The Zelevinsky–Aubert duality for classical groups

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Abstract

In 1980, Zelevinsky [14] studied the representation theory of p-adic general linear groups. He introduced an involution on the Grothendieck group of smooth representations of finite length, which exchanges the trivial representation with the Steinberg representation. In fact, he conjectured that it preserves the irreducibility. Aubert [5] extended this involution to p-adic reductive groups, which is now called the Zelevinsky–Aubert duality. It is expected that this duality preserves the unitarity. In this article, based on the joint work with Alberto Mínguez [3], we give an algorithm to compute the Zelevinsky–Aubert duality for odd special orthogonal groups or symplectic groups.

1 The Zelevinsky–Aubert duality

Let G be a split connected reductive group over a p-adic field F. We identify G with $G(F)$. Denote by $\mathrm{Rep}(G)$ the set of equivalence classes of complex smooth representations of G of finite length, and by $\mathrm{Irr}(G)$ the subset consisting of irreducible representations. Fix a rational Borel subgroup of G. For a standard parabolic subgroup $P = MN$ of G, let $\mathrm{Ind}_P^G(\pi_M) \in \mathrm{Rep}(G)$ be the (normalized) parabolically induced representation of $\pi_M \in \mathrm{Rep}(M)$, and $\mathrm{Jac}_P(\pi) \in \mathrm{Rep}(M)$ be the Jacquet module of $\pi \in \mathrm{Rep}(G)$ along P. We denote by $\pi \mapsto [\pi]$ the semisimplification. When Π is of finite length, let $\mathrm{soc}(\Pi)$ denote the *socle* of Π, i.e., the maximal semisimple subrepresentation of Π.

For $\pi \in \mathrm{Rep}(G)$, define

$$D_G(\pi) := \sum_{P=MN} (-1)^{\dim A_M} [\mathrm{Ind}_P^G(\mathrm{Jac}_P(\pi))],$$

where $P = MN$ runs over all standard parabolic subgroups of G, and A_M is the maximal split torus of the center of M. Aubert [5] showed that if π is irreducible, then there exists a sign $\epsilon \in \{\pm 1\}$ such that $\hat{\pi} = \epsilon \cdot D_G(\pi)$ is also an irreducible representation. We call the map $\pi \mapsto \hat{\pi}$ the *Zelevinsky–Aubert duality*. It satisfies the following important properties ([5, Théorème 1.7]):

1. The dual of $\hat{\pi}$ is equal to π, i.e., the map $\pi \mapsto \hat{\pi}$ is an involution.
2. If π is supercuspidal, then $\hat{\pi} = \pi$.
3. The duality commutes with Jacquet functors.

It is predicted that if π is irreducible and unitary, then so is $\hat{\pi}$. In particular, this duality would produce many unitary representations. For $G = \mathrm{GL}_n(F)$, Mœglin–Waldspurger

[12] gave an algorithm to compute $\hat{\pi}$. Together with Tadić's result [13] on the unitary dual of $\mathrm{GL}_n(F)$, it shows this prediction in this case.

For a classical group G, Jantzen [8] gave an algorithm to compute $\hat{\pi}$ when π is in the "half-integral case". In the joint work with Alberto Mínguez [3], we extend this algorithm to all cases. Our main result is roughly stated as follows.

Theorem 1.1 ([3, Algorithm 4.1])**.** *We give an explicit algorithm to compute $\hat{\pi}$ when $G = \mathrm{Sp}_{2n}(F)$ or $G = \mathrm{SO}_{2n+1}(F)$.*

2 The case for $\mathrm{GL}_n(F)$

Before starting the explanation for the classical group case, let us consider the general linear group case.

