations, Galois representations, and its related topics", held at Kyoto in January 2021. We thank Takuya Yamauchi for his kind invitation to deliver a talk at this conference. Being based on a talk, the style of this note will be somewhat informal.

The restriction problem referred to in the title is that arising in the Gross-Prasad conjecture. Let $G_n = \mathrm{GL}_n$, U_n or SO_n over a local field F. For $\pi \in \mathrm{Irr}(G_n)$ and $\sigma \in \mathrm{Irr}(G_{n-1})$, define a branching multiplicity

$$m(\pi, \sigma) = \dim \mathrm{Hom}_{G_{n-1}}(\pi, \sigma).$$

One has the following multiplicity-at-most-one result, due to Aizenbud-Gourevitch-Rallis-Schiffman [AGRS] and Sun-Zhu [SZ]:

Theorem 1.1.

$$m(\pi, \sigma) \leq 1.$$

The GP conjecture [GGP1] proposes a determination of $m(\pi, \sigma)$ when π and σ belong to generic L-packets.

In the case of $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$, one has:

Theorem 1.2. *If π and σ are both generic (i.e. supports nonzero Whittaker functionals), then*

$$m(\pi, \sigma) = 1.$$

In particular, the theorem covers every tempered irreducible representation of GL_n. This leads to the natural question:
Question: What if $\pi \otimes \sigma$ is not generic?

In the recent paper [GGP2], Gross, Prasad and I proposed an answer to this question for representations in the automorphic spectrum. The purpose of this note is explain this conjecture. After some preliminaries in §2, we discuss the case of GL in §3 before moving to the classical groups in §4.

2. Preliminaries

In this section, we introduce some basic notions needed to formulate our conjecture.

2.1. **Automorphic Spectrum.** The automorphic spectrum of $\mathrm{GL}_n(F)$ is the subset of the unitary dual consisting of those unitary representations of $\mathrm{GL}_n(F)$ which occur in the spectral decomposition of

$$L^2_\chi(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A}))$$

for some global field k with $F = k_v$ for some place v. Here χ denotes a fixed (but arbitrary) unitary central character.

For $G = \mathrm{GL}_n$, denote the automorphic spectrum by

$$\widehat{G}^{\mathrm{temp}} \subset \widehat{G}^{\mathrm{aut}} \subset \widehat{G}^{\mathrm{unit}} \quad \text{(Unitary dual)}.$$

Arthur's conjecture give a description of the elements of $\widehat{G}^{\mathrm{aut}}$ in terms of the notion of A-parameters.

2.2. **Arthur Parameters.** A local A-parameter for GL_n over a p-adic field F is a conjugacy class of maps

$$\psi : \left(W_F \times \mathrm{SL}_2(\mathbb{C})^D\right) \times \mathrm{SL}_2(\mathbb{C})^A \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

such that

$$\psi(W_F) \text{ is bounded in } \mathrm{GL}_n(\mathbb{C}).$$

One may write the irreducible decomposition of ψ as:

$$\psi = \bigoplus_i M_i \otimes [a_i] \otimes [b_i]$$

where M_i an irreducible representation of W_F (with bounded determinant) and $[a]$ denotes the irreducible representation of SL_2 of dim. a. Sometimes, we will also consider the decomposition $\psi = \oplus_d M_d \otimes [d]$, as $[d]$ runs over the irreducible representations of $\mathrm{SL}_2(\mathbb{C})^A$.

The restriction of ψ to $\mathrm{SL}_2(\mathbb{C})^A$ corresponds to a unipotent conjugacy class in $\mathrm{GL}_n(\mathbb{C})$, i.e. a partition of n. We call this the *type of* ψ. More precisely, if $\psi = \oplus_d M_d \otimes [d]$, then the type of ψ is the partition

$$(d^{\dim M_d})_{d \geq 1} = (1^{\dim M_1}, 2^{\dim M_2}, ...).$$

For example: if ψ is trivial on SL_2^A, then ψ is simply a tempered L-parameter, and its type is trivial.

One can view the maximum d occurring in $\psi = \oplus_d M_d \otimes [d]$ as a measure of nontemperedness of ψ.

