EISENSTEIN COHOMOLOGY AND CM CONGRUENCES

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Abstract. This note announces results on congruences between base changed Eisenstein cohomology classes and cuspidal cohomology classes over imaginary quadratic fields generalizing [Ber09].

1. Introduction

This note is based on my talk given at the RIMS conference on "Automorphic forms, automorphic representations, Galois representations, and its related topics" on 28 January 2021 and reports on ongoing joint work with Adel Betina (University of Vienna).

For $K/\mathbf{Q}$ imaginary quadratic let $\psi : \mathbf{A}_K^*/K^* \to \mathbf{C}^*$ be a Hecke character with $\psi_\infty(z) = z^{-1}$. Define $\psi^c(x) := \psi(\overline{x})$ and write $\psi^- := \psi/\psi^c$.

Consider p split in $K/\mathbf{Q}$. Under some other mild conditions Hida proved in [Hid82] that if $p \mid L^{\rm int}(1, \psi^-)$ then there exists a classical cuspidal eigenform f of weight 2 without CM such that $f \equiv f_\psi \mod p$ (where the latter denotes the CM form associated to ψ). Base changing to K this gives a congruence between the cuspidal Bianchi form $\mathrm{BC}(f)$ and an Eisenstein series with Hecke eigenvalues given by $\psi + \psi^c$. Since the p-adic Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ associated to f stays irreducible under restriction to $G_K := \mathrm{Gal}(\overline{K}/K)$ one can apply Ribet's lattice construction from [Rib76] to construct extensions of $\tilde{\psi}^c$ by $\tilde{\psi}$ (where the tilde denotes the corresponding p-adic Galois characters). In particular, one obtains that p divides the order of a Selmer group associated to $\tilde{\psi}^-$, in accordance with the Bloch-Kato conjecture for these Galois characters. This approach led to the proof of the anticyclotomic main conjecture (see e.g. [HT94]); for recent investigations of such CM congruences see [CWEHar] Theorem 4.2.2, and Corollary A.2.5.

We prove the Eisenstein congruences over the imaginary quadratic fields directly, which also allows us to treat the case of p inert in $K/\mathbf{Q}$. The integral structure we use is that of the Betti cohomology of the 3-manifolds $\Gamma\backslash\mathbf{H}_3$. Following Harder [Har87] we construct suitable Eisenstein cohomology classes in $H^1(\Gamma\backslash\mathbf{H}_3, \overline{\mathbf{Q}}_p)$ for a pair of Hecke characters (ϕ_1, ϕ_2). The integrality properties of these classes, in particular, lower bounds on their denominators were studied in [Ber08]. In [Ber09] this was used to prove congruences between the Eisenstein and cuspidal cohomology classes in terms of $L(0, \phi_1/\phi_2)$. Due to the existence of torsion in $H^2_c(\Gamma\backslash\mathbf{H}_3, \overline{\mathbf{Z}}_p)$ [Ber09] proved unconditional results only when ϕ_1/ϕ_2 was unramified. In this note we explain how in the *anticyclotomic* case $\phi_2 = \psi$ and $\phi_1 = \phi_2^c \cdot |\cdot|$ (when the

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Eisenstein cohomology class corresponds to the base change of the classical CM-form associated to ψ) we can treat ψ of any conductor and therefore recover and generalize Hida's result for the Bloch-Kato conjecture for $\tilde{\psi}^-$.

2. Notation

We consider $K \neq \mathbf{Q}(i), \mathbf{Q}(\sqrt{-3})$ an imaginary quadratic field with class number 1 (for simplicity in this note). Let $p > 3$ split or inert in $K/\mathbf{Q}$. We fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$.

