THE LANGLANDS-RAPOPORT CONJECTURE

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Abstract. We give a brief introduction to the Langlands-Rapoport conjecture, which describes the mod p points of Shimura varieties. We overview known results and explain what is missing to deal with the general case.

1. Hasse-Weil zeta functions of smooth projective varieties

Let E be a number field and let X be a smooth projective variety over E. The Hasse-Weil zeta function of X is (see [26]), at least conjecturally, a meromorphic function $\zeta_X(s)$ on the complex numbers which encodes deep global arithmetic information about X. For example the conjecture of Birch and Swinnerton-Dyer predicts for an elliptic curve X over $\mathbb{Q}$ that its zeta function knows the Mordell-Weil rank of X. The zeta function is defined as an Euler product over all primes $\mathfrak{p}$ of E

$$\zeta_X(s) = \prod_{\mathfrak{p}} Z_{X,\mathfrak{p}}(s).$$

of local Zeta functions $Z_{X,\mathfrak{p}}(s)$. For primes $\mathfrak{p}$ where X has good reduction, the local zeta function $Z_{X,\mathfrak{p}}(s)$ encodes information about the number of points of the special fiber $X_{\mathfrak{p}}$ over finite fields. The local zeta functions at places of bad reduction are harder to define, and it is not always clear what arithmetic information they encode, except in special cases.

Let S be the set of places of bad reduction. The partial product

$$\zeta_{X,S}(s) = \prod_{\mathfrak{p} \notin S} Z_{X,\mathfrak{p}}(s)$$

converges absolutely for $\mathrm{Re}(s) > 1 + d$, where d is the dimension of X (see Section 1.2 of [26]) and defines a holomorphic function. Proving this absolute convergence relies on the Hasse-Weil bounds for the number of points of X over finite fields. It now makes sense to conjecture that $\zeta_{X,S}(s)$ has analytic continuation to a meromorphic function on $\mathbb{C}$, which is one half of the Hasse-Weil conjecture. It will follow that $\zeta_X(s)$ has meromorphic continuation to $\mathbb{C}$, because the local Euler factors $Z_{X,\mathfrak{p}}(s)$ for places $\mathfrak{p}$ of bad reduction are meromorphic functions by construction.

The other half of the Hasse-Weil conjecture is a functional equation for $\zeta_X(s)$. Just as with the functional equation for the Riemann zeta function, this is best stated in terms of a completed zeta function. Define

$$\xi_X(s) = A^{s/2} \zeta_X(s) \cdot \prod_{v \in \Sigma_E^{\infty}} \Gamma_{X,v}(s).$$

Here $A \in \mathbb{Q}_{>0}$ is the conductor of X (see Section 4.1 of [26]), the symbol Σ_E^{∞} denotes the infinite places of E and $\Gamma_{X,v}(s)$ denotes the Gamma-factor of X at an infinite place (see Section 3 of [26]). We can now state a formal conjecture, in which we will implicitly assume that X is equidimensional.

Conjecture 1.0.1. *The function $\xi_X(s)$ has meromorphic continutation to all of $\mathbb{C}$ and satisfies*

$$\xi_X(s) = \pm\xi_X(d+1-s),$$

where d is the dimension of X.

This conjecture is completely open in general, and most of the known results all follow the same strategy: Show that $\zeta(X,s)$ is equal to a product of "automorphic L-functions", and prove that these automorphic L-functions have meromorphic continuation and a functional equation. Unfortunately, it is far beyond the scope of this article to define automorphic representations and automorphic L-functions, so we'll settle for an example:

Example 1.0.2. Let X be the elliptic curve over $\mathbb{Q}$ defined by the equation $y^2+y=x^3-x^2$, then the Hasse-Weil zeta function of X is equal to

$$\zeta_X(s) = \frac{\zeta(s)\zeta(s-1)}{\mathcal{L}_f(s)},$$

where $\zeta(s) = \zeta_{\mathrm{Spec}\,\mathbb{Q}}(s)$ is the usual Riemann-zeta function, and $\mathcal{L}_{f,s}$ is the L-function of the modular form

$$f(z) = \prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11})^2 \in S_2[\Gamma_0(11)].$$

The Hasse-Weil conjecture for $\zeta_X(s)$ now follows from the functional equation and meromorphic continuation for $\zeta(s)$ and $\mathcal{L}_{f,s}$.

