MODULAR FORMS ON EXCEPTIONAL GROUPS

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Abstract. The purpose of this note is to give an update about some of our results on the so-called modular forms on quaternionic exceptional groups.

1. Introduction

Suppose G is a reductive adjoint exceptional group over $\mathbf{Q}$, with $G(\mathbf{R})$ quaternionic. This means that $G(\mathbf{R})$ is one of the adjoint groups $G_{2,2}, D_{4,4}, F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}$, where the notation means that in case $G(\mathbf{R})$ is of type G_2 then it is of rank two, and otherwise it is of rank four. These groups possess special cohomological automorphic forms, called the *quaternionic modular forms*. They were singled out for study by Gross-Wallach [GW96] and Gan-Gross-Savin [GGS02].

One feature of quaternionic modular forms is that they possess a robust Fourier expansion and Fourier coefficients, similar to the holomorphic Siegel modular forms; this is the main result of [Pol20a], which follows an earlier result of Wallach [Wal03] in the same direction. With this semi-classical notion of Fourier expansion in hand, one can ask about the Fourier coefficients of some special quaternionic modular forms. Below we describe this Fourier expansion, and some results about the Fourier coefficients of special modular forms, from our papers [Pol20b, Pol21b, Pol20c]. These papers include results about

(1) [Pol20b]: The minimial modular form on E_8, and a distinguished modular form on E_6;
(2) [Pol20c]: The next-to-minimal modular form on E_8;
(3) [Pol21b]: Some cuspidal modular forms on G_2.

2. The quaternionic exceptional groups

We begin by describing the internal structure of the quaternionic exceptional groups. A reference for this section is [Pol20a, sections 2,3,4] and the course notes [Pol21a].

Assume J is rational cubic norm structure, with $J \otimes \mathbf{R}$ having a positive-definite trace form. We assume the reader is familiar with these notions. For details, see the above references. In particular, it follows that $J \otimes \mathbf{R}$ is one of the following:

(1) $\mathbf{R}$;
(2) $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$;
(3) $H_3(C)$, the Hermitian 3×3 matrices with values in the positive composition algebra C, where $C = \mathbf{R}, \mathbf{C}, \mathbf{H}, \Theta$ is the real numbers, the complex numbers, Hamilton's quaternions, or the octonions with positive-definite norm form;
(4) $\mathbf{R} \times V$, where V is a quadratic space of signature $(1, n-3)$.

Out of J one can create a rational reductive adjoint group that has $G(\mathbf{R})$ quaternionic. We will briefly review this construction. It is exactly the positive-definiteness condition on $J \otimes \mathbf{R}$ that corresponds to the fact that $G(\mathbf{R})$ is quaternionic. The groups one obtains for $G(\mathbf{R})$ are $G_{2,2}, D_{4,4}, F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}$, and a group isogenous to $\mathrm{SO}(4, n)$ in the corresponding order for J from above. For example, from $J = \mathbf{R}$ one obtains a group of type G_2 and from $J = H_3(\mathbf{C})$ one obtains a group of type E_6.

Thus, suppose J is as above a rational cubic norm structure. Set $W_J = \mathbf{Q} \oplus J \oplus J^\vee \oplus \mathbf{Q}$; we put on W_J Freudenthal's symplectic form and quartic form. See [Pol20a, section 3] or [Pol21a, chapter 3] for this. Let H_J denote the group that preserves these forms up to similitude, and $\nu : H_J \to \mathrm{GL}_1$ this similitude. Thus if $g \in H_J$, $\langle \, , \, \rangle$ is Freudenthal's symplectic form and $q : W_J \to \mathbf{Q}$ is Freudenthal's quartic form, then $\langle gv, gw \rangle = \nu(g)\langle v, w\rangle$ and $q(gv) = \nu(g)^2 q(v)$ for all $v, w \in W_J$. Now, if H_J^1 is the subgroup of H_J with similitude equal to 1 and $\mathfrak{h}(J)^0$ its Lie algebra, then in fact $\mathfrak{g}(J) \simeq \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0 \oplus V_2 \otimes W_J$, and this is a $\mathbf{Z}/2\mathbf{Z}$-grading. The Lie bracket in terms of this grading can be found in [Pol20a, section 4] or [Pol21a, chapter 4].

