Diffusion wave phenomena and L^p decay estimates of solutions of compressible viscoelastic system

Yusuke Ishigaki
Department of Mathematics,
Tokyo Institute of Technology,
Tokyo 152-8550, Japan,
E-mail:ishigaki.y.aa@m.titech.ac.jp

1 Introduction

This article is the summary of [9]. We consider the system for a motion of compressible viscoelastic fluids:

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1}$$
$$\rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \mathrm{div} + \nabla p(\rho) = \beta^2 \mathrm{div}(\rho F^\top F), \tag{1.2}$$
$$\partial_t F + v \cdot \nabla F = \nabla v F \tag{1.3}$$

in $\mathbb{R}^3$. Here $\rho = \rho(x,t)$, $v = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, and $F = (F^{jk}(x,t))_{1\leq j,k\leq 3}$ denote the unknown density, the velocity field, and the deformation tensor, respectively, at position $x \in \mathbb{R}^3$ and time $t \geq 0$; $P = P(\rho)$ is the given pressure; ν and ν' are the viscosity coefficients satisfying

$$\nu > 0, \ 2\nu + 3\nu' \geq 0;$$

$\beta > 0$ is the strength of the elasticity. In particular, if we set $\beta = 0$, the system (1.1)–(1.3) becomes the usual compressible Navier-Stokes equation. We assume that $P'(1) > 0$, and we set $\gamma = \sqrt{P'(1)}$.

The system (1.1)–(1.3) is considered under the initial condition

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0). \tag{1.4}$$

We also impose the following conditions for ρ_0 and F_0:

$$\rho_0 \mathrm{det} F_0 = 1, \tag{1.5}$$
$$\sum_{m=1}^{3}(F_0^{ml}\partial_{x_m}F_0^{jk} - F_0^{mk}\partial_{x_m}F_0^{jl}) = 0, \ j,k,l = 1,2,3, \tag{1.6}$$
$$\mathrm{div}(\rho_0{}^\top F_0) = 0. \tag{1.7}$$

It follows from [5, Appendix A] and [18, Proposition.1] that the quantities (1.5)–(1.7) are invariant for $t \geq 0$:

$$\rho \det F = 1, \tag{1.8}$$

$$\sum_{m=1}^{3}(F^{ml}\partial_{x_m}F^{jk} - F^{mk}\partial_{x_m}F^{jl}) = 0,\ \ j,k,l = 1,2,3. \tag{1.9}$$

$$\mathrm{div}(\rho^{\top}F) = 0. \tag{1.10}$$

The purpose of this article is to study the large time behavior of solutions of the problem (1.1)–(1.7) around a motionless state $(1, 0, I)$, where I is the 3×3 identity matrix. Especially, we are interested how the elastic force $\beta^2\mathrm{div}(\rho F^{\top}F)$ works.

The system (1.1)–(1.3) is obtained from motion of compressible viscoelastic fluid in macroscopic scale.

In the case $\beta = 0$, the large time behavior of the solutions around $(\rho, v) = (1, 0)$ has been investigated so far. Matsumura and Nishida [15] proved the global in time existence of the solutions of the problem (1.1)–(1.4) provided that the initial perturbation is sufficiently small in $H^3 \cap L^1$, and derived the following L^2-decay estimates:

$$\|\nabla^k(\phi(t), m(t))\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}},\ \ k = 0, 1,$$

where $(\phi, m) = (\rho - 1, \rho v)$. Hoff and Zumbrun [2] established the following L^p $(1 \leq p \leq \infty)$ decay estimates in $\mathbb{R}^n$, $n \geq 2$:

$$\|(\phi(t), m(t))\|_{L^p} \leq \begin{cases} C(1+t)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{n-1}{4}\left(1-\frac{2}{p}\right)}L(t), & 1 \leq p < 2, \\ C(1+t)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}, & 2 \leq p \leq \infty, \end{cases}$$

where $L(t) = \log(1+t)$ when $n = 2$, and $L(t) = 1$ when $n \geq 3$. Furthermore, Hoff-Zumbrun[2] derived the following L^p $(1 \leq p \leq \infty)$ decay estimates and asymptotic properties:

$$\|(\phi(t), m(t))\|_{L^p} \leq \begin{cases} C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)}, & 1 \leq p < 2, \\ C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, & 2 \leq p \leq \infty. \end{cases}$$

$$\begin{aligned} &\left\|\left((\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t}\mathcal{P}(\xi)\hat{m}_0\right)\right)\right)\right\|_{L^p} \quad 2 \leq p \leq \infty, \\ &\quad \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)}, \end{aligned}$$

where $(\phi(t), m(t)) = (\rho(t) - 1, \rho(t)v(t))$ and $\mathcal{P}(\xi) = I - \frac{\xi^{\top}\xi}{|\xi|^2}$, $\xi \in \mathbb{R}^3$. Here the symbol $\hat{\cdot}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ denotes the inverse