2.3. **L-parameter associated to an A-parameter.** To each A-parameter ψ, we can associate an L-parameter

$$\phi_\psi : W_F \times \mathrm{SL}_2^D \longrightarrow W_F \times \mathrm{SL}_2^D \times \mathrm{SL}_2^A \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

given by

$$\phi_\psi(w, g) = \psi\left(w, g, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

The map $\psi \mapsto \phi_\psi$ gives an injection

$$\{\text{tempered } L\text{-parameters}\} \hookrightarrow \{A\text{-parameters}\} \hookrightarrow \{L\text{-parameters}\}$$

whose image (i.e. the L-parameters of the form ϕ_ψ) are *L-parameters of Arthur type*.

2.4. **Arthur Packets.** Arthur's conjecture postulates that to each ψ, one can associate a finite (multi-)set Π_ψ of irreducible unitary representations (in the automorphic spectrum). One basic property that Π_ψ should have is

$$\Pi_\psi \supset \Pi^L_{\phi_\psi} \quad \text{(L-packet associated to } \phi_\psi).$$

For GL_n, it turns out that

$$\Pi_\psi = \Pi^L_{\phi_\psi}$$

and in particular is a singleton. Hence

$$\widehat{\mathrm{GL}_n}^{\mathrm{aut}} = \{\Pi_\psi : \psi \text{ an } A\text{-parameter}\}.$$
$$= \bigsqcup_{\text{type } P} \widehat{\mathrm{GL}_n}_P^{\mathrm{aut}} \quad \text{(finite partition)}.$$

The representations in Π_ψ are said to be of Arthur type and can be constructed as follows.

2.5. **Representations of Arthur type.** Suppose first that ψ is irreducible:

$$\psi = \rho \otimes [a] \otimes [b].$$

Let

$$\mathrm{St}(\rho, a) = \text{unique irred. submodule of } \pi_\rho |-|^{(a-1)/2} \times ... \times \pi_\rho |-|^{-(a-1)/2}$$

be the generalized Steinberg representation with L-parameter $\rho \otimes [a]$. Then Π_ψ is the unique irreducible quotient $\mathrm{Speh}(\rho, a, b)$ of

$$\mathrm{St}(\rho, a)|-|^{(b-1)/2} \times ..... \times \mathrm{St}(\rho, a)|-|^{-(b-1)/2}.$$

If $\psi = \oplus_i \psi_i$ with ψ_i irreducible, then

$$\Pi_\psi = \Pi_{\psi_1} \times .... \times \Pi_{\psi_r} \quad \text{(parabolic induction)}.$$

As an example, $\psi = [n]$ gives trivial representation of $\mathrm{GL}_n(F)$.

2.6. **Unitary Restriction Problem.** We are essentially ready to formulate our conjecture. Before that, it is useful to recall a result of Clozel and Venkatesh which concerns a unitary versiopn of our restriction problem.

More precisely, consider the direct integral decomposition of Π_ψ restricted to GL_{n-1}:

$$\Pi_\psi|_{\mathrm{GL}_{n-1}} = \int_{\widehat{\mathrm{GL}_{n-1}}^{\mathrm{unit}}} m(\sigma) \cdot \sigma \, d\mu_\psi(\sigma)$$

One is interested in understanding the support of the spectral measure $d\mu_\psi$:

$$\mathrm{supp}(d\mu_\psi) \subset \widehat{\mathrm{GL}_{n-1}}^{\mathrm{unit}}.$$

Theorem 2.1 (Burger-Sarnak)**.**

$$\mathrm{supp}(d\mu_\psi) \subset \widehat{\mathrm{GL}_{n-1}}^{\mathrm{aut}}.$$

Theorem 2.2 (Clozel)**.** *All representations in* $\mathrm{supp}(d\mu_\psi)$ *are of the same type* Q_ψ*. Moreover,* Q_ψ *depends only on the type* P *of* ψ*.*

Clozel's theorem implies there is a map

$$f : \{\text{Partitions of } n\} \longrightarrow \{\text{Partitions of } n-1\}$$

such that any Arthur type representation of GL_n of type P is supported on Arthur type representations of type $f(P)$ when restricted to GL_{n-1}. Venkatesh [V] explicated this map:

Theorem 2.3 (Venkatesh)**.**

$$f(n_1, n_2, .., n_r) = (n_1 - 1, n_2 - 1, ..., n_r - 1, \quad 1, ...1)$$

Here we discard those $n_i - 1$ which are 0, and we add the appropriate number of 1's. Indeed, Venkatesh considered a number of such problems: restriction from GL_n to GL_m, induction from GL_m to GL_n and tensor product of two representations of GL_n. Since the unitary restriction problem is a coarser version of the restriction problem in smooth representation theory, Venkatesh's theorem serves as a first approximation to the answer for the smooth restriction problem.