For a general variety X the approach sketched above seems hopeless, because it is not clear 'where' the automorphic representations should come from. This is different for Shimura varieties, because automorphic representations are closely related to Shimura varieties. For example it follows from work of Eichler [3] and Shimura [28] that the Hasse-Weil zeta functions of modular curves are products of L-functions of modular forms. However, their approach does not easily generalise to higher-dimensional Shimura varieties.

In [17], Langlands outlines a three-part approach to prove that the Hasse-Weil zeta functions of Shimura varieties are related to L-functions of automorphic representations. The first and third part are 'a matter of harmonic analysis', we refer the reader to [31] for an introduction. The second part is about describing the mod p points of suitable (smooth) integral models of Shimura varieties. His original strategy is only suitable for studying the local zeta functions at places of good reduction, but it was generalised by Rapoport and Kottwitz to include places of parahoric[1] bad reduction [22].

Disregarding the very interesting and very complicated harmonic analysis that will no doubt have to be used, computing the (semisimple) [2] local zeta functions of Shimura varieties at primes of parahoric bad reduction requires two ingredients: The first is constructing reasonable integral models and describing their singularities, or rather computing the (semisimple) trace of Frobrenius on the sheaf of nearby cycles. The integral models were constructed by Kisin-Pappas [11] and the recent work of Haines-Richarz [5] solves the problem of understanding the nearby cycles. The second ingredient is describing the mod p-points of these Kisin-Pappas integral models,

[1] This means that the level at p is a parahoric subgroup.

[2] The semisimple local zeta function is a variant of the local zeta function defined in [2] from which the usual local zeta function can be recovered, if one assumes the weight-monodromy conjecture.

which is done in [?Zhou] and in my PhD thesis [8]. A conjectural description of the mod p points of (conjectural) integral models of Shimura varieties was first given by Langlands in [16] and was later refined by Langlands-Rapoport [18] and by Rapoport [22] to include the case of parahoric bad reduction.

2. The Langlands-Rapoport conjecture

The Langlands-Rapoport conjecture gives a conjectural description of the $\overline{\mathbb{F}}_p$-points of conjectural integral models of a Shimura variety associated to a Shimura datum (G, X). Stating the conjecture is technically quite involved and we will postpone that to Chapter **??**. The goal of this section is to show that beneath all the technicalities lies a beautiful motivic story. We will start by discussing mod p points on the modular curve.

2.1. **The modular curve.** There are great and detailed explanations of Langlands-Rapoport for the modular curve elsewhere (e.g. [23]), here we will only give a basic overview. Consider the tower of schemes

$$\{Y_N/\mathbb{Z}_{(p)}\}_N$$

where N runs over positive integers coprime to p ordered by divisibility and Y_N is the moduli space of elliptic curves E together with an isomorphism of group schemes $E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$. This is most easily done by considering all N at once, or working with the inverse limit. Define

$$Y(\overline{\mathbb{F}}_p) := \varprojlim_{(N,p)=1} Y_N(\overline{\mathbb{F}}_p),$$

which has a natural action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ that extends to an action of $\mathrm{GL}_2(\mathbb{A}_f^p)$ via Hecke correspondences. We want a 'group theoretic' description of $Y(\overline{\mathbb{F}}_p)$, which takes this action into account. We will give a description in two steps:

(1) Divide elliptic curves into isogeny classes and classify them (Honda-Tate theory).

(2) Count elliptic curves inside a fixed isogeny class.