Let ρ be a supercuspidal representation of $\mathrm{GL}_d(F)$. A *segment* is a set

$$[x, y]_\rho = \{\rho|\cdot|^x, \rho|\cdot|^{x-1}, \dots, \rho|\cdot|^y\},$$

where $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Z}_{\geq 0}$. Denote by $\Delta_\rho[x, y]$ (resp. $Z_\rho[y, x]$) the unique irreducible subrepresentation (resp. quotient) of the parabolically induced representation

$$\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y := \mathrm{Ind}_P^{\mathrm{GL}_{d(x-y+1)}(F)}(\rho|\cdot|^x \boxtimes \cdots \boxtimes \rho|\cdot|^y).$$

Then $\Delta_\rho[x, y]$ is an essentially discrete series representation, which is called a *(generalized) Steinberg representation*. On the other hand, If $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, then $Z_\rho[y, x] = |\det|^{\frac{x+y}{2}}$ is a character of $\mathrm{GL}_{x-y+1}(F)$.

Proposition 2.1 ([14, Propositions 3.4, 9.5])**.** *When P_d is a standard parabolic subgroup with Levi $\mathrm{GL}_d(F) \times \mathrm{GL}_{d(x-y)}(F)$, we have*

$$\mathrm{Jac}_{P_d}(\Delta_\rho[x, y]) = \rho|\cdot|^x \boxtimes \Delta_\rho[x-1, y],$$
$$\mathrm{Jac}_{P_d}(Z_\rho[y, x]) = \rho|\cdot|^y \boxtimes Z_\rho[y+1, x].$$

Here, we set $\Delta_\rho[y-1, y] := \mathbf{1}_{\mathrm{GL}_0(F)}$ and $Z_\rho[x+1, x] := \mathbf{1}_{\mathrm{GL}_0(F)}$.

Now we introduce the following notion.

Definition 2.2 (Jantzen [6], Mínguez [11])**.** *The k-th* left ρ-derivative *of $\tau \in \mathrm{Rep}(\mathrm{GL}_n(F))$ is a semisimple representation $L_\rho^{(k)}(\tau)$ satisfying*

$$[\mathrm{Jac}_{P_{dk}}(\tau)] = \rho^k \boxtimes L_\rho^{(k)}(\tau) + \sum_i \tau_i' \boxtimes \tau_i'',$$

where $\tau_i' \boxtimes \tau_i'' \in \mathrm{Irr}(\mathrm{GL}_{dk}(F) \times \mathrm{GL}_{n-dk}(F))$ is such that $\tau_i' \not\cong \rho^k$. Here, $\rho^k = \rho \times \cdots \times \rho$ (k times).

When $L_\rho^{(k)}(\tau) \neq 0$ but $L_\rho^{(k+1)}(\tau) = 0$, call $L_\rho^{(k)}(\tau)$ the *highest ρ-derivative*. Similarly, one can define the *right ρ-derivative* $R_\rho^{(k)}(\tau)$.

Theorem 2.3 (Jantzen [6], Mínguez [11]). *Let τ be an irreducible representation of $\mathrm{GL}_n(F)$.*

1. *The highest ρ-derivative $L_\rho^{(k)}(\tau)$ is irreducible.*
2. *τ can be recovered from $L_\rho^{(k)}(\tau)$ by*
$$\tau = \mathrm{soc}\left(\rho^k \times L_\rho^{(k)}(\tau)\right).$$
3. *There is an explicit formula for the highest ρ-derivative $L_\rho^{(k)}(\tau)$.*
4. *There is an explicit formula for $\mathrm{soc}(\rho^r \times \tau)$, which is irreducible.*

The analogous statements for $R_\rho^{(k)}$ also hold.

The derivatives are related to the Zelevinsky duality as follows.

Proposition 2.4 (Aubert [5]). *For $\tau \in \mathrm{Irr}(\mathrm{GL}_n(F))$, we have*
$$L_\rho^{(k)}(\hat{\tau}) = R_\rho^{(k)}(\tau)\widehat{\ }.$$

Using this proposition together with the explicit formulas for derivatives and socles, one can compute $\hat{\tau}$ by induction on n.