3. Conjecture and Theorem: GL_n case

In our paper [GGP2], Gross, Prasad and I made the following conjecture

Conjecture 3.1. *Let ψ (resp. ψ') be an A -parameter of GL_n (resp. GL_{n-1}). Then*

$$m(\Pi_\psi, \Pi_{\psi'}) = 1 \Longleftrightarrow (\psi, \psi') \textit{ is a relevant pair of A-parameters}$$

We verified our conjecture when, for example, $\psi|_{\mathrm{SL}_2^D}$ and $\psi'|_{\mathrm{SL}_2^D}$ are trivial. Subsequently, Max Gurevich [G] showed

$$m(\Pi_\psi, \Pi_{\psi'}) = 1 \Longrightarrow (\psi, \psi') \text{ is relevant },$$

and shortly after, Kei Yuen Chan [C] showed

Theorem 3.2 (KY Chan)**.** *The above Conjecture holds.*

In this section, we shall explain the notion of "relevance" which occurs in the conjecture and various ways of understanding it.

3.1. **Relevant A-parameters.** Given A-parameters ψ and ψ' of GL_n and GL_{n-1}, we say (ψ, ψ') are relevant if we may write

$$\psi = \left(\bigoplus_{i \in I} \phi_i \otimes [b_i]\right) \oplus \left(\bigoplus_{j \in J} \rho_j \otimes [c_j - 1]\right)$$

and

$$\psi' = \left(\bigoplus_{i \in I} \phi_i \otimes [b_i - 1]\right) \oplus \left(\bigoplus_{j \in J} \rho_j \otimes [c_j]\right).$$

with $b_i, c_j \geq 1$.

For example:

- when all b_i and $c_j = 1$: the pair (ψ, ψ') is always relevant; this is the tempered/generic case;
- when all $c_j = 1$: here ψ' is related to ψ as in Venkatesh's theorem above.

In general, we may think of ψ and ψ' as "within distance 1 of each other".

Here is another formulation. Write

$$\psi = \bigoplus_d M_d \otimes [d] \quad \text{and} \quad \psi' = \bigoplus_d N_d \otimes [d].$$

Then (ψ, ψ') is relevant if there are decompositions of WD_F-modules:

$$M_d = M_d^+ \oplus M_d^- \quad \text{and} \quad N_d = N_d^+ \oplus N_d^-$$

such that

$$M_d^+ = N_{d+1}^- \quad \text{and} \quad M_d^- = N_{d-1}^+.$$

This formulation is more convenient for extending the notion of relevance to classical groups.

Examples:

(1) When $\psi = [n]$, Π_ψ is the trivial representation of GL_n, so we know its restriction to GL_{n-1}. The only ψ' such that (ψ, ψ') is relevant is

$$\psi' = [n-1].$$

(2) Now take $\psi' = [n-1]$, so that $\Pi_{\psi'}$ is the trivial representation of GL_{n-1}. The only ψ's for which (ψ, ψ') is relevant are:
   - $\psi = [n]$
   - $\psi = [n-2] + \rho$, with 2-dim tempered ρ.

   This corresponds to the fact that "GL_{n-1}-distinguished nontrivial reps of GL_n are theta lifts from GL_2".

3.2. **Relevance and Correlator.** Zhiwei Yun has given a more conceptual formulation of the notion of relevance. Given

$$\psi : WD_F \times \mathrm{SL}_2^A \longrightarrow \mathrm{GL}(V) \quad \text{and} \quad \psi' : WD_F \times \mathrm{SL}_2^A \longrightarrow \mathrm{GL}(W),$$

the maximal torus of SL_2^A gives a $\mathbb{C}^\times$-action and hence a WD_F-stable $\mathbb{Z}$-grading on V and W. Then $\mathrm{End}(V)$, $\mathrm{End}(W)$, $\mathrm{Hom}(V,W)$ and $\mathrm{Hom}(W,V)$ also inherit a grading.

The unipotent elements

$$e = d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(V) \quad \text{and} \quad e' = d\psi' \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(W).$$

are elements of degree 2.