Fourier transform. The authors of [2] showed that the hyperbolic aspect of sound wave makes the decay rate of the solution slower than the heat kernel when $1 \leq p < 2$. On the other hand, if $2 < p \leq \infty$, the compressible part of the solution $(\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t}\hat{\mathcal{P}}(\xi)\hat{m}_0\right)\right)$ converges to 0 faster than the heat kernel.

We next give the development of the works in the case $\beta > 0$. The local in time existence of the strong solution of the initial value problem (1.1)–(1.7) was shown by Hu and Wang [4]. The global existence of the strong solution of the initial value problem (1.1)–(1.7) was proved by Hu and Wang [5], Qian and Zhang [18], and Hu and Wu [6], provided that the initial perturbation $(\rho_0 - 1, v_0, F_0 - I)$ is sufficiently small. Hu-Wu[6] and Li-Wei-Wao[12] established the following L^p $(2 \leq p \leq \infty)$ decay estimates:

$$\|u(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)},$$

where $u(t) = (\phi(t), w(t), G(t)) = (\rho(t), w(t), F(t)) - (1, 0, I)$. This shows the diffusive aspect of the system (1.1)–(1.3) at least. However the hyperbolic aspects of elastic shear wave and sound wave does not appear. We will clarify the diffusion wave phenomena caused by interaction of three properties; sound wave, viscous diffusion and elastic shear wave and improve the results obtained in [6, 12]. We also refer to [3, 14, 25] in recent progresses.

In view of the results in [2], it is expected that the system (1.1)–(1.3) has the diffusion wave phenomena affected by the interaction of the sound wave, viscous diffusion and elastic shear wave. In fact, we characterize above phenomena by showing that if the initial perturbation $u_0 = (\rho_0 - 1, v_0, F_0 - I)$ is sufficiently small in $L^1 \cap H^3$, then the global strong solution satisfies the following L^p decay estimate

$$\|(\rho(t) - 1, v(t), F(t) - I)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)}, \ 1 < p \leq \infty, \ t \geq 0.$$

This result improves the decay rate of the L^p norm of the perturbation u obtained in [6, 12] for $p > 2$. Moreover the above decay rate might be optimal.

We give an outline of the proof of the main result. Since the constraints (1.8)–(1.10) are nonlinear, straightforward application of the semigroup theory does not work well. To overcome this obstacle, we construct a nonlinear transform which makes the constraint (1.10) a linear one. We first give a displacement vector $\tilde{\psi} = x - X \in \mathbb{R}^3$ used in [19, 22], where $X = X(x,t)$ is the inverse of the material coordinate. We next make use of the transform $\psi = \tilde{\psi} - (-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))$. Here $h(\nabla\tilde{\psi})$ is a function satisfying $h(\nabla\tilde{\psi}) = O(|\nabla\tilde{\psi}|^2), |\nabla\psi| \ll 1$, and $(-\Delta)^{-1}$ is the nonlocal transform defined as $(-\Delta)^{-1} = \mathcal{F}^{-1}|\xi|^{-2}\mathcal{F}$. Here $\mathcal{F}$ denotes the Fourier transform. This follows that the constraint (1.10) becomes the linear condition $\phi + \mathrm{tr}(\nabla\psi) = \phi + \mathrm{div}\psi = 0$. Furthermore, the decay estimate of the L^p $(1 < p \leq \infty)$ norm of $u = (\phi, w, G)$ is obtained from $U = (\phi, w, \nabla\psi)$. Consequently, the L^p decay estimate can be obtained by employing the following integral equation

$$U(t) = e^{-tL}U(0) + \int_0^t e^{-(t-s)L}N(U)\mathrm{d}s.$$

Here L is the linearized operator around $(1, 0, I)$; $N(U) = (N_1(U), N_2(U), N_3(U))$ is a nonlinearity such that $N_1 + \mathrm{tr}N_3 = 0$. Indeed we find that $e^{-tL}U(0)$ and $\int_0^t e^{-(t-s)L}N(U)\mathrm{d}s$ decay as $t \to \infty$ in L^p for $p > \frac{5}{4}$, provided that the linear constraints for U and $N(U)$ hold.