The Lie algebra $\mathfrak{g}(J)$ also has a 5-step $\mathbf{Z}$-grading, which we will require. It is as follows. Let e, f the standard basis of V_2, and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathfrak{sl}_2$ the usual $\mathfrak{sl}_2$-triple. Then

- In degree 2: $\mathbf{Q}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$
- In degree 1: $e \otimes W_J$
- In degree 0: $\mathbf{Q}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \oplus \mathfrak{h}(J)^0$
- In degree -1: $f \otimes W_J$
- In degeee -2: $\mathbf{Q}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

We define G_J to be the rational reductive adjoint group associated to the Lie algebra $\mathfrak{g}(J)$. Let P_J be the parabolic subgroup of G_J whose Lie algebra consists of the elements in non-negative degrees in the above $\mathbf{Z}$-grading. As a Levi subgroup, one may take H_J, which has Lie algebra the elements in degree 0. The unipotent radical N_J of P_J has Lie algebra consisting of the elements of positive degree. We set $Z = [N_J, N_J]$ which is the subgroup of P_J whose Lie algebra consists of the elements of degree two. Note that N_J/Z can be identified with W_J via the exponential map.

The maximal compact subgroup K_J of $G_J(\mathbf{R})$ is isomorphic to $(\mathrm{SU}(2) \times L_J^0)/\mu_2$, where L_J^0 is a compact group whose Lie algebra is isomorphic to $\mathfrak{h}(J)^0$ over $\mathbf{C}$. We remark that the symmetric space $G_J(\mathbf{R})/K_J$ does not have complex structure.

3. The definition of quaternionic modular forms

We now give the definition of quaternionic modular forms. We will define them on the simply-connected quaternionic groups. Our definition is essentially a paraphrase of the one in [GGS02].

Thus suppose G'_J is the simply-connected $\mathbf{Q}$-group associated to the Lie algebra $\mathfrak{g}(J)$, where J is a cubic norm structure with $J \otimes \mathbf{R}$ having positive-definite trace form. Then $G'_J(\mathbf{R})$ is connected, and the maximal compact subgroup K'_J of $G'_J(\mathbf{R})$ has a surjection to $\mathrm{SU}(2)/\mu_2$ which we fix in [Pol20a].

For a positive integer ℓ, denote by V_ℓ the representation $Sym^{2\ell}(\mathbf{C}^2)$ of K'_J, so that the action factors through the surjection to $\mathrm{SU}(2)/\mu_2$.

Definition 1. A modular form for G'_J of weight ℓ is a moderate growth automorphic form φ on $G'_J(\mathbf{Q})\backslash G'_J(\mathbf{A})$, valued in V_ℓ, that satisfies the following properties:

(1) $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in G'_J(\mathbf{A})$ and all $k \in K'_J$;
(2) $D_\ell \varphi(g) \equiv 0$, for a certain linear differential operator D_ℓ that we are about to define.

To define the differential operator D_ℓ, we first recall the Cartan decomposition of $\mathfrak{g}(J) \otimes \mathbf{R}$. Over $\mathbf{C}$, we have $\mathfrak{g}(J) \otimes \mathbf{C} = \mathfrak{k}_J \oplus \mathfrak{p}_J$, with $\mathfrak{p}_J = V_2 \boxtimes W_J$ as a representation of K'_J. Note that $\mathfrak{p}_J \simeq \mathfrak{p}_J^\vee$ and $V_\ell \otimes \mathfrak{p}_J^\vee \simeq Sym^{2\ell+1}(V_2) \boxtimes W_J \oplus Sym^{2\ell-1}(V_2) \boxtimes W_J$. Denote by pr the K'_J-equivariant projection $V_\ell \otimes \mathfrak{p}_J^\vee \to Sym^{2\ell-1}(V_2) \boxtimes W_J$.