Lemma 3.3. *The pair (ψ, ψ') is relevant if and only if there exists degree 1 elements $T \in \mathrm{Hom}_{WD_F}(V, W)$ and $S \in \mathrm{Hom}_{WD_F}(W, V)$ such that*

$$e = S \circ T \quad \textit{and} \quad e' = T \circ S.$$

The map T in the above lemma is called a correlator between ψ and ψ'.

3.3. **Relevance and Moment Map.** One can reformulate Yun's interpretation in more geometric terms, in terms of a moment map arising in theta correspondence. The group $\mathrm{GL}(V) \times \mathrm{GL}(W)$ acts on the symplectic variety

$$T^*(V^* \otimes W) = (V^* \otimes W) \times (V \otimes W^*).$$

This action is Hamiltonian and thus give rise to moment maps:

$$\begin{tikzcd} & \mathrm{Hom}(V,W) \times \mathrm{Hom}(W,V) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathfrak{gl}(V) & & \mathfrak{gl}(W) \end{tikzcd}$$

where

$$p(T,S) = S \circ T \quad \text{and} \quad q(T,S) = T \circ S.$$

Hence (ψ, ψ') is relevant if and only if there exists a WD_F-invariant degree 1 element (T,S) such that $p(T,S) = e$ and $q(T,S) = e'$.

3.4. **Philosophy of Sakellaridis-Venkatesh.** We can view the previous discussion through the philosophy of Sakellaridis-Venkatesh [SV] in the relative Langlands program. In the relative Langlands program, one considers the following

Problem: If $X = H \backslash G$ is a G-spherical variety, describe

- the decomposition of $L^2(X)$;
- the subset $\mathrm{Irr}_X(G) = \{\pi \in \mathrm{Irr}(G) : \pi \hookrightarrow C^\infty(X)\}$.

The monograph [SV] formulates a conjectural answer:

- there is a dual group $X^\vee$ equipped with $\iota : X^\vee \times \mathrm{SL}_2 \to G^\vee$;
- there is a $\mathbb{Z}$-graded finite-dimensional representation V_X of $X^\vee$,

so that

- representations in $L^2(X)$ or $\mathrm{Irr}_X(G)$ are those whose A-parameters factor through ι;
- the spectral measure of $L^2(X)$ is described by the L-function associated to V_X.

In our case, $X = (GL_n \times \mathrm{GL}_{n-1})/\,\mathrm{GL}_{n-1}^{\Delta}$ and $X^\vee = G^\vee = \mathrm{GL}_n \times \mathrm{GL}_{n-1}$

Recently, Ben-Zvi-Sakellaridis-Venkatesh reformulated the above as a duality between certain Hamiltonian G-varieties. More precisely:

- Starting from a Hamiltonian G-variety Y, one can attach a dual Hamiltonian $G^\vee$-variety $Y^\vee$;
- A Hamiltonian G-variety can be quantized to a unitary rep. Π_Y of G;
- the spectral decomposition of Π_Y, or the description of

$$\mathrm{Irr}_Y(G) = \{\pi \in \mathrm{Irr}(G) : \mathrm{Hom}_G(\Pi_Y^\infty, \pi) \neq 0\},$$

  is governed by the symplectic geometry of $Y^\vee$.

One thus have a picture:

$$\begin{tikzcd}
\{\text{Hamiltonian } G\text{-var. } Y\} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \{\text{Hamiltonian } G^\vee\text{-var. } Y^\vee\} \arrow[d, dash] \arrow[dl, dash] \\
\{\text{Unitary rep. } \Pi_Y \text{ of } G\} & \{\text{Unitary rep. } \Pi_{Y^\vee} \text{ of } G^\vee\}
\end{tikzcd}$$

where the vertical arrows denote quantization, the horizontal arrows denote the purported duality of Hamiltonian varieties and the diagonal arrows indicate that the spectral problem in the bottom row is governed by the symplectic geometry of the Hamiltonian variety in the top row.

Let us give some examples of quantization here:

- If $Y = T^*(X)$ (X a G-variety), then $\Pi_Y = L^2(X)$.
- If $Y = \mathcal{O}^* \hookrightarrow \mathfrak{g}^*$ a coadjoint orbit, the $\pi_Y \in \mathrm{Irr}(G)$ (Orbit method).
- If $Y = W = X + X^*$ (symplectic space with Lagrangian X) with $G = \mathrm{Sp}(W)$, $\Pi_Y = L^2(X)$ is a Weil representation of $\widetilde{\mathrm{Sp}}(W)$.