2.1.1. *The structure of isogeny classes.* If we fix an elliptic curve $E_0/\overline{\mathbb{F}}_p$, then its isogeny class $\mathscr{I}_\phi \subset Y(\overline{\mathbb{F}}_p)$ has a description in terms of linear- and semi-linear algebra. Let V^pE_0 be the rational prime-to-p adic Tate module of E_0, in other words it is

$$\left(\prod_{\ell \neq p} T_\ell E_0\right) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $T_\ell E_0 = \varprojlim_m E_0[\ell^m](\overline{\mathbb{F}}_p)$. Since each $T_\ell E_0$ is a free $\mathbb{Z}_\ell$ module of rank two, we see that V^pE_0 is a free module of rank two over the ring

$$\mathbb{A}_f^p := \left(\prod_{\ell \neq p} \mathbb{Z}_\ell\right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let V_pE_0 be $T_pE_0[1/p]$, where T_pE_0 is the covariant Dieudonné module of E_0. In short, T_pE_0 is a free $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$-module Λ of rank two equipped with a Frobenius semi-linear map $F : \Lambda \to \Lambda$ satisfying $p\Lambda \subset F\Lambda \subset \Lambda$. An isogeny $f : E_0 \to E$ induces bijections $V^pE_0 \simeq V_pE$ and $V_pE_0 \simeq$

V_pE, but the lattices inside will be different; in fact f will be determined by the induced lattices in V^pE_0 and V_pE_0. We define

$$X^p(\phi) = \{\hat{\mathbb{Z}}^p\text{-lattices } \Lambda^p \subset V^pE_0 \text{ together with an isomorphism } \Lambda^p \simeq (\hat{\mathbb{Z}}^p)^{\oplus 2}\}$$
$$X_p(\phi) = \{\text{Dieudonné-lattices in } V_pE_0\}.$$

Then $X^p(\phi)$ is a $\mathrm{GL}_2(\mathbb{A}_f^p)$-torsor and there is a $\mathrm{GL}_2(\mathbb{A}_f^p)$-equivariant map

$$X^p(\phi) \times X_p(\phi) \to \mathscr{I}_\phi,$$

sending a pair of lattices (Λ^p, Λ_p) corresponding to an isogeny $f : E_0 \to E$ to the elliptic curve E together with its trivialisation. This induces an isomorphism

$$\mathscr{I}_\phi \simeq I_\phi(\mathbb{Q})\backslash \left(X^p(\phi) \times X_p(\phi)\right),$$

where $I_\phi(\mathbb{Q})$ is the set of self quasi-isogenies of E_0.

2.1.2. *Classification of isogeny classes.* Classical Honda-Tate theory describes isogeny classes of abelian varieties over $\mathbb{F}_q$ in terms of q-Weil numbers. Equivalently, we can describe isogeny classes of elliptic curves E by the characteristic polynomial of Frob_q acting on the ℓ-adic Tate module of E for some $\ell \neq p$ (this is an element of $\mathbb{Z}[X]$ independent of ℓ). This gives us a (semisimple) conjugacy class of matrices in $\mathrm{GL}_2(\mathbb{Q})$ and its stabiliser is an inner form of the group I_ϕ of self quasi-isogenies of E. Another perspective is that the isogeny class of E determines a two-dimensional pure motive (say with numerical equivalence) over $\mathbb{F}_q$, or a motive with GL_2-structure.

2.1.3. *Conclusion.* In conclusion, we see that the mod p points on the modular curve can be described as

$$\varprojlim_{(N,p)=1} Y_N(\overline{\mathbb{F}}_p) \simeq \coprod_\phi I_\phi(\mathbb{Q}\backslash X_p(\phi) \times X^p(\phi), \tag{2.1.1}$$

equivariant for the action of $\mathrm{GL}_2(\mathbb{A}_f^p)$, where ϕ ranges over the set of isogeny classes of elliptic curves over $\overline{\mathbb{F}}_p$. Moreover it turns out that the action of Frobenius on the left hand side corresponds to the action of a certain operator Φ on $X_p(\phi)$. The Langlands-Rapoport conjecture for a general Shimura variety has the same shape as (2.1.1).