Now, let $\{X_\gamma\}_\gamma$ be a basis of $\mathfrak{p}_J$ and $\{X_\gamma^\vee\}_\gamma$ the dual basis of $\mathfrak{p}_J^\vee$. Define $\widetilde{D}_\ell \varphi$ as

$$\widetilde{D}_\ell(\varphi) = \sum_\gamma X_\gamma \varphi \otimes X_\gamma^\vee.$$

Here $X_\gamma\varphi$ denotes the right regular action; $\widetilde{D}_\ell(\varphi)$ is a function on $G'_J(\mathbf{A})$ valued in $V_\ell \otimes \mathfrak{p}_J^\vee$ and is independent of the choice of basis $\{X_\gamma\}_\gamma$. Finally, we define $D_\ell = \mathrm{pr} \circ \widetilde{D}_\ell$. Thus $D_\ell(\varphi)$ is, by its definition, valued in $Sym^{2\ell-1}(V_2) \boxtimes W_J$.

For representation-theoretic reasons, one should think of the annihilation equation $\mathcal{D}_\ell\varphi \equiv 0$ as an analogue of the Cauchy-Riemann equations to the (non-complex) symmetric space $X_J = G'_J(\mathbf{R})/K'_J$. That the modular forms satisfy this equation gives them their rigid structure and is the crucial part of the definition.

Following [GGS02], one can make an essentially equivalent definition in terms of discrete series representations of the Lie group $G'_J(\mathbf{R})$: For ℓ sufficiently large depending on G'_J, there is a discrete series representation π_ℓ of the Lie group $G'_J(\mathbf{R})$ whose minimal K'_J-type is V_ℓ. In this framework, a modular form of weight ℓ is a $(Lie(G'_J(\mathbf{R})), K'_J)$-module map $\pi_\ell \to \mathcal{A}(G'_J(\mathbf{A}))$ from π_ℓ to the space of automorhpic forms on $G'_J(\mathbf{A})$. In other words, one considers minimal K'_J-type vectors in automorphic representations $\pi = \otimes_v \pi_v$ on $G'_J(\mathbf{A})$, when the archimedean component π_∞ is the discrete series representation π_ℓ. In terms of Definition 1, the two conditions of that definition are another way of saying that the K'_J-type V_ℓ occurs in π and that it is the minimal K'_J-type at the archimedean place.

4. The Fourier expansion of quaternionic modular forms

We now describe the Fourier expansion of quaternionic modular forms, following [Pol20a].

Suppose $\chi : N_J(\mathbf{R}) \to \mathbf{C}^\times$ is a unitary character. One can consider all functions $F_\chi : G'_J(\mathbf{R}) \to V_\ell$ satisfying

(1) $F_\chi(ng) = \chi(n)F_\chi(g)$ for all $n \in N_J(\mathbf{R})$, $g \in G'_J(\mathbf{R})$
(2) $F_\chi(gk) = k^{-1} \cdot F_\chi(g)$ for all $g \in G'_J(\mathbf{R})$ and $k \in K'_J$
(3) $D_\ell F_\chi(g) \equiv 0$.

There is a notion of positive semi-definiteness of such characters of $N_J(\mathbf{R})$, which we denote by $\chi \geq 0$. See [Pol20a] for this.

One has the following theorem.

Theorem 2 ([Pol20a]). *Suppose $\chi : N_J(\mathbf{R}) \to \mathbf{C}^\times$ is a nontrivial unitary character.*

(1) *If χ is not positive semi-definite, then the only moderate growth function F_χ as above is the 0 function.*
(2) *If χ is positive semi-definite, then there is unique* **C***-line of moderate growth functions $F_\chi : G'_J(\mathbf{R}) \to V_\ell$ satisfying the three properties above.*

When χ is a generic character of N_J and ℓ is sufficiently large, this was proved by Wallach [Wal03].

When χ is trivial, one can also explicitly describe the space of all such functions F_χ. They are closely related to holomorphic functions on the symmetric space for $H_J(\mathbf{R})$, which is a Hermitian tube domain. See [Pol20a] for this.

In fact, the main theorem of *loc cit* gives an **explicit function** W_χ in the above **C**-line. This explicit function is defined in terms of the K-Bessel functions. Using it, we can describe the Fourier coefficients of a modular form, as follows.

Thus suppose $\varphi : G'_J(\mathbf{Q})\backslash G'_J(\mathbf{A}) \to V_\ell$ is a weight ℓ modular form. Let $\varphi_Z(g) = \int_{Z(\mathbf{Q})\backslash Z(\mathbf{A})} \varphi(zg)\, dz$ denote the constant term of φ along $Z = [N_J, N_J]$. One has the following immediate corollary of Theorem 2.