It is reasonable to expect that the decomposition of Π_Y is controlled by the underlying geometry of Y, e.g. through its moment map

$$\mu_Y : Y \longrightarrow \mathfrak{g}^*.$$

Indeed, in the unitary setting, one would expect the spectral support of Π_Y to be described by the image of the moment map μ_Y, through the orbit method. As another example, one has the process of symplectic reduction:

$$\text{Given } \mathcal{O}^* \subset \mathfrak{g}^*,\ \mu^{-1}(\mathcal{O}^*)//G \text{ is a symplectic variety}$$

The quantization of this process is the extraction of the multiplicity space of $\pi_{\mathcal{O}^*}$ in the spectral decomposition of Π_Y.

What I find intriguing about the BZSV-philosophy are:

- Instead of using the geometry of Y, they propose that the spectral decomposition of Π_Y is related to the geometry of $Y^\vee$.
- They highlighted that period or branching problems come in pairs: Π_Y and $\Pi_{Y^\vee}$.

One can thus examine the period problems that have been studied in the literature and try to identify their other half. What new insights does one get?

The above discussion is largely conjectural, because of the following:

Problems:

- there is as yet no clear geometric definition of which class of Hamiltonian G-variety to consider for the duality theory;
- there is as yet no natural geometric construction of $Y \mapsto Y^\vee$.

There is one case when one has a good idea what $Y^\vee$ should be. When X is a G-spherical variety, let $Y = T^*(X)$. Then the quantization of Y is

$$\Pi_Y = L^2(X) \quad \text{and} \quad \Pi_Y^\infty = C_c^\infty(X).$$

In this case, $Y^\vee$ is something like

$$Y^\vee = G^\vee \times_{X^\vee} V_X$$

at least when $\iota(\mathrm{SL}_2)$ is trivial.

3.5. **Our example.** In our case, $X = (GL_n \times \mathrm{GL}_{n-1})/\,\mathrm{GL}_{n-1}^{\Delta}$,

$$X^\vee = G^\vee = \mathrm{GL}_n \times \mathrm{GL}_{n-1} \quad \text{and} \quad V_X = V^* \otimes W \times V \otimes W^*.$$

So

$$Y = T^*(X) \quad \text{and} \quad Y^\vee = V_X = V^* \otimes W \times V \otimes W^* = T^*(V^* \otimes W).$$

Our conjecture can be viewed as giving a precise formulation of the expectation:

The symplectic geometry of $Y^\vee$ (via its moment map) governs the decomposition of the quantization of Y.

We may exchange the role of Y and $Y^\vee$. The quantization of $Y^\vee$ is

$$L^2(V^* \otimes W) = \text{a Weil representation of } \mathrm{GL}(V) \times \mathrm{GL}(W).$$

So we expect the theta correspondence to be governed by the symplectic geometry of Y! Hence, in the sense of the BZSV-philosophy,

Gross-Prasad periods is dual to theta correspondence.

4. **Classical Groups**

For the rest of this note, we consider the case of $G \times H = \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$, though one can consider the general case: $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$. So $G^\vee = \mathrm{Sp}_{2n}(\mathbb{C})$ and $H^\vee = \mathrm{SO}_{2n}(\mathbb{C})$. Hence an A-parameter for G is a symplectic representation

$$\psi : WD_F \times \mathrm{SL}_2^A \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$$

whereas an A-parameter for H is an orthogonal representation

$$\psi' : WD_F \times \mathrm{SL}_2^A \longrightarrow \mathrm{O}_{2n}(\mathbb{C}).$$

We may write

$$\psi = \bigoplus_d M_d \otimes [d] \quad \text{and} \quad \psi' = \bigoplus_d N_d \otimes [d].$$