2.2. **The Langlands-Rapoport conjecture in general.** Let (G, X) be a Shimura datum, let p be a prime number, let $U_p \subset G(\mathbb{Q}_p)$ be a parahoric subgroup. Consider the tower of Shimura varieties $\{\mathbf{Sh}_{G,U^pU_p}\}_{U^p}$ over the reflex field E with its action of $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$, where U^p varies over compact open subgroups of $G(\mathbb{A}_f^p)$ and where Z_G is the center of the algebraic group G. Then we conjecture that this tower has a $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$-equivariant extension to a tower of flat (normal) schemes $\{\mathscr{S}_{G,U^pU_p}\}_{U^p}$ over $\mathcal{O}_{E_{(v)}}$. When U_p is hyperspecial, the integral model should be smooth and satisfy a certain extension property, which determines it uniquely if it exists (c.f. [20]). Recent work [21] of Pappas defines a notion of canonical integral models when U_p is an arbitrary parahoric and (G, X) is of Hodge type, and proves that they are unique if they exist. Moreover, there should be a bijection

$$\varprojlim_{U^p} \mathscr{S}_{U^pU_p}(\overline{\mathbb{F}}_p) \simeq \varprojlim_{U^p} \coprod_{[\phi]} I_\phi(\mathbb{Q})\backslash X^p(\phi) \times X_{p,U_p}(\phi)/U^p$$

compatible with the action of $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$. Here $X^p(\phi)$ is a $G(\mathbb{A}_f^p)$-torsor as before, and we are left to explain the indexing set ϕ, the sets $X_p(\phi)$ and the groups $I_\phi(\mathbb{Q})$. The indexing set should be a generalisation of the notion of isogeny class; we'll explore this in the next section.

2.3. **Mod p isogeny classes on general Shimura varieties.** We would like to say that the mod p isogeny classes on the special fiber of a Shimura variety associated to a Shimura datum (G, X) are isogeny classes of "abelian varieties with G-structure" or "motives with (G, X)-structure". There are various ways of making this precise, the simplest one works only for Shimura varieties of Hodge type.

Suppose that (G, X) is of Hodge type and let $i : (G, X) \hookrightarrow (\mathrm{GSp}_{2g}, S^{\pm})$ be a Hodge embedding. Then an abelian variety with G structure over $\overline{\mathbb{F}}_p$ is a g-dimensional abelian variety A together with a finite collection of tensors

$$\{s_{\alpha,\ell}\}_{\alpha \in C} \in V_\ell(A)^{\otimes}$$

for all ℓ (including $\ell = p$) such that the stabiliser of the tensors in $\mathrm{GL}\, V_\ell(A)$ is given by $G_{\mathbb{Q}_\ell}$ (note that the indexing set C is independent of ℓ). Here $V_\ell(A)^{\otimes}$ is the direct sum of all modules obtained from the rational ℓ-adic Tate-module (or rational Dieudonné-module if $\ell = p$) $V_\ell(A)^{\otimes}$ using the operations of duals, tensor products, symmetric powers and exterior powers. This is the kind of "abelian variety with G-structure" that one actually gets from a $\overline{\mathbb{F}}_p$-point on the special fiber of the Kisin-Pappas integral models of Hodge type Shimura varieties. In fact there will a finite field $\mathbb{F}_q$ such that the abelian variety is defined over $\mathbb{F}_q$ and such that the tensors are Galois invariant.

This notion of "abelian variety with G-structure" is not well behaved because we are not asking for any compatibility between tensors for different ℓ. Indeed, the Tate conjecture for motives over finite fields predicts that our tensors come from algebraic cycles and we would obviously like to say that the $s_{\alpha,\ell}$ are the ℓ-adic realisations the same cycle s_α. Another issue is that we would like to get rid of the choice of Hodge embedding.