Corollary 3. *For every nontrivial character $\chi : N_J(\mathbf{Q})\backslash N_J(\mathbf{A}) \to \mathbf{C}^\times$ there are locally constant functions $c_\varphi(\chi) : G_J(\mathbf{A}_f) \to \mathbf{C}$ so that*

$$\varphi_Z(g_f g_\infty) = \varphi_{N_J}(g_f g_\infty) + \sum_{\chi \neq 1, \chi \geq 0} c_\varphi(\chi)(g_f) W_\chi(g_\infty)$$

for $g_f \in G'_J(\mathbf{A}_f)$ and $g_\infty \in G'_J(\mathbf{R})$. Here φ_{N_J} is the constant term of φ along N_J. Moreover, the constant term φ_{N_J} can be related to holomorphic modular forms on H_J.

If $R \subseteq \mathbf{C}$ is a ring, one says that the modular form φ has Fourier coefficients in R if the following conditions hold:

(1) The locally constant function $c_\varphi(\chi)$, when restricted to $H_J(\mathbf{A}_f)$, are valued in R for every nontrivial character χ of $N_J(\mathbf{Q})\backslash N_J(\mathbf{A})$;
(2) The classical Fourier coefficients of the holomorphic modular form associated to φ_{N_J} has Fourier coefficients in R.

We remark that, using the aforemented result [Wal03] of Wallach, Gan-Gross-Savin had previously succeeded in defining Fourier coefficients of modular forms on G_2 associated to generic characters χ.

It is a perhaps surprising fact that, with the above definition, there exists modular forms on quaternionic exceptional groups with Fourier coefficients in small rings, such as $\mathbf{Z}, \mathbf{Q}, \overline{\mathbf{Q}}$.

The idea of the proof of Theorem 2 is simple: One considers a general function F_χ as above. When one writes down, in explicit detail, the differential equation $D_\ell F_\chi = 0$, and considers the two other equivariance properties, one finds that there is a unique moderate growth solution, up to constant multiple if χ is positive semi-definite. And if χ is not positive semi-definite, one finds that there are no moderate growth solutions.

5. Some singular and distinguished modular forms

In this section we describe our results on singular and distinguished modular forms on quaternionic exceptional groups. These results can be found in the papers [Pol20b, Pol20c].

First let us recall the definition of singular and distinguished Siegel modular forms. A Siegel modular form $f(Z)$ of genus n with Fourier expansion $f(Z) = \sum_{T \geq 0} a_f(T) e^{2\pi i \operatorname{tr}(TZ)}$ is said to be **singular** if $a_f(T) \neq 0$ implies $\det(T) = 0$. In other words, f is said to be singular if its nonzero Fourier coefficients are supported on matrices T with rank strictly less than n. The Siegel modular form f is said to be **distinguished** if the following two properties hold: 1) there exists T with $\det(T) \neq 0$ and $a_f(T) \neq 0$; 2) there exists a square class $\kappa \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2$ so that if $a_f(T) \neq 0$ and $\det(T) \neq 0$ then $\det(T) \equiv \kappa$ modulo $(\mathbf{Q}^\times)^2$. The papers [Pol20b, Pol20c] contain results about certain special quaternionic modular forms which are *singular* and *distinguished*, a notion we are about to define for the groups G'_J.

Note that the characters χ of $N_J(\mathbf{Q})\backslash N_J(\mathbf{A})$ are parametrized by elements of $(N_J/Z)(\mathbf{Q})^\vee = W_J(\mathbf{Q})^\vee \simeq W_J(\mathbf{Q})$ where we have used that W_J is isomorphic to its dual by virtue of Freudenthal's symplectic pairing. For $\omega \in W_J(\mathbf{Q})$ and φ a quaternionic modular form on G'_J, we will consequently write $a_\varphi(\omega)$ for its Fourier coefficients.