Example 2.5. *The Zelevinsky dual of $Z_\rho[y, x]$ is $\Delta_\rho[x, y]$ since*
$$L^{(1)}_{\rho|\cdot|^x}(\Delta_\rho[x, y]) = \Delta_\rho[x-1, y], \quad R^{(1)}_{\rho|\cdot|^x}(Z_\rho[y, x]) = Z_\rho[y, x-1].$$

Remark 2.6.
- *Mœglin–Waldspurger [12] gave a different algorithm to compute the Zelevinsky dual $\hat{\tau}$.*
- *The idea to use derivatives for an algorithm to compute $\hat{\tau}$ is due to Jantzen [6].*
- *There are three explicit formulas for the highest derivatives and socles given by Jantzen [6], Mínguez [11], and Lapid–Mínguez [9].*
- *We will use the explicit formula of Lapid–Mínguez [9], which uses the notion of* best matching functions.

3 The case for classical groups

Now we explain the classical group case. Let $G = \mathrm{Sp}_{2n}(F)$ or $G = \mathrm{SO}_{2n+1}(F)$ (split). Basically, we try to use the same strategy as in the general linear group case.

Let P_d is a parabolic subgroup of G with Levi $\mathrm{GL}_d(F) \times G_0$. For $\tau \boxtimes \pi_0 \in \mathrm{Rep}(\mathrm{GL}_d(F) \times G_0)$, we denote the parabolically induced representation by
$$\tau \rtimes \pi_0 := \mathrm{Ind}_{P_d}^{G}(\tau \boxtimes \pi_0).$$

On the other hand, for $\pi \in \mathrm{Rep}(G)$, the Jacquet module along P_d is denoted by $\mathrm{Jac}_{P_d}(\pi)$.

Fix an irreducible supercuspidal representation ρ of $\mathrm{GL}_d(F)$.

Definition 3.1. *The k-th ρ*-derivative *of π is a semisimple representation $D_\rho^{(k)}(\pi)$ satisfying*

$$[\mathrm{Jac}_{P_{dk}}(\pi)] = \rho^k \boxtimes D_\rho^{(k)}(\pi) + \sum_i \tau_i \boxtimes \pi_i,$$

where $\tau_i \boxtimes \pi_i \in \mathrm{Irr}(\mathrm{GL}_{dk}(F) \times G_0)$ is such that $\tau_i \not\cong \rho^k$.

If $D_\rho^{(k)}(\pi) \neq 0$ but $D_\rho^{(k+1)}(\pi) = 0$, we call $D_\rho^{(k)}(\pi)$ the *highest ρ-derivative.*

As in the $\mathrm{GL}_n(F)$-case, derivatives are compatible with the Zelevinsky–Aubert duality.

Proposition 3.2 (Aubert [5])**.** *We have*

$$D_\rho^{(k)}(\hat{\pi}) = D_{\rho^\vee}^{(k)}(\pi)\hat{}.$$

Here, $\rho^\vee$ denotes the contragredient of ρ.

To obtain an algorithm for $\hat{\pi}$, we would like to answer the following questions:

- Is the highest derivative $D_\rho^{(k)}(\pi)$ irreducible?
- Can π be recovered from $D_\rho^{(k)}(\pi)$?
- Can we establish explicit formulas for the correspondences $\pi \leftrightarrow D_\rho^{(k)}(\pi)$?

If these questions were all affirmative, the same algorithm as the $\mathrm{GL}_n(F)$-case would work.

These questions are affirmative if ρ is not self-dual.

Proposition 3.3 (Jantzen [7], Lapid–Tadić [10], A.–Mínguez [3])**.** *Suppose that ρ is not self-dual. Let π be an irreducible representation of G.*

1. *The highest derivative $D_\rho^{(k)}(\pi)$ is irreducible.*
2. *The socle* $\mathrm{soc}(\rho^r \rtimes \pi)$ *is irreducible.*
3. *They are related as*

$$\pi = \mathrm{soc}\left(\rho^k \rtimes D_\rho^{(k)}(\pi)\right).$$

Hence, we need explicit formulas for $D_\rho^{(k)}(\pi)$ and for $\mathrm{soc}(\rho^r \rtimes \pi)$.