Put $G = \mathrm{Res}_{K/\mathbf{Q}}(GL_2)$ and write B for its Borel subgroup and T for its maximal split torus. Let $K_\infty = U(2) \cdot \mathbf{C}^* \subset \mathrm{GL}_2(\mathbf{C})$. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G/\mathbf{Q})$ is a $\mathbf{Q}$-vector space and we define $\mathfrak{g}_\infty = \mathfrak{g} \otimes_{\mathbf{Q}} \mathrm{Res}_{K/\mathbf{Q}}$. It carries a positive semidefinite K_∞-invariant form, the Killing form

$$\langle X, Y \rangle = \frac{1}{16}\mathrm{trace}(\mathrm{ad}X \cdot \mathrm{ad}Y),$$

and with respect to this form we have an orthogonal decomposition $\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{p}$, where $\mathfrak{k}_\infty = \mathrm{Lie}(K_\infty)$ and

$$\mathfrak{p} = \mathbf{R}H \oplus \mathrm{Res}_{K/\mathbf{Q}}\, E_1 \oplus \mathrm{Res}_{K/\mathbf{Q}}\, E_2 := \mathbf{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbf{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbf{R}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Put $S_\pm := 1/2\left(\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes_{\mathbf{R}} 1 - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes_{\mathbf{R}} i\right) \in \mathfrak{p}_{\mathbf{C}}$.

3. Eisenstein cohomology

In this section we briefly recap the theory of Eisenstein cohomology classes. For more detail please see [Har87] and [Ber08].

For $K \subset G(\mathbf{A}_f)$ compact open with $\det(K_f) = \hat{\mathcal{O}}_K^*$ we have

$$X_{K_f} = G(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty = X_\Gamma = \Gamma\backslash \mathbf{H}_3$$

for $\Gamma = K_f \cap G(\mathbf{Q})$ and there is a long exact sequence of Betti cohomology groups

$$\ldots \to H^1_c(X_\Gamma, \mathcal{O}) \to H^1(X_\Gamma, \mathcal{O}) \overset{\mathrm{res}}{\to} H^1(\partial \overline{X}_\Gamma, \mathcal{O}) \overset{\partial}{\to} H^2_c(X_\Gamma, \mathcal{O}) \to \ldots, \tag{3.1}$$

where $\overline{X}_\Gamma$ is the Borel-Serre compactification and $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbf{Q}_p$.

Theorem 3.1 (Harder [Har87] Theorem 1)**.** *We have an isomorphism of $G(\mathbf{A}_f)$-modules*

$$H^1(\partial \overline{X}_{K_f}, \overline{\mathbf{Q}}) = \bigoplus_{\phi} V_\phi^{K_f} \oplus V_{w_0.\phi}^{K_f}, \tag{3.2}$$

where the sum is over all $\phi = (\phi_1, \phi_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ with $\phi_\infty(z) = (z, z^{-1})$ and $w_0.\phi = (\phi_2|\cdot|, \phi_1|\cdot|^{-1})$ and

$$V_\phi = \mathrm{Ind}_{B(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \overline{\mathbf{Q}}_\phi = \{\Psi : G(\mathbf{A}_f) \to \overline{\mathbf{Q}} | \Psi(bg) = \phi(b)\Psi(g) \forall b \in B(\mathbf{A}_f)\}.$$

We write $\mathfrak{M}_i$ for the conductor of $\phi_i, i = 1, 2$ and put $\mathfrak{M} = \mathfrak{M}_1\mathfrak{M}_2$. By Casselman [Cas73] we know that

$$V_\phi^{K_1(\mathfrak{M})} = \Psi_\phi^{\mathrm{new}} \cdot \overline{\mathbf{Q}}, \tag{3.3}$$

where

$$K^1(\mathfrak{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_K), d-1, c \equiv 0 \mod \mathfrak{M} \right\}$$

and $\Psi_\phi^{\text{new}} = \prod_v \Psi_v^{\text{new}}$ with

(3.4)

$$\Psi_v^{\text{new}}(g) = \begin{cases} \phi_{1,v}(\pi_v^{-r})\phi_{1,v}(a)\phi_{2,v}(d) & \text{if } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix} k, k \in K^1(\pi_v^s \mathcal{O}_{K_v}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_v^r \| \mathfrak{M}_2$ and $\pi_v^s \| \mathfrak{M}_1\mathfrak{M}_2$.