2.3.1. *Motives.* Let $\mathcal{C}_q$ be the category of motives with numerical equivalence over $\mathbb{F}_q$, see [25] for an introduction. A priory this is just a pseudo-abelian category with a tensor product, but it follows from [9] that this is actually a semisimple abelian category. Moreover it follows from [10] that it is a semisimple Tannakian category, see Section 1 of [19]. If we assume the Tate conjecture in the form of Conjecture 1.14 of [19], then for $\ell \neq p$ there is a fully faithful tensor functor

$$\mathcal{C}_q \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \mathbb{V}_\ell(\mathbb{F}_q)$$

to the category of semisimple continuous representations of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$ on finite dimensional vector spaces over $\mathbb{Q}_\ell$, given by ℓ-adic étale cohomology. Similarly if we assume the crystalline version of the Tate conjecture, then there are fully faithful tensor functors

$$\mathcal{C}_q \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{V}_p(\mathbb{F}_q)$$

to the category of F-isocrystals over $W(\mathbb{F}_q)[1/p]$. When we pass to the category $\mathcal{C}$ of motives over $\overline{\mathbb{F}}_p$, we get fully faithful tensor functors

$$\omega_\ell : \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \mathbb{V}_\ell(\overline{\mathbb{F}}_p)$$
$$\omega p : \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{V}_p(\overline{\mathbb{F}}_p),$$

where $\mathbb{V}_p(\overline{\mathbb{F}}_p)$ is the category of isocrystals over $W(\overline{\mathbb{F}}_p)[1/p]$ and the category $\mathbb{V}_\ell(\overline{\mathbb{F}}_p)$ is the category of "germs of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-representations". Its objects are equivalence classes of Galois representations of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$ for some n, with equivalence given by $\rho \sim \rho'$ if there is some open subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on which they agree. The morphisms are given by

$$\hom(\rho,\rho') = \varinjlim_n \hom_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^n)}(\rho,\rho').$$

2.3.2. *Motives with G-structure.* If $G/\mathbb{Q}$ is an algebraic group then a motive with G-structure is an exact tensor-functor

$$\alpha : \mathrm{Rep}_{\mathbb{Q}}(G) \to \mathcal{C}$$

such that for all ℓ the following diagram commutes

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_\ell}(G) & \xrightarrow{\alpha_\ell} & \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \\ \downarrow & & \downarrow \omega_\ell \\ \mathrm{Vect}_{\mathbb{Q}_\ell} & = & \mathrm{Vect}_{\mathbb{Q}_\ell} . \end{array}$$

In other words, for each representation V of G we get a motive $\alpha(V)$ such that the ℓ-adic étale cohomology of $\alpha(V)$ is of dimension equal to $\mathrm{Dim}\, V$, and the same for the crystalline cohomology.

Associated to a motive with G-structure is the composition

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{V}_p(\overline{\mathbb{F}}_p).$$

This is an isocrystal with G-structure in the sense of Kottwitz [14], and these are classified up to isomorphism by the set $B(G)$. If we are also given a Shimura datum (G, X), then it makes sense to ask that the element of $B(G)$ we obtain is admissible with respect to this Shimura datum, in other words, to ask that it lies in $B(G, X) \subset B(G)$. Let us call a motive with G-structure *admissible* (with respect to (G, X)) if this is the case. When (G, X) is the Siegel Shimura datum, this comes down to asking that the isocrystal (with alternating form) comes from a p-divisible group (with a polarisation).