As a parametrization of $\mathrm{Irr}(G)$, we will use the *Langlands classification.* It says that any $\pi \in \mathrm{Irr}(G)$ can be written as

$$\begin{aligned}\pi &= L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi(\phi, \eta)) \\ &:= \mathrm{soc}\left(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi(\phi, \eta)\right),\end{aligned}$$

where

- ρ_i is unitary supercuspidal, and $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$;
- (ϕ, η) is a tempered L-parameter.
- $\pi(\phi, \eta)$ is the irreducible tempered representation corresponding to (ϕ, η).

Here, a *tempered L-parameter* is a pair (ϕ, η) such that

- ϕ is an orthogonal or symplectic representation of Weil–Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$ such that $\phi(W_F)$ is bounded;
- η is a character of the component group of ϕ.

We can decompose

$$\phi = \bigoplus_{i=1}^{s} \rho'_i \boxtimes S_{a_i},$$

where ρ'_i is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{d_i}(F)$ which is identified with an irreducible bounded representation of Weil group W_F, and S_d is the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d. If all irreducible components $\rho'_i \boxtimes S_{a_i}$ are self-dual of the same type as ϕ, then η is determined by $\{\eta(\rho'_i \boxtimes S_{a_i})\}_{1 \leq i \leq s} \in \{\pm 1\}^s$. By the local Langlands correspondence established by Arthur [1], any irreducible tempered representation of G is written as $\pi(\phi, \eta)$ for some tempered L-parameter (ϕ, η).

From now on, we fix an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(F)$, and consider the $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ for $x \in \mathbb{R}$. Jantzen [8] suggested an algorithm to compute the highest $\rho|\cdot|^x$-derivative of π for $x \neq 0$. The first step is to rewrite

$$\pi = \mathrm{soc}\left(\tau \rtimes \pi_A\right),$$

where

- $\tau = \mathrm{soc}(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r])$ with $[x_i, y_i]_{\rho_i} \neq [x-1, -x]_\rho$ for any i;
- $\pi_A = L(\Delta_\rho[x-1, -x]^t; \pi(\phi, \eta))$ with $t \geq 0$ such that $t = 0$ unless $x > 0$.

This algorithm was completed by the author [2] using Jantzen's formulas for $R^{(a)}_{\rho|\cdot|^{-x}}(\tau)$ and $L^{(k)}_{\rho|\cdot|^x}(\tau)$ ([6, Propositions 2.1.4, 2.4.3]). The key observation is that π_A is of Arthur type (see [2, Example 3.12, Proposition 3.13]). However, by using formulas for $R^{(a)}_{\rho|\cdot|^{-x}}(\tau)$ and $L^{(k)}_{\rho|\cdot|^x}(\tau)$ established by Lapid–Mínguez [9, Theorem 5.11], Jantzen's algorithm is sublimated into a formula.

Theorem 3.4 ([3, Proposition 6.1, Theorem 7.1])**.** *Assume that $\rho|\cdot|^x$ is not self-dual . Let $\pi \in \mathrm{Irr}(G)$. Then we have explicit formulas for the highest $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ and for the socle $\mathrm{soc}((\rho|\cdot|^x)^r \rtimes \pi)$ in terms of best matching functions and A-parameters.*

Therefore, the computation of $\hat{\pi}$ is reduced to the case where π is irreducible such that

$$D^{(1)}_\rho(\pi) \neq 0 \implies \rho \text{ is self-dual.} \qquad (*)$$

From now on, we assume that ρ is self-dual. In this case, the ρ-derivatives $D^{(k)}_\rho(\pi)$ are difficult, and π cannot be recovered from $D^{(k)}_\rho(\pi)$ in general.

Example 3.5. *If σ is supercuspidal such that $\rho \rtimes \sigma = \pi_+ \oplus \pi_-$, then*

$$D^{(1)}_\rho(\pi_+) = D^{(1)}_\rho(\pi_-) = \sigma.$$

However it is easy to see that $\hat{\pi}_+ = \pi_-$ by definition (see [5, Corollaire 1.10]).

As in this example, we can always compute $\hat{\pi}$ if π is tempered and satisfies $(*)$.