Employing relative Lie algebra cohomology Harder constructs particular boundary classes in

$$H^1(\partial \overline{X}_{K^1(\mathfrak{M})}, \mathbf{C}) \simeq H^1(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/\mathbf{C}^* K^1(\mathfrak{M})))$$

realizing the isomorphism (3.2):

Define

$$\omega_\phi : \mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, V_\phi^{K^1(\mathfrak{M})}) \simeq V_\phi^{K^1(\mathfrak{M})} \otimes \check{\mathfrak{p}}_{\mathbf{C}}$$

as

$$\begin{aligned} (3.5)\quad \omega_\phi(g) &:= \omega_\phi(b_\infty k_\infty \cdot g_f) = \\ &= (\phi_\infty)(b_\infty) \cdot \Psi_\phi^{\text{new}}(g_f) \begin{cases} k_\infty^{-1}.(\check{S}_+) & \text{if } \phi_\infty(z) = (z, z^{-1}), \\ k_\infty^{-1}.((-\check{S}_-) & \text{if } \phi_\infty(z) = (\overline{z}, \overline{z}^{-1}). \end{cases} \end{aligned}$$

Here K_∞ acts on $\mathfrak{p}_{\mathbf{C}}$ by the adjoint action. We write $[\omega_\phi] \in H^1(\partial \overline{X}_{K^1(\mathfrak{M})}, \mathbf{C})$ for the corresponding cohomology class.

Using meromorphic continuation one can define

$$\mathrm{Eis}(\omega_\phi)(g) = \sum_{\gamma \in B(\mathbf{Q})\backslash G(\mathbf{Q})} \omega_\phi(\gamma g).$$

[Har87] Theorem 2 shows that this defines a holomorphic closed form in

$$H^1(X_{K^1(\mathfrak{M})}, \mathbf{C}) \simeq H^1(\mathfrak{g}_\infty, K_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/\mathbf{C}^* K^1(\mathfrak{M})))$$

and we write $[\mathrm{Eis}(\omega_\phi)]$ for the corresponding cohomology class.

Lemma 3.2. *For $v \nmid \mathfrak{M}_1\mathfrak{M}_2$ the class $[\mathrm{Eis}(\omega_\phi)]$ is an eigenvector for T_{π_v} with eigenvalue $\phi_1(\pi_v)|\pi_v|_v^{-1} + \phi_2(\pi_v)$*

4. Constant term of Eisenstein class

Harder showed in [Har87] that the image of $[\mathrm{Eis}(\omega_\phi)]$ under

$$H^1(X_\Gamma, \mathbf{C}) \stackrel{\text{res}}{\to} H^1(\partial \overline{X}_\Gamma, \mathbf{C})$$

is given by

$$\text{res}([\mathrm{Eis}\omega_\phi)]) = \Psi_\phi^{\text{new}} + * \frac{L(-1, \phi_1/\phi_2)}{L(0, \phi_1/\phi_2)} T_\phi(\Psi_\phi^{\text{new}}), \tag{4.1}$$

where $*$ is some non-zero factor and $T_\phi : V_{\phi_f}^{K_f} \to V_{w_0 \cdot \phi_f}^{K_f}$ is an intertwining operator.

By combining [Har87] Theorem 2 and [Sch02] Proposition 2.2.2 we get:

Theorem 4.1 (Berger-Betina). *We have*

$$\mathrm{res}(\mathrm{Eis}(\omega_\phi)) = \omega_\phi + c(\phi)\omega_{w_0.\phi},$$

where

$$c(\phi) = -\frac{2\pi}{\sqrt{d_K}} \frac{L(-1, \phi_1/\phi_2)}{L(0, \phi_1/\phi_2)} \cdot \prod_{v|\mathfrak{M}_1\mathfrak{M}_2} c_v(\phi)$$

and

$$c_v(\phi, 0) = \epsilon_v(0, \phi_1/\phi_2, \psi) \cdot \frac{\epsilon_v(1/2, \phi_2^{-1}, \psi)}{\epsilon_v(3/2, \phi_1^{-1}, \psi)}$$

where

$$\epsilon_v(s, \lambda, \psi) := \begin{cases} 1 & \text{if } \lambda|_{\mathcal{O}^*_{K_v}} = 1 \\ \int_{\varpi^{-c(\lambda)}\mathcal{O}^*_{K_v}} |x|^{-s}\lambda^{-1}(x)\psi(x)dx & \text{if } \lambda|_{\mathcal{O}^*_{K_v}} \neq 1 \end{cases}$$

with $\psi : K\backslash\mathbf{A}_K \to \mathbf{C}^*$ *the standard additive character given by* $\psi_{\mathbf{Q}} \circ \mathrm{Tr}_{K/\mathbf{Q}}$.

Using standard properties of epsilon factors and the functional equation for the Hecke L-values (and noting that $\overline{\phi_1/\phi_2} = (\phi_1/\phi_2)^c$ for this choice of (ϕ_1, ϕ_2)) we calculate:

Proposition 4.2 (Berger-Betina). *For* $\phi_2 : K^*\backslash\mathbf{A}_K^* \to \mathbf{C}^*$ *with* $\phi_{2,\infty}(z) = z^{-1}$ *put* $\phi_1 := \phi_2^c \cdot |\cdot|$. *Then*

$$\prod_{v|\mathfrak{M}_1\mathfrak{M}_2} c_v(\phi) = \frac{\pm 1}{\sqrt{\mathrm{Nm}(\mathfrak{M})}}$$

so

$$c(\phi) = -\frac{L(0, \overline{\phi_1/\phi_2})}{L(0, \phi_1/\phi_2)} W(\phi_1/\phi_2) \cdot \sqrt{\mathrm{Nm}(\mathfrak{M})} \cdot \frac{\pm 1}{\sqrt{\mathrm{Nm}(\mathfrak{M})}} = \pm 1,$$

and

$$\mathrm{res}(\mathrm{Eis}(\omega_\phi)) = \omega_\phi \pm \omega_{\phi^c}.$$

5. Denominator of Eisenstein class

For $\mathcal{O}$ the ring of integers in a sufficiently large finite extension E of $\mathbf{Q}_p$ (containing e.g. the values of $\phi_i|_{\mathbf{A}^*_{K,f}}, i = 1, 2$) and $\mathfrak{p}$ its maximal ideal one can prove the following p-integrality statement for the boundary class, as in [Ber08] Lemma 15 but without the assumption that $\mathfrak{M}_1\mathfrak{M}_2$ is coprime to p (note that $\phi_1(\mathfrak{M}_2) \in \mathcal{O}^*$ if $\mathfrak{p} \nmid \mathfrak{M}_2\mathcal{O}$):

Lemma 5.1.

$$\phi_1(\mathfrak{M}_2) \cdot [\omega_\phi] \in H^1(\partial\overline{X}_\Gamma, \mathcal{O})\backslash\mathfrak{p}H^1(\partial\overline{X}_\Gamma, \mathcal{O}).$$

Harder proved that $[\mathrm{Eis}(\omega_\phi)] \in H^1(X_\Gamma, E)$ see e.g. [Ber08] Proposition 13. We define the denominator of $[\mathrm{Eis}(\omega_\phi)]$ as follows:

$$\delta([\mathrm{Eis}(\omega_\phi)]) := \{a \in \mathcal{O} | a \cdot [\mathrm{Eis}(\omega_\phi)] \in H^1(X_\Gamma, \mathcal{O}) \hookrightarrow H^1(X_\Gamma, E)\}$$

By analogous calculations to those proving Proposition 18 in [Ber08] we can evaluate our Eisenstein cohomology class on the homology class corresponding to the modular symbol connecting the cusps 0 and ∞:

Proposition 5.2.

$$\int_0^\infty \mathrm{Eis}(\omega_\phi)\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\frac{H}{2}\right)\frac{dt}{t} = \frac{1}{2}\frac{L(0, \phi_1)L(0, \phi_2^{-1})}{L(0, \phi_1/\phi_2)}.$$

As the proposition calculates the evaluation pairing of the Eisenstein cohomology class against an integral cycle its value provides a lower bound on the denominator:

$$\delta([\mathrm{Eis}(\omega_\phi)]) \subseteq \left(\frac{1}{2}\frac{L(0,\phi_1/\phi_2)}{L(0,\phi_1)L(0,\phi_2^{-1})}\right).$$

Let $\Omega \in \mathbf{C}$ be the complex period of a Néron differential of an elliptic curve E defined over some number field such that E has complex multiplication by $\mathcal{O}_K$, E has good reduction at the place above p and $\overline{\omega}$ is a non-vanishing invariant differential on the reduced curve $\overline{E}$. We define

$$L^{\mathrm{int}}(1,\phi_2) := L^{\mathrm{int}}(0,\phi_1) := \frac{L(0,\phi_1)}{\Omega}$$

and

$$L^{\mathrm{int}}(1,\phi_2^-) := L^{\mathrm{int}}(0,\phi_1/\phi_2) := \frac{L(0,\phi_1/\phi_2)}{\Omega^2}.$$

We refer the reader to [Ber08] Theorem 3 for results on the p-integrality of these normalisations.

Theorem 5.3 (Berger-Betina). *For $\phi_2 : K^*\backslash\mathbf{A}_K^* \to \mathbf{C}^*$ with $\phi_{2,\infty}(z) = z^{-1}$ put $\phi_1 := \phi_2^c \cdot |\cdot|$ and assume $L^{\mathrm{int}}(1,\phi_2) \in \mathcal{O}^*$. Then*

$$\delta([\mathrm{Eis}(\omega_\phi)]) \subseteq \left(L^{\mathrm{int}}(1,\phi_2^-)\right).$$

6. Eisenstein congruences

To recap, we have constructed $[\mathrm{Eis}(\omega_\phi)] \in H^1(X_\Gamma, E)$ with

$$\mathrm{res}([\mathrm{Eis}(\omega_\phi)]) = [\omega_\phi] \pm [\omega_{\phi^c}] \in H^1(\partial\overline{X}_\Gamma, \mathcal{O})$$

if $\mathfrak{p} \nmid \mathfrak{M}_2$, and $\delta \cdot [\mathrm{Eis}(\omega_\phi)] \in H^1(X_\Gamma, \mathcal{O})/\mathrm{tors}$ for $(\delta) = \delta([\mathrm{Eis}(\omega_\phi)]) \subset \mathcal{O}$.

By the long exact sequence (3.1) we know

$$\delta \cdot \partial(\mathrm{res}([\mathrm{Eis}(\omega_\phi)]) = 0 \in H_c^2(X_\Gamma, \mathcal{O}).$$

However, $H_c^2(X_\Gamma, \mathcal{O})_{\mathrm{tors}}$ is often non-trivial, see the discussion in [Ber09] 4.4.

As explained in [Ber09] Proposition 9, if there exists $c_\phi \in H^1(X_\Gamma, \mathcal{O})$ with

$$\mathrm{res}(c_\phi) = \mathrm{res}([\mathrm{Eis}(\omega_\phi)]) \in H^1(\partial\overline{X}_\Gamma, \mathcal{O})$$

then $d_\phi := \delta \cdot (c_\phi - [\mathrm{Eis}(\omega_\phi)]) \in H_c^1(X_\Gamma, \mathcal{O})$ satisfies

$$d_\phi \equiv [\mathrm{Eis}(\omega_\phi)]\delta \mod \delta.$$

We describe in the following how we generalize the result on a p-integral lift of $\mathrm{res}([\mathrm{Eis}(\omega_\phi)])$ for ϕ_1/ϕ_2 unramified ([Ber09] Proposition 12) to Hecke characters ϕ_2 and $\phi_1 = \phi_2^c \cdot |\cdot|$ without restrictions on their conductors.