Definition 4. A modular form φ on $G'_J(\mathbf{A})$ is said to be **singular** if its Fourier coefficients satisfy $a_\varphi(\omega) \neq 0$ implies $q(\omega) = 0$ for all $\omega \in W_J(\mathbf{Q})$. The modular form φ is said to be **distinguished** if

(1) there exists $\omega \in W_J(\mathbf{Q})$ with $q(\omega) \neq 0$ so that $a_\varphi(\omega) \neq 0$;
(2) there exists a (unique) square class $\kappa \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2$ so that if $a_\varphi(\omega) \neq 0$ and $q(\omega) \neq 0$ then $q(\omega) \equiv \kappa$ modulo $(\mathbf{Q}^\times)^2$.

Thus singular and distinguished quaternionic modular forms are analogous to singular and distinguished Siegel modular forms.

To describe the special modular forms we construct, we begin with a family of Eisenstein series. Thus let ℓ be a positive even integer. In [Pol20b] we study a family of V_ℓ-valued Eisenstein series $E_J(g,s;\ell)$ on the groups $G'_J = E_8$. These Eisenstein series are associated to the induced representation (non-normalized induction) $Ind_{P_J}^{G'_J}(\delta_{P_J}^{s/29})$, where δ_{P_J} is the modulus character of P_J.

Proposition 5. *Suppose $\ell \geq 30$. Then the sum defining the Eisenstein series $E_J(g,s=\ell+1;\ell)$ converges absolutely. The value $E_J(g;s=\ell+1;\ell)$ is a modular form on G'_J of weight ℓ.*

When $\ell = 4,8$ the Eisenstein series $E_J(g,s;\ell)$ are of interest at the special point $s=\ell+1$, even though this point is outside the range of absolute convergence:

Theorem 6. *Let $E_J(g,s;\ell)$ be the Eisenstein series on $G'_J = E_8$ just described.*

1. *(Gan* [Gan00]*, Gan-Savin* [GS05]*) The Eisenstein series $E_J(g,s;4)$ is regular at $s=5$. The value $\theta_{min}(g) = E_J(g,s=5;4)$ is a square integrable automorphic form, which is a quaternionic modular form of weight* 4*.*
2. *(P.* [Pol20c]*) The Eisenstein series $E_J(g,s;8)$ is regular at $s=9$. The value $\theta_{ntm}(g) = E_J(g,s=9;8)$ is a square integrable automorphic form, which is a quaternionic modular form of weight* 8*.*

To describe the special properties of these objects, we recall the notion of **rank** of elements of W_J. Let $(\ ,\ ,\ ,\ )$ denote a polarization of the quartic form q on W_J, so that it is a symmetric four-linear form on W_J.

Definition 7. All elements of W_J have rank at most 4. An element w of W_J has rank at most 3 if $q(w)=0$. An element w of W_J has rank at most two if $(w,w,w,x)=0$ for all $x \in W_J$. The element w has rank at most one if $(w,w,x,y)=0$ for all $y \in W_J$ and all $x \in W_J$ with $\langle w,x\rangle = 0$. Finally, 0 is the unique element of rank zero.

Note that the rank is an invariant the H_J orbits on W_J.

The following result is an application of theorems of Savin, in the context of Theorem 6.

Theorem 8 (Savin)**.** *The modular forms θ_{min} and θ_{ntm} are singular. Moreover,*

1. [Sav94] *The Fourier coefficients of θ_{min} satisfy $a_{\theta_{min}}(\omega) \neq 0$ implies ω has rank at most one.*
2. [Pol20c, Appendix] *The Fourier coefficients of θ_{ntm} satisfy $a_{\theta_{ntm}}(\omega) \neq 0$ implies ω has rank at most two.*

The results of Savin from [Sav94] and [Pol20c, Appendix] are really about the wavefront set of certain spherical subrepresentations of degenerate principal series. We have simply stated the consequence of these results for θ_{min} and θ_{ntm} for our own convenience.

The modular forms θ_{min} and θ_{ntm} are anologues to $E_{8,4}$ of the singular modular forms constructed by Henry Kim on $E_{7,3}$ [Kim93]. In fact, Kim's singular modular forms on $GE_{7,3} = H_J(\mathbf{R})$ appear in the constant terms to H_J of θ_{min} and θ_{ntm}.

One can in fact say more about the Fourier coefficients of these special modular forms:

Theorem 9 ([Pol20b],[Pol20c])**.** *The singular modular forms θ_{min} and θ_{ntm} have rational Fourier coefficients.*

We now briefly indicate the steps in the proof of this theorem, in the case of θ_{ntm}. The following steps constitute the paper [Pol20c].