This article is organized as follows. In Section 2 we state the main result of this article on the L^p decay estimates. In Section 3 we give an outline of the proof of the main result.

2 Main Result

In this section we summerize the results in [9].

We set $u(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t), F(t) - I)$. Then $u(t)$ satisfies the following initial value problem

$$\begin{cases} \partial_t\phi + \mathrm{div}w = g_1, \\ \partial_t w - \nu\Delta w - \tilde{\nu}\nabla\mathrm{div}w + \gamma^2\nabla\phi - \beta^2\mathrm{div}G = g_2, \\ \partial_t G - \nabla w = g_3, \\ \nabla\phi + \mathrm{div}^\top G = g_4, \\ u|_{t=0} = u_0 = (\phi_0, w_0, G_0). \end{cases} \tag{2.1}$$

Here $g_j, j = 1, 2, 3, 4$, denote the nonlinear terms;

$$\begin{aligned} g_1 &= -\mathrm{div}(\phi w), \\ g_2 &= -w \cdot \nabla w + \frac{\phi}{1+\phi}(-\nu\Delta w - \tilde{\nu}\nabla\mathrm{div}w + \gamma^2\nabla\phi) - \frac{1}{1+\phi}\nabla Q(\phi) \\ &\quad -\frac{\beta^2\phi}{1+\phi}\mathrm{div}G + \frac{\beta^2}{1+\phi}\mathrm{div}(\phi G + G^\top G + \phi G^\top G), \\ g_3 &= -w \cdot \nabla G + \nabla w G, \\ g_4 &= -\mathrm{div}(\phi^\top G), \end{aligned}$$

where

$$Q(\phi) = \phi^2 \int_0^1 P''(1+s\phi)\mathrm{d}s, \ \nabla Q = O(\phi)\nabla\phi$$

for $|\phi| \ll 1$.

We recall the L^2 decay estimates obtained in [12].

Proposition 2.1. ([12]) *Let $u_0 \in H^N$, $N \geq 3$. There is a positive number ϵ_0 such that if u_0 satisfies $\|u_0\|_{L^1} + \|u_0\|_{H^3} \leq \epsilon_0$, then there exists a unique solution $u(t) \in C([0,\infty); H^N)$ of the problem* (2.1)*, and $u(t) = (\phi(t), w(t), G(t))$ satisfies*

$$\|u(t)\|_{H^N}^2 + \int_0^t (\|\nabla\phi(s)\|_{H^{N-1}}^2 + \|\nabla w(s)\|_{H^N}^2 + \|\nabla G(s)\|_{H^{N-1}}^2)\mathrm{d}s \leq C\|u_0\|_{H^N}^2,$$

$$\|\nabla^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|u_0\|_{L^1} + \|u_0\|_{H^N})$$

for $k = 0, 1, 2, \ldots, N-1$ and $t \geq 0$.

We next state the main result of this article which reflects an effect of hyperbolic aspects of diffusion waves.

Theorem 2.2. (i) *Let $2 \leq p \leq \infty$. Assume that ϕ_0, G_0, and F_0^{-1} satisfy $\nabla\phi_0 - \mathrm{div}^\top(I+G_0)^{-1} = 0$ and $F_0^{-1} = \nabla X_0$ for some vector field X_0. There is a positive number ϵ such that if $u_0 = (\phi_0, w_0, G_0)$ satisfies $\|u_0\|_{H^3} \leq \epsilon$ and $u_0 \in L^1$, then there exists a unique solution $u(t) \in C([0,\infty); H^3)$ of the problem* (2.1)*, and $u(t) = (\phi(t), w(t), G(t))$ satisfies*

$$\|u(t)\|_{L^p} \leq C(p)(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)}(\|u_0\|_{L^1} + \|u_0\|_{H^3})$$

uniformly for $t \geq 0$. Here $C(p)$ is a positive constant depending only on p.