Lemma 6.1 ([Ber09] Lemma 16). *Suppose that we have an orientation reversing involution ι on X_Γ such that*

$$H^1(X_\Gamma, \mathcal{O}) \stackrel{\mathrm{res}}{\to} H^1(\partial\overline{X}_\Gamma, \mathcal{O})^\epsilon \subset H^1(\partial\overline{X}_\Gamma, \mathcal{O}),$$

where the superscript $\epsilon = \pm 1$ indicates the ϵ-eigenspace for ι.

Then the restriction map is surjective onto $H^1(\partial\overline{X}_\Gamma, \mathcal{O})^\epsilon$.

Proof. It suffices to prove the surjectivity of

$$H^1(X_\Gamma, \mathcal{O}/\mathfrak{p}^n) \stackrel{\text{res}}{\to} H^1(\partial \overline{X}_\Gamma, \mathcal{O}/\mathfrak{p}^n)^\epsilon$$

for each n.

We take the opportunity to repeat a correction to the proof of [Ber09] Lemma 16: Instead of the Poincaré pairing invoked there we need to use the Pontryagin duality pairing (which is non-degenerate also on torsion).

We have the following diagram:

$$\begin{array}{ccccc}
H^2_c(X, E)^{\pm} & \times & H^1(X_\Gamma, E/\mathcal{O})^{\mp} & \longrightarrow & E/\mathcal{O} \\
\uparrow \partial & & \downarrow \text{res} & & \\
H^1(\partial X_\Gamma, \mathcal{O})^{\pm} & \times & H^1(\partial X_\Gamma, E/\mathcal{O})^{\mp} & \longrightarrow & E/\mathcal{O} \\
\uparrow \text{res} & & \downarrow \partial & & \\
H^1(X_\Gamma, \mathcal{O})^{\pm} & \times & H^2_c(X_\Gamma, E/\mathcal{O})^{\mp} & \longrightarrow & E/\mathcal{O}
\end{array}$$

Here the vertical sequences are exact, the horizontal pairings are perfect and res and ∂ are adjoint, i.e.

$$\langle \text{res}(x), y\rangle = \langle x, \partial(y)\rangle$$

for all $x \in H^1(X_\Gamma, \mathcal{O}), y \in H^1(\partial X_\Gamma, E/\mathcal{O})$. As the pairings are induced from the cup product and evaluation on the fundamental class (which is inverted by the involution ι) the $+1$- and -1-eigenspaces are dual to each other.

The adjointness of res and ∂ and the perfectness of the pairings implies for all n that

$$\text{im}(H^1(X_\Gamma, \mathcal{O}/\varpi^n) \stackrel{\text{res}}{\to} H^1(X_\Gamma, \mathcal{O}/\varpi^n)) = \text{im}(\text{res})^\perp.$$

The assumption of the Lemma implies

$$H^1(\partial \overline{X}_\Gamma, \mathcal{O}/\varpi^n)^\epsilon \subset \text{im}(\text{res})^\perp,$$

so together this proves

$$H^1(\partial \overline{X}_\Gamma, \mathcal{O}/\varpi^n)^\epsilon \subset \text{im}(\text{res}).$$

□

Theorem 6.2 (Serre [Ser70] Théorème 9)**.** *For* $\Gamma = \text{SL}_2(\mathcal{O}_K)$ *and* ι *induced by* $(z, r) \in \mathbf{H}_3 \mapsto (\overline{z}, r)$ *one has*

$$H^1(X_\Gamma, \mathcal{O}) \stackrel{\text{res}}{\to} H^1(\partial \overline{X}_\Gamma, \mathcal{O})^- \subset H^1(\partial \overline{X}_\Gamma, \mathcal{O}).$$