1. We develop a theory of "modular forms" on groups of type $\mathrm{SO}(3,n)$, $n \geq 4$, similar to the theory of modular forms on the quaternionic exceptional groups, and a theory of Fourier coefficients for these objects;

(2) We prove that a certain family of Eisenstein series on the groups $\mathrm{SO}(3,n)$ that are modular forms have rational Fourier coefficients;
(3) We prove that rank one and rank two Fourier coefficients of θ_{ntm} are equal to the Fourier coefficients of one of the above Eisenstein series, thus are rational;
(4) Applying the above result of Gordan Savin, the rank three and rank four Fourier coefficients of θ_{ntm} are 0, thus giving the full rationality.

The following corollary of Theorem 9 is proved in [Pol20b]. It follows almost immediately from the theory of Fourier coefficients on the groups G'_J with results on *arithmetic invariant theory* from [Pol18].

Corollary 10 ([Pol20b])**.** *There are nonzero singular and distinguished modular forms on $E_{7,4}$ and $E_{6,4}$, respectively. In more detail:*

(1) *Fix a quaternion algebra B over $\mathbf{Q}$ that is ramified at infinity. Let $G'_{H_3(B)}$ be the simply-connected group of type $E_{7,4}$ constructed out of B. The group $G'_{H_3(B)}$ embeds in $G'_J = E_{8,4}$. Let θ_B be the pullback of θ_{min} under this embedding. Then θ_B is a weight four modular form with rational Fourier coefficents. These Fourier coefficients satisfy $a_{\theta_B}(\omega) \neq 0$ implies ω has rank at most two.*
(2) *Fix an imaginary quadratic field K. Let $G'_{H_3(K)}$ be the simply-connected group of type $E_{6,4}$ constructed out of K. The group $G'_{H_3(K)}$ embeds in $G'_J = E_{8,4}$. Let θ_K be the pullback of θ_{min} under this embedding. Then θ_K is a weight four modular form with rational Fourier coefficients. Moreover, it is distinguished: if $a_{\theta_K}(\omega) \neq 0$ and ω is rank four then $q(\omega) \equiv [K]$ in $\mathbf{Q}^\times/(\mathbf{Q}^\times)^2$, where $[K]$ is the square class corresponding to K.*

6. Cuspidal modular forms on G_2

The modular forms θ_{min} and θ_{ntm} are not cuspidal. Thus, one can ask if there exists cuspidal modular forms with algebraic Fourier coefficients. On G_2, we have answered this question positively.

Theorem 11. [Pol21b] *Let $w \geq 16$ be even. Then there are nonzero cuspidal modular forms on G_2 of weight w with all Fourier coefficients in $\overline{\mathbf{Q}}$.*

The outline of the proof of Theorem 11 is as follows.

(1) First, we consider the theta lift, from Sp_4 to $\mathrm{SO}(4,4)$. If f is an automorphic form on Sp_4, let $\theta(f)$ be its theta lift to $\mathrm{SO}(4,4)$. Using very special data for the Weil representation that we write down, we show that if f is a Siegel modular form, then $\theta(f)$ is a quaternionic modular form on $\mathrm{SO}(4,4)$.
(2) Next, we prove a simple formula for the Fourier coefficients of $\theta(f)$ in terms of those of f. In particular, if f is a level one Siegel modular form with Fourier coefficients in the subring R of $\mathbf{C}$, then we prove that the quaternionic Fourier coefficients of $\theta(f)$ are all valued in R.
(3) One next pulls back $\theta(f)$ to G_2, under the embedding $G_2 \subseteq \mathrm{SO}(4,4)$. One shows that the Fourier coefficients of the pullback are **finite sums** of the Fourier coefficients of $\theta(f)$, and thus again live in R.
(4) By a result of Rallis, $\theta(f)$ is cuspidal, and using the robust theory of quaternionic Fourier coefficients, it is easy to check that its pullback to G_2 remains cuspidal.

By taking the Siegel modular form f to have Fourier coefficients in $\overline{\mathbf{Q}}$, which can be done because GSp_4 has a Shimura variety, one obtains cusp forms on G_2 with all algebraic Fourier coefficients.

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