(ii) *Let $1 < p < 2$. Assume that ϕ_0, G_0, and F_0^{-1} satisfy $\nabla\phi_0 - \mathrm{div}^\top(I + G_0)^{-1} = 0$ and $F_0^{-1} = \nabla X_0$ for some vector field X_0. There is a positive number ϵ_p such that if $u_0 = (\phi_0, w_0, G_0)$ satisfies $\|u_0\|_{H^3} \leq \epsilon_p$ and $u_0 \in L^1$, then there exists a unique solution $u(t) \in C([0,\infty); H^3)$ of the problem* (2.1)*, and $u(t) = (\phi(t), w(t), G(t))$ satisfies*

$$\|u(t)\|_{L^p} \leq C(p)(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}(\|u_0\|_{L^1} + \|u_0\|_{L^p} + \|u_0\|_{H^3})$$

uniformly for $t \geq 0$. Here $C(p)$ is a positive constant depending only on p.

Remark 2.3. Since $\frac{1}{2}\left(1 - \frac{2}{p}\right) > 0$ for $2 < p \leq \infty$, Theorem 2.2 (i) follows that the L^p norm of the perturbation $u = (\phi, w, G)$ tends to 0 faster than the heat kernel as $t \to \infty$. This gives the hyperbolic aspect of the elastic force $\beta^2\mathrm{div}(\rho F^\top F)$ which leads to the improvement of the result in [12].

3 Proof of Theorem 2.2

In this section, we give an outline of the proof of Theorem 2.2. Since the global in time existence and the L^2 decay estimates of higher order derivatives are proved by Proposition 2.1, it suffice to investigate the L^p decay estimates.

We first rewrite the problem (2.1) into a specific form to prove Theorem 2.2.

Let $x = x(X,t)$ be the material coordinate defined as the solution of the flow map:

$$\begin{cases} \dfrac{dx}{dt}(X,t) = v(x(X,t),t), \\ x(X,0) = X, \end{cases}$$

and we denote its inverse by $X = X(x,t)$. According to [1, 22], F is given by $F = \frac{\partial x}{\partial X}$. It is shown in [19] that its inverse F^{-1} is written as $F^{-1}(x,t) = \nabla X(x,t)$ if F_0^{-1} has the form $F_0^{-1} = \nabla X_0$. We set $\tilde{\psi} = x - X$. Then $\tilde{\psi}$ solves

$$\partial_t \tilde{\psi} - v = -v \cdot \nabla \tilde{\psi},$$

and satisfies

$$G = \nabla \tilde{\psi} + h(\nabla \tilde{\psi}), \tag{3.1}$$

where $h(\nabla\tilde{\psi}) = (I - \nabla\tilde{\psi})^{-1} - I - \nabla\tilde{\psi}$.

We note that (3.1) is equivalent to

$$\nabla\tilde{\psi} = I - (I+G)^{-1}. \tag{3.2}$$

We have the following estimates for G and $\nabla\tilde{\psi}$.

Lemma 3.1. *Assume that G and $\tilde{\psi}$ satisfy* (3.1)*. There is a positive number δ_0 such that if $\|G\|_{H^3} \le \min\{1, \delta_0\}$, the following inequalities hold:*

$$C^{-1}\|\nabla\tilde{\psi}\|_{L^p} \le \|G\|_{L^p} \le C\|\nabla\tilde{\psi}\|_{L^p}, \ 1 \le p \le \infty, \tag{3.3}$$

$$\|\nabla^2\tilde{\psi}\|_{L^2} \le C\|\nabla G\|_{L^2}, \tag{3.4}$$

$$\|\nabla^3\tilde{\psi}\|_{L^2} \le C(\|\nabla G\|_{H^1}^2 + \|\nabla^2 G\|_{L^2}), \tag{3.5}$$

$$\|\nabla^4\tilde{\psi}\|_{L^2} \le C(\|\nabla G\|_{H^1}\|\nabla^2 G\|_{H^1} + \|\nabla^3 G\|_{L^2}). \tag{3.6}$$

See [9, Lemma 4.2.] for a proof of Proposition 3.1.