Let $\mathbf{T}$ (resp. $\mathbf{T}^0$) be the $\mathcal{O}-$algebra generated by the Hecke operators T_v for $v \nmid \mathfrak{M}_1\mathfrak{M}_2$ acting on $H^1(X_\Gamma, \mathcal{O})$ (resp. $H^1_!(X_\Gamma, \mathcal{O}) := \text{im}(H^1_c(\overline{X}_\Gamma, \mathcal{O}) \to H^1(\overline{X}_\Gamma, \mathcal{O}))$. Let $\mathfrak{m} \subset \mathbf{T}$ be the maximal ideal containing the Eisenstein ideal J generated by $\{T_{\pi_v} - \phi_2(\pi_v) - \phi_2^c(\pi_v)\}$ (and $J^0 \subset \mathbf{T}^0$ defined analogously).

In the following we assume $p \nmid \#(\mathcal{O}/\mathfrak{M}_2)^*$. We write φ_2 for the Hecke character with $\varphi_{2,\infty}(z) = z^{-1}$ such that the p-adic Galois characters $\tilde{\varphi}_2$ and $\tilde{\phi}_2^c$ are congruent modulo $\mathfrak{p}$ and let $\tilde{J}$ be the ideal generated by $\{T_{\pi_v} - \varphi_2(\pi_v) - \varphi_2^c(\pi_v)\}$.

Theorem 6.3 (Berger -Betina)**.** *For* ι *induced by* $(z, r) \in \mathbf{H}_3 \mapsto (\overline{z}, r)$ *one has*

$$H^1(X_\Gamma, \mathcal{O})_\mathfrak{m} \stackrel{\text{res}}{\to} H^1(\partial \overline{X}_\Gamma, \mathcal{O})^{\pm}_\mathfrak{m} \subset H^1(\partial \overline{X}_\Gamma, \mathcal{O})_\mathfrak{m}.$$

Sketch of proof. Under our assumptions we have

$$H^1(\partial\overline{X}_\Gamma, \mathcal{O})_\mathfrak{m} \subset H^1(\partial\overline{X}_\Gamma, E)[J] \oplus H^1(\partial\overline{X}_\Gamma, E)[\tilde{J}].$$

By Theorem 3.1 and Casselman (3.3) we know $H^1(\partial\overline{X}_\Gamma, E)[J] = \omega_\phi \cdot E \oplus \omega_{\phi^c} \cdot E$. Since $\iota(\omega_\phi) = \omega_{\phi^c}$ Proposition 4.2 proves

$$H^1(X_\Gamma, E)[J] \stackrel{\text{res}}{\to} H^1(\partial\overline{X}_\Gamma, E)^\pm.$$

The analogous statement (with the same sign) applies to $H^1(X_\Gamma, E)[\tilde{J}]$. □

Together with the analogue of Lemma 6.1 this allows us to prove

$$H^1(X_\Gamma, \mathcal{O})_\mathfrak{m} \stackrel{\text{res}}{\to} H^1(\partial\overline{X}_\Gamma, \mathcal{O})^\pm_\mathfrak{m}$$

is surjective. As $\text{res}([\text{Eis}(\omega_\phi)]) = [\omega_\phi] \pm [\omega_{\phi^c}] \in H^1(\partial\overline{X}_\Gamma, \mathcal{O})^\pm_\mathfrak{m}$ this provides us with an integral lift to $H^1(X_\Gamma, \mathcal{O})$ and we can apply [Ber09] Proposition 9 to prove:

Theorem 6.4 (Berger-Betina)**.** *Assume* $\mathfrak{p} \nmid \mathfrak{M}_2\#(\mathcal{O}/\mathfrak{M}_2)^*$. *If* $L^{\text{int}}(1, \phi_2) \in \mathcal{O}^*$ *then we have an* $\mathcal{O}$*-algebra surjection*

$$\mathbf{T}^0/J^0 \twoheadrightarrow \mathcal{O}/L^{\text{int}}(1, \phi_2^-).$$

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