Proposition 3.6 ([3, Proposition 5.4]). *Suppose that* $\pi = \pi(\phi, \eta)$ *is tempered and satisfies* $(*)$. *Let* $\rho_1, \dots, \rho_r$ *be the irreducible representations of* W_F *appearing in* ϕ *with even multiplicities, and set* $y_i = \max\{\frac{d_i - 1}{2} \mid \rho_i \boxtimes S_{d_i} \subset \phi\}$. *Assume that* $y_1 \geq \cdots \geq y_t > 0 = y_{t+1} = \cdots = y_r$. *Then*

$$\hat{\pi} = L(\Delta_{\rho_1}[0, -y_1], \dots, \Delta_{\rho_t}[0, -y_t]; \pi(\phi', \eta')),$$

where

$$\phi' = \phi - \bigoplus_{i=1}^{t} \rho_i \boxtimes (S_1 \oplus S_{2y_i+1})$$

and $\eta'(\rho \boxtimes S_d) \neq \eta(\rho \boxtimes S_d) \iff \rho \in \{\rho_1, \dots, \rho_r\}$.

The remaining case is when π is non-tempered and satisfies $(*)$. The key idea to deal with this case is *to define new derivatives.*

Definition 3.7. *Define the k-th* $\Delta_\rho[0,-1]$-derivative $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ *and the k-th* $Z_\rho[0,1]$-derivative $D^{(k)}_{Z_\rho[0,1]}(\pi)$ *as the semisimple representations satisfying*

$$[\mathrm{Jac}_{P_{2dk}}(\pi)] = \Delta_\rho[0,-1]^k \boxtimes D^{(k)}_{\Delta_\rho[0,-1]}(\pi) + Z_\rho[0,1]^k \boxtimes D^{(k)}_{Z_\rho[0,1]}(\pi) + (\mathit{others}).$$

One can define the notions of the *highest* $\Delta_\rho[0,-1]$*-derivative* and the *highest* $Z_\rho[0,1]$*-derivative.* These derivatives are the substitute for the ρ-derivatives.

Proposition 3.8 ([3, Propositions 3.7, 3.9]). *Suppose that* π *is irreducible and satisfies* $(*)$.

1. *The highest* $\Delta_\rho[0,-1]$*-derivative* $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ *(resp. the highest* $Z_\rho[0,1]$*-derivative* $D^{(k)}_{Z_\rho[0,1]}(\pi)$*) is irreducible.*
2. *The socle* $\mathrm{soc}(\Delta_\rho[0,-1]^r \rtimes \pi)$ *(resp.* $\mathrm{soc}(Z_\rho[0,1]^r \rtimes \pi)$*) is irreducible.*
3. $\pi \cong \mathrm{soc}(Z_\rho[0,1]^k \rtimes D^{(k)}_{Z_\rho[0,1]}(\pi))$.
4. $D^{(k)}_{Z_\rho[0,1]}(\hat{\pi}) = D^{(k)}_{\Delta_\rho[0,-1]}(\pi)^{\wedge}$.

Moreover, we have:

Theorem 3.9 ([3, Proposition 3.8, Theorem 8.1, Corollary 8.2]). *Suppose that* π *satisfies* $(*)$. *Then we have explicit formulas for*

- *the highest* $\Delta_\rho[0,-1]$*-derivative* $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$*;*
- *the highest* $Z_\rho[0,1]$*-derivative* $D^{(k)}_{Z_\rho[0,1]}(\pi)$*;*
- *the socle* $\mathrm{soc}(Z_\rho[0,1]^r \rtimes \pi)$

in terms of matching functions and A-parameters.

Actually, we only assume a weaker condition than $(*)$ in Proposition 3.8 and Theorem 3.9.

In conclusion, we obtain an algorithm to compute $\hat{\pi}$ as follows.