2.3.3. *Circumventing the Tate conjecture.* Assuming the Tate conjecture, Milne [19] gives an explicit description of the category $\mathcal{C}$ with its tensor structure. In fact this description is so explicit that it is possible to write down a Tannakian category $\tilde{\mathcal{C}}$ together with faithful tensor functors

$$\tilde{\omega}_\ell : \tilde{\mathcal{C}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to V_\ell(\overline{\mathbb{F}}_p)$$

for all ℓ *without* assuming the Tate conjecture. If the Tate conjecture does hold, then there is an equivalence of categories $\mathcal{C} \simeq \tilde{\mathcal{C}}$, compatible with all the tensor functors. The category $\tilde{\mathcal{C}}$ is exactly the category of representations of the pseudo-motivic groupoid $\mathfrak{P}$ introduced in Chapter **??**. Let us call $\tilde{\mathcal{C}}$ the category of *fake motives* over $\overline{\mathbb{F}}_p$; the notion of fake motive with G-structure and admissible fake motive with G-structure is now obvious.

2.3.4. *Back to the Langlands-Rapoport conjecture.* Let (G, X) be a Shimura datum such that Z_G^0 satisfies the Serre condition, i.e., such that Z_G^0 is isogenous to a product $T_1 \times T_2$ where $T_1/\mathbb{Q}$ is a split torus and where T_2 is a torus with $T_2(\mathbb{R})$ compact. This condition automatically holds when (G, X) is of Hodge type because then $Z_G^0/w(\mathbb{G}_m)(\mathbb{R})$ is compact, where $w : \mathbb{G}_m \to G$ is the weight homomorphism obtained from X.

Under these assumptions, the indexing set of the Langlands-Rapoport conjecture is closely related to the set of equivalence classes of admissible fake motives with G-structure. In fact there are precisely two extra conditions we have to put on an admissible fake motive with G-structure in order for it to give rise to an admissible morphism (which are the objects that index the isogeny classes). To explain these, let us furthermore assume that G^{der} is simply connected.

- The first condition has to do with the induced motive with G^{ab}-structure. To be precise, it should agree with the one coming via "reduction modulo p" of the CM motive associated to the CM torus $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$, see Section 4 of [19]).
- One condition at infinity, having to do with fully faithful tensor functors (the first is constructed by Milne assuming the Tate conjecture)

$$\omega_\infty : \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{R} \to V_\infty(\overline{\mathbb{F}}_p)$$
$$\tilde{\omega}_\infty : \tilde{\mathcal{C}} \otimes_{\mathbb{Q}} \mathbb{R} \to V_\infty(\overline{\mathbb{F}}_p).$$

Here $V_\infty(\overline{\mathbb{F}}_p)$ is the $\mathbb{R}$-linear Tannakian category of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces equipped with a τ-linear map F respecting the grading such that F^2 acts as $(-1)^m$ on the m-th graded piece, and τ is the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

For a general Shimura datum (G, X), we have to replace the pseudo-motivic Galois gerb $\mathfrak{P}$ with the quasi-motivic Galois gerb $\mathfrak{Q}$, which comes equipped with a map $\mathfrak{Q} \to \mathfrak{P}$. In other words we are replacing the category $\tilde{\mathcal{C}}$ with a category $\mathcal{D}$, which comes with a natural functor $\tilde{\mathcal{C}} \to \mathcal{D}$.

Remark 2.3.5. The perspective in Chapter **??** is in terms of Galois gerbs and morphisms of Galois gerbs rather than Tannakian categories and functors between them. The reason for this shift is that Kisin's paper [12] is written entirely in the former perspective. Since our proofs are merely generalisations of his to ramified groups, we often refer to his work for certain details and arguments, and its therefore natural to adopt his notation and perspective.

2.4. **Affine Deligne-Lusztig varieties.** In order to generalise the sets $X_p(\phi)$, we recall that in the case of the modular curve the set $X_p(\phi)$ is a subset of the space of all "$\breve{\mathbb{Z}}_p$-lattices" in V_pE_0, which can be identified with

$$\mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p).$$

In other words, the set of Dieudonné lattices in $V_PE_0 \simeq \breve{\mathbb{Q}}_p^{\oplus 2}$ sits inside the set of all lattices in $\breve{\mathbb{Q}}_p^{\oplus 2}$. The condition that a lattice $\Lambda \subset \breve{\mathbb{Q}}_p^{\oplus 2}$ is a Dieudonné lattice is the condition that