Based on Lemma 3.1, we consider $\tilde{\psi}$ instead of G. In terms of $\tilde{U} = (\phi, w, \nabla\tilde{\psi})$, the problem (2.1) is transformed into

$$\begin{cases} \partial_t \phi + \mathrm{div} w = f_1, \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \Delta \tilde{\psi} = f_2, \\ \partial_t \nabla \tilde{\psi} - \nabla w = f_3, \\ \nabla \phi + \nabla \mathrm{div} \tilde{\psi} = f_4, \\ \tilde{U}|_{t=0} = \tilde{U}_0 = (\phi_0, w_0, \nabla \tilde{\psi}_0). \end{cases} \tag{3.7}$$

Here $f_j, j = 1, 2, 3, 4$, denote the nonlinear terms;

$$\begin{aligned} f_1 &= g_1, \\ f_2 &= g_2 + \beta^2 \mathrm{div} h(\nabla \tilde{\psi}), \\ f_3 &= -\nabla(w \cdot \nabla \tilde{\psi}), \\ f_4 &= -\mathrm{div}^\top (\phi \nabla \tilde{\psi} + (1+\phi) h(\nabla \tilde{\psi})). \end{aligned}$$

We next introduce ψ by $\psi = \tilde{\psi} - (-\Delta)^{-1} \mathrm{div}^\top (\phi \nabla \tilde{\psi} + (1+\phi) h(\nabla \tilde{\psi}))$, where $(-\Delta)^{-1} = \mathcal{F}^{-1} |\xi|^{-2} \mathcal{F}$, and set $\Psi = \nabla \psi$. By this transformation, the nonlinear constraint $\nabla \phi + \nabla \mathrm{div} \tilde{\psi} = f_4$ is transformed into the linear constraint $\phi + \mathrm{div} \psi = 0$; and the problem (3.7) is rewritten as

$$\begin{cases} \partial_t \phi + \mathrm{div} w = N_1(U), \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} \Psi = N_2(U), \\ \partial_t \Psi - \nabla w = N_3(U), \\ \phi + \mathrm{tr} \Psi = 0, \ \Psi = \nabla \psi, \\ U|_{t=0} = U_0 = (\phi_0, w_0, \Psi_0). \end{cases} \tag{3.8}$$

Here $N_j(U), j = 1, 2, 3$, denote the nonlinear terms;

$$\begin{aligned} N_1(U) &= f_1, \\ N_2(U) &= f_2 - \beta^2 \mathrm{div}^\top (\phi \nabla \tilde{\psi} + (1+\phi) h(\nabla \tilde{\psi})), \\ N_3(U) &= -\nabla(w \cdot \nabla \tilde{\psi}) - \nabla(-\Delta)^{-1} \nabla \mathrm{div}(\phi w) - \nabla(-\Delta)^{-1} \nabla \mathrm{div}(w \cdot \nabla \tilde{\psi}). \end{aligned}$$

We note that N_1 and N_3 satisfy $N_1 + \mathrm{tr} N_3 = 0$. The relations between ψ and $\tilde{\psi}$ are given as follows.

Lemma 3.2. (i) *Let $\tilde{U}_0$ and U_0 be the ones as in* (3.7) *and* (3.8)*, respectively. If ϕ_0 and $\tilde{\psi}_0$ satisfy $\nabla \phi_0 + \nabla \mathrm{div} \tilde{\psi}_0 = 0$, then it holds $U_0 = \tilde{U}_0 = (\phi_0, w_0, \nabla \tilde{\psi}_0)$.*

(ii) *There is a positive number* δ_0 *such that the following assertion holds true. Let*

$$\phi \in C([0,\infty);H^3),\ \ \psi \in C([0,\infty);H^4).$$

If $\|\phi\|_{C([0,\infty);H^3)}+\|\psi\|_{C([0,\infty);H^4)} \le \delta_0$*, then there uniquely exists* $\tilde\psi \in C([0,\infty);H^4)$ *such that*

$$\begin{aligned} &\|\tilde\psi\|_{C([0,\infty);H^4)} \le \sqrt{\delta_0}, \\ &\tilde\psi = \psi + (-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde\psi + (1+\phi)h(\nabla\tilde\psi)). \end{aligned} \tag{3.9}$$