Algorithm 3.10. *Let π be an irreducible representation of G.*

Step 1 *If there exists an irreducible supercuspidal representation ρ of $\mathrm{GL}_d(F)$ such that ρ is not self-dual and $\pi_0 := D_\rho^{(k)}(\pi) \neq 0$ for $k > 0$, then use*

$$\hat{\pi} = \mathrm{soc}\left((\rho^\vee)^k \rtimes \hat{\pi}_0\right)$$

to reduce the computation of $\hat{\pi}$ to the one of $\hat{\pi}_0$.

Step 2 *Otherwise, and if π is not tempered, one can find an irreducible self-dual supercuspidal representation ρ of $\mathrm{GL}_d(F)$ such that $\pi_0 := D^{(k)}_{\Delta_\rho[0,-1]}(\pi) \neq 0$ for $k > 0$. Use*

$$\hat{\pi} = \mathrm{soc}\left(Z_\rho[0,1]^k \rtimes \hat{\pi}_0\right)$$

to reduce the computation of $\hat{\pi}$ to the one of $\hat{\pi}_0$.

Step 3 *Otherwise, and if π is tempered, then we have an explicit formula for $\hat{\pi}$.*

In the following example, we set $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, and drop ρ in the notation. Let $\pi(x_1^{\eta_1}, \dots, x_r^{\eta_r}) = \pi(\phi, \eta)$ with $\phi = \oplus_{i=1}^r S_{2x_i+1}$ and $\eta(S_{2x_i+1}) = \eta_i \in \{\pm 1\}$.

Example 3.11. *Let us consider an irreducible tempered representation $\pi(0^-, 0^-, 1^+, 2^-, 2^-)$ of $\mathrm{Sp}_{14}(F)$. Then*

$$\begin{array}{ccc}
\pi(0^-,0^-,1^+,2^-,2^-) & & L(|\cdot|^{-2}, |\cdot|^{-1}, \Delta[0,-2]; \pi(0^-,0^-,1^+)) \\
\downarrow D^{(1)}_{|\cdot|^2} & & \uparrow \mathrm{soc}(|\cdot|^{-2} \rtimes -) \\
L(\Delta[1,-2]; \pi(0^-,0^-,1^+)) & & L(|\cdot|^{-1}, \Delta[0,-2]; \pi(0^-,0^-,1^+)) \\
\downarrow D^{(1)}_{|\cdot|^1} & & \uparrow \mathrm{soc}(|\cdot|^{-1} \rtimes -) \\
L(\Delta[0,-2]; \pi(0^-,0^-,1^+)) & & L(\Delta[0,-2]; \pi(0^-,0^-,1^+)) \\
\downarrow D^{(1)}_{\Delta[0,-1]} & & \uparrow \mathrm{soc}(Z[0,1] \rtimes -) \\
L(|\cdot|^{-2}; \pi(0^-,0^-,1^+)) & & L(\Delta[0,-2]; \pi(0^+)) \\
\downarrow D^{(1)}_{|\cdot|^{-2}} & & \uparrow \mathrm{soc}(|\cdot|^{2} \rtimes -) \\
\pi(0^-,0^-,1^+) & \xrightarrow{\pi \mapsto \hat{\pi}} & L(\Delta[0,-1]; \pi(0^+))
\end{array}$$

so that we have

$$\hat{\pi}(0^-,0^-,1^+,2^-,2^-) = L(|\cdot|^{-2}, |\cdot|^{-1}, \Delta[0,-2]; \pi(0^-,0^-,1^+)).$$

Remark 3.12. *1. As in this example, even if we start with a tempered representation, we need to compute $\Delta_\rho[0,-1]$-derivatives and $Z_\rho[0,1]$-socles in general.*

2. In the latest paper [4], we refine Mœglin's explicit construction of local A-packets as an application of $Z_\rho[0,1]$-derivatives.

3. *In that work, a more reasonable formula for* $\hat{\pi}$ *for* π *of* Arthur type *was proven ([4, Theorem 6.2]).*

4. *Although the explicit formulas for derivatives and for socles are complicated, it would be easy to write a computer program.*

Acknowledgement

We would like to thank Alberto Mínguez for giving useful comments. The author was supported by JSPS KAKENHI Grant Number 19K14494.

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