$$p\Lambda \subset F\sigma^*\Lambda \subset \Lambda,$$

here $F : \sigma^*\breve{\mathbb{Q}}_p^{\oplus 2} \simeq \breve{\mathbb{Q}}_p^{\oplus 2}$ is the map coming from the fact that V_pE_0 is an isocrystal. Let us write $F = b \otimes \sigma$ as a σ-linear map, with $b \in \mathrm{GL}_2(\breve{\mathbb{Q}}_p)$. Then a lattice Λ is a Dieudonné-lattice if under the relative position map

$$\mathrm{Inv} : \mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p) \times \mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p) \to \Big(\mathrm{GL}_2(\breve{\mathbb{Z}}_p)\backslash\,\mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p)\Big) \simeq \mathbb{Z}^2/S_2$$

the image $\mathrm{Inv}(\Lambda, b\Lambda) = (1, 0)$. If we think of $\mathbb{Z}^2/S_2$ as the set of conjugacy classes of cocharacters of GL_2, then the element $(1, 0)$ corresponds precisely to the inverse of the Hodge cocharacter associated to the Shimura datum of the modular curve.

It is important to note that $X_p(\phi)$, unlike $X^p(\phi)$ depends on the isogeny class of E_0, or rather it depends on the isogeny class of $E_0[p^\infty]$ or equivalently on the σ-conjugacy class of b. For a connected reductive group $G/\mathbb{Q}_p$ and choice of parahoric $\mathcal{G}$, we consider

$$G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p),$$

which we think of as the space of '$\mathcal{G}$-lattices' inside the standard G-isocrystal given by $b \in B(G, X)$. To define the affine Deligne-Lusztig variety we again have a relative position map (see Section **??**)

$$\mathrm{Rel} : G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \times G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \to \Big(\mathcal{G}(\breve{\mathbb{Z}}_p)\backslash G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)\Big) \simeq W_{\mathcal{G}}\backslash \tilde{W}/W_{\mathcal{G}},$$

and $X_p(\phi)$ will be a subset of $G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$ defined by a condition on $\mathrm{Rel}(g, bg)$. It remains for us to define this condition, which will take the form of a finite set, called the admissible set

$$\mathrm{Adm}(\mu)_{\mathcal{G}} \subset \mathcal{G}(\breve{\mathbb{Z}}_p)\backslash G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \simeq W_{\mathcal{G}}\backslash \tilde{W}/W_{\mathcal{G}}$$

When $\mathcal{G}$ is hyperspecial then $W_{\mathcal{G}}\backslash \tilde{W}/W_{\mathcal{G}}$ is just the set of conjugacy classes of cocharacters of G, and the admissible set will consists of a single element corresponding to the inverse of the Hodge cocharacter associated to the Shimura datum. When $\mathcal{G}$ is a general parahoric subgroup, the admissible set will have more than one element and we will define it in Section **??**. This definition is motivated by considerations from local harmonic analysis, see [4] for an introduction.

3. Known- and unknown results

3.1. Known results. Kottwitz describes the mod p points of Shimura varieties of PEL type A and C, at primes $p > 2$ of hyperspecial good reduction in [15]. His description of the isogeny classes is essentially the same, but his classification of the isogeny classes takes a slightly different form than the one in the Langlands-Rapoport conjecture.

Kisin [12] proves a slightly weaker version of the Langlands-Rapoport conjecture for abelian type Shimura varieties under the assumption that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension, and that U_p is hyperspecial. An important idea in his proof is to show that both admissible morphisms and isogeny classes 'come from special points'. He deduces the former from Satz 5.3 of [18] and the latter is deduced, after a lengthy dévissage from the abelian type to the Hodge type case, from uniformisation of isogeny classes (we'll discuss his strategy for proving uniformisation of isogeny classes when we discuss our own proof strategy).