4.1. **Relevance and Moment Map.** The notion of relevance of the pair (ψ, ψ') can be imported verbatim from the GL case, provided we use the appropriate formulation. More precisely, with ψ and ψ' decomposed as above, we say that (ψ, ψ') is relevant if there are decompositions of WD_F-modules:

$$M_d = M_d^+ \oplus M_d^- \quad \text{and} \quad N_d = N_d^+ \oplus N_d^-$$

such that

$$M_d^+ = N_{d+1}^- \quad \text{and} \quad M_d^- = N_{d-1}^+.$$

We can also formulate this notion in terms of a moment map. With $G^\vee = \mathrm{Sp}(W)$ and $H^\vee = \mathrm{SO}(V)$, $G^\vee \times H^\vee$ acts naturally on the symplectic variety $W^* \otimes V$, giving the moment map diagram:

$$\begin{tikzcd} & W^* \otimes V \arrow[dl, "p"'] \arrow[dr, "q"] & \\ e \in \mathfrak{sp}(W) & & e' \in \mathfrak{so}(V) \end{tikzcd}$$

The pair of A-parameters (ψ, ψ') is relevant if and only if there exists a degree 1 element (correlator) $T \in \mathrm{Hom}_{WD_F}(W, V)$ such that

$$T^* \circ T = e \quad \text{and} \quad T \circ T^* = e'$$

where $T^* \in \mathrm{Hom}_{WD_F}(V^*, W^*) \cong \mathrm{Hom}_{WD_F}(V, W)$ is the adjoint map.

4.2. **A-Packets.** Given A-parameter ψ of $G = \mathrm{SO}_{2n+1}$, Arthur associates an A-packet Π_ψ of finitely many unitary reps in the automorphic spectrum. Though one knows much less about Π_ψ compared to the case of GL_n, one does know the following:

- If ϕ_ψ is the L-parameter associated to ψ, then $\Pi^L_{\phi_\psi} \subset \Pi_\psi$.
- Let
$$A_\psi = \pi_0(Z_{G^\vee}(\psi)).$$
Then Π_ψ is a representation of $G \times A_\psi$:
$$\Pi_\psi = \bigoplus_{\eta \in \mathrm{Irr}(A_\psi)} \eta \otimes \pi_\eta$$
where π_η is a unitary rep. of G of finite length (maybe 0). Moeglin has shown that Π_ψ is multiplicity-free in the p-adic case.

The complications for A-packets of classical groups include

- Π_ψ is not a singleton, in general.
- while
$$\widehat{G}^{\mathrm{aut}} = \bigcup_\psi \Pi_\psi,$$
these unions are not disjoint. So a representation $\pi \in \widehat{G}^{\mathrm{aut}}$ may have multiple types.

For example, a nongeneric supercuspidal rep. belongs to a tempered L-packet but may also belong to a nontempered A-packet, such as a Saito-Kurokawa A-packet for SO_5.

However, a result of Moeglin says that an unramified rep. in $\widehat{G}^{\mathrm{aut}}$ has a well-defined type.

4.3. **Questions.** Given $\pi \in \mathrm{Irr}(G)$ and $\sigma \in \mathrm{Irr}(H)$ of Arthur type, we would like to determine

$$m(\pi, \sigma) = \dim \mathrm{Hom}_H(\pi, \sigma).$$

We may consider this at various levels:

- For which A-parameters (ψ, ψ') is
$$m(\psi, \psi') = \dim \mathrm{Hom}_H(\Pi_\psi, \Pi_{\psi'}) \neq 0?$$

- If $m(\psi,\psi')$ is nonzero, for which (η,η') is
$$m(\pi_\eta,\sigma_{\eta'})\neq 0?$$
- Determine $m(\pi_\eta,\sigma_{\eta'})$ precisely.

Given the results in the GL setting, one might expect the following implication:
$$m(\psi,\psi')\neq 0 \Longrightarrow\ (\psi,\psi') \text{ is relevant.}$$
It turns out that this is not true.

4.4. **A Counterexample.** To give a counterexample to the above naive expectation, we will consider A-parameters ψ and ψ' which are trivial on W_F, so they are just representations of $\mathrm{SL}_2^D\times\mathrm{SL}_2^A$. For any subset $J\subset\{1,2,3,...,n\}$, set
$$\psi_J=\bigoplus_{i\notin J}[2i]\otimes[1]\oplus\bigoplus_{j\in J}[1]\otimes[2j].$$
This is an A-parameter for some SO_{2N+1}. We will take
$$\psi'=([1]+[3]+...+[2n-1])\otimes[1].$$
so that ψ' is a discrete series L-parameter for some SO_{2M}.