In the parahoric case, uniformisation of isogeny classes was proven by Zhou in [30], under the assumption that $G_{\mathbb{Q}_p}$ is residually split. We remind the reader that split implies residually split implies quasi-split and that residually split + unramified implies split.

In my PhD thesis [8], we prove the Langlands-Rapoport conjecture for Shimura varieties of abelian type under the assumption that $G_{\mathbb{Q}_p}$ is quasi-split and unramified. We can also deal with quasi-split and tamely ramified groups, as long as the group is not of type A and the Shimura varieties in question are proper, see Theorem 5.4.1 of op. cit. for the precise technical assumptions.

There is also important work of Reimann [24], which not only proves the Langlands-Rapoport conjecture for certain quaternionic Shimura varieties but also corrects the definition of the quasi-motivic Galois gerb given by Langlands-Rapoport.

3.2. The methods of Kisin and Zhou. Both Kisin and Zhou employ the proof strategy: They first prove uniformisation of isogeny classes and then use that to prove that every isogeny class contains the reduction of a special point. In the quasi-split case, every admissible morphism also comes from a special point of the Shimura variety and from there it is not too complicated to show that isogeny classes are in bijection with admissible morphisms up to conjugacy.

To prove uniformisation of isogeny classes, both Kisin and Zhou argue as follows: The integral models $\mathscr{S}_G$ of Hodge type Shimura varieties come equipped, by construction, with finite maps $\mathscr{S}_G \to \mathscr{S}_{\mathrm{GSp}}$ to Siegel modular varieties. Given a point $x \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$, classical Dieudonné theory produces a map

$$X_p(\phi) \to \mathscr{S}_{\mathrm{GSp}}(\overline{\mathbb{F}}_p)$$

and it suffices to show that it factors through $\mathscr{S}_G$. A deformation theoretic result shows that it suffices to prove this result for one point on each connected component of $X_p(\phi)$, and therefore we need to understand these connected components. In the hyperspecial case, this is done in [1], and in the parahoric case this is done in [7], under the assumption that $G_{\mathbb{Q}_p}$ is residually split. The main obstruction to extend Zhou's methods beyond the residually split case, is that we do not understand connected components of affine Deligne-Lusztig varieties for more general groups.

3.3. **The methods of the author's PhD thesis.** In my PhD thesis [8], we prove cases of the LR conjecture (unramified groups) but we do not address connected components of affine Deligne-Lusztig varieties. Instead, we prove the theorem at parahoric level by reducing to the case of a hyperspecial parahoric, where we can use Kisin's result. In the general quasi-split case we reduce to the case of a very special parahoric, which is dealt with by Rong Zhou's results ([30] and Appendix A of [8]. Our argument makes crucial use of moduli spaces of mixed characteristic shtukas (see [27, 29]) and the incarnation of special fibers of local models as subvarieties of *mixed characteristic* affine Grassmannians (see [6]).

3.4. **The non quasi-split case.** Almost nothing is known about the Langlands-Rapoport conjecture when the group $G_{\mathbb{Q}_p}$ is not quasi-split except in some PEL cases (e.g. [2]). The techniques described above are not sufficient to handle the non quasi-split case, for the following reasons:

- We do not understand the connected components of affine Deligne-Lusztig varieties for non quasi-split groups. We cannot circumvent this by reducing to when the parahoric is very special, because very special parahorics do not exist for non quasi-split groups.
- Even if we could prove uniformisation of isogeny classes, it is *not* true that every isogeny class contains the reduction of a special point. This means we have no way of proving that the special fiber of our Shimura variety contains 'enough' isogeny classes.

However, in a recent paper [13] Kisin and Zhou manage to construct Kottwitz triples associated to isogeny classes in the special fibers of Shimura varieties. They do this by a clever "reduction to the quasi-split case" argument. I've tried to use the same argument to prove the Langlands-Rapoport conjecture for non quasi-split groups, but it doesn't seem to be possible.

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