We shall show that there is a supercuspidal representation
$$\pi\in\bigcap_J\Pi_{\psi_J},\quad\text{such that}\quad m(\pi,\psi')=1.$$
Hence, $m(\psi_J.\psi')\neq 0$ for any J, but
$$(\psi_J,\psi')\text{ is relevant}\Longleftrightarrow J=\emptyset\text{ or }\{1\}.$$

How does one show the existence of π? Let $\Delta:\mathrm{SL}_2^D\to\mathrm{SL}_2^D\times\mathrm{SL}_2^A$ be the diagonal map and set $\psi_J^\Delta=\psi_J\circ\Delta$. Observe:
$$\psi_J^\Delta=\rho:=\bigoplus_{j=1}^n[2j]\quad\text{(a discrete series L-parameter indept. of } J)$$
We now use two results of Moeglin:

- The L-packet of Ψ_ρ has a unqiue supercuspidal member π.
- π lies in any A-packet Π_ψ for which $\psi^\Delta=\rho$.

This gives the desired supercuspidal $\pi\in\bigcap_J\Pi_{\psi_J}$. To show $m(\pi,\psi')=1$, one applies the GP conjecture (proved by Waldspurger) to the tempered parameters (ρ,ψ').

4.5. **The Conjecture for L-packets of Arthur Type.** In view of this beautiful but unfortunate counterexample, what can we expect in the case of classical groups? Given A-parameter (ψ,ψ'), consider its associated L-packet $\Pi^L_{\phi_\psi}\times\Pi^L_{\phi_{\psi'}}$. Then we conjectured the following in [GGP2]:

Conjecture 4.1.

$$m(\phi_{\psi}, \phi_{\psi'}) \neq 0 \Longleftrightarrow \ (\psi, \psi') \textit{ is relevant,}$$

in which case

$$m(\phi_{\psi}, \phi_{\psi'}) = 1$$

Moreover, the unique representation in the L-packet with nonzero contribution is given as a character of the component group by the same recipe as in the generic case of [GGP1].

Thus, this conjecture may be viewed as the natural extension of Gross-Prasad conjecture from the class of tempered L-packets to the larger class of L-packets of Arthur type.

4.6. **The Conjecture for A-packets.** What can we expect for the whole A-packet? We believe:

- If ψ and ψ' are both trivial on SL_2^D, then
$$m(\psi, \psi') \neq 0 \Longleftrightarrow \ (\psi, \psi') \text{ is relevant.}$$
- If π and σ are reps of Arthur type, then $m(\pi, \sigma) \neq 0$ implies
$$\text{there exists relevant } (\psi, \psi') \text{ such that } \pi \in \Pi_{\psi} \text{ and } \sigma \in \Pi_{\psi'}$$

Observe that this conjecture is still far from answering the list of questions we listed earlier. So much work remains to be done fora full understanding!

4.7. **Example: Automorphic Descent.** We conclude this note with an explanation of how the theory of automorphic descent (by Ginzburg-Rallis-Soudry) and its extension (twisted automorphic descent by Jiang-Zhang [JZ]) is explained by our conjectures.

Start with an elliptic global L-parameter for, say SO_{2n+1}:

$$\Pi = \oplus_i \Pi_i \quad \text{with } L(1, \Pi_i, \wedge^2) = \infty \text{ for each } i.$$

The goal of automorphic descent is to produce the generic cuspidal representation in the associated L-packet, and more generally the whole L-packet. For this, one may think of Π as a cuspidal representation of $\prod_i \mathrm{GL}_{n_i}$, which is a Levi factor M of a parabolic subgroup of SO_{4n}. From this, one constructs the descent as follows:

- consider iterated residue at $(1/2, ..., 1/2)$ of the Eisenstein series of SO_{4n} associated to the induced rep Π on M. The residue is a square-integrable automorphic form $\mathrm{Res}(\Pi)$ with A-parameters
$$\Pi \otimes [2] = \bigoplus_i \Pi_i \otimes [2].$$
- Consider the Bessel descent to SO_{2n+1} and ask: for which elliptic A-parameter Σ of SO_{2n+1} is
$$\mathrm{Bessel}(\mathrm{Res}(\Pi), \Sigma) \neq 0?$$

The global analog of our conjecture says that the A-parameters of $\mathrm{Res}(\Pi)$ and Σ should be relevant for this nonvanishing to happen. This can hold only if $\Sigma = \Pi$ and explains why automorphic descent produces elements in the L-packet associated to Π.

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