(iii) *Let* $1 < p < \infty$*. There is a positive number* δ_p *such that if* $\|\phi\|_{C([0,\infty);H^3)}+\|\nabla\tilde\psi\|_{C([0,\infty);H^3)} \le \min\{\delta_0,\delta_p\}$*, the following inequalities hold for* $t \ge 0$*:*

$$C_p^{-1}\|\nabla\tilde\psi(t)\|_{L^p} \le \|\nabla\psi(t)\|_{L^p} \le C_p\|\nabla\tilde\psi(t)\|_{L^p}. \tag{3.10}$$

(iv) *There is a positive number* δ_1 *such that if* $\|\phi\|_{C([0,\infty);H^3)}+\|\nabla\tilde\psi\|_{C([0,\infty);H^3)} \le \min\{\delta_0,\delta_1\}$*, the following inequalities hold for* $t \ge 0$*:*

$$\begin{aligned} \|\nabla\tilde\psi(t)\|_{L^\infty} \le\ & C(\|\phi(t)\|_{L^\infty} + \|\nabla\psi(t)\|_{L^\infty}) \\ & +C(\|\nabla\phi(t)\|_{H^1} + \|\nabla^2\tilde\psi(t)\|_{H^1})^2, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \|\nabla^2\psi(t)\|_{L^2} \le\ & C\|\nabla^2\tilde\psi(t)\|_{L^2} \\ & +C(\|\phi(t)\|_{H^2} + \|\nabla\tilde\psi(t)\|_{H^2})\|\nabla\tilde\psi(t)\|_{H^2}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \|\nabla^3\psi(t)\|_{L^2} \le\ & C(1+\|\phi(t)\|_{H^2} + \|\nabla\tilde\psi(t)\|_{H^2})\|\nabla^3\tilde\psi(t)\|_{L^2} \\ & +C(\|\nabla\phi(t)\|_{H^1} + \|\nabla^2\tilde\psi(t)\|_{H^1})\|\nabla\tilde\psi(t)\|_{H^2}, \end{aligned} \tag{3.13}$$

$$\|\nabla^4\psi(t)\|_{L^2} \le C\|\nabla^4\tilde\psi(t)\|_{L^2} + C(\|\phi(t)\|_{H^3} + \|\nabla\tilde\psi(t)\|_{H^3})^2. \tag{3.14}$$

See [9, Lemma 4.] for a proof of Proposition 3.2.

Remark 3.3. Due to the restriction $p > 1$ in Lemma 3.2 (iii), the case $p = 1$ is removed in Theorem 2.2.

We rewite the problem (3.8) into the following form:

$$\begin{cases} \partial_t U + LU = N(U), \\ \phi + \mathrm{div}\psi = 0, \\ U|_{t=0} = U_0, \end{cases} \tag{3.15}$$

where

$$L = \begin{pmatrix} 0 & \mathrm{div} & 0 \\ \gamma^2\nabla & -\nu\Delta - \tilde\nu\nabla\mathrm{div} & -\beta^2\mathrm{div} \\ 0 & -\nabla & 0 \end{pmatrix},\ \ N(U) = \begin{pmatrix} N_1(U) \\ N_2(U) \\ N_3(U) \end{pmatrix}.$$

We introduce the low-high frequency decomposition of $U(t)$. Let $\hat{\varphi}_1, \hat{\varphi}_\infty \in C^\infty(\mathbb{R}^3)$ be cut-off functions such that

$$\hat{\varphi}_1(\xi) = \begin{cases} 1 & |\xi| \le \frac{M_1}{2}, \\ 0 & |\xi| \ge \frac{M_1}{\sqrt{2}}, \end{cases} \quad \hat{\varphi}_1(-\xi) = \hat{\varphi}_1(\xi),$$

$$\hat{\varphi}_\infty(\xi) = 1 - \hat{\varphi}_1(\xi),$$

where

$$M_1 = \min\left\{ \frac{\beta}{\nu}, \frac{\sqrt{\beta^2 + \gamma^2}}{\nu + \tilde{\nu}} \right\}.$$

We define the operators P_1 and P_∞ on L^2 by

$$P_1 u = \mathcal{F}^{-1}(\hat{\varphi}_1 \hat{u}), \;\; P_\infty u = \mathcal{F}^{-1}(\hat{\varphi}_\infty \hat{u}) \text{ for } u \in L^2.$$

The solution $U(t)$ of (3.15) is decomposed as

$$U(t) = U_1(t) + U_\infty(t), \;\; U_1(t) = P_1 U(t), \;\; U_\infty(t) = P_\infty U(t).$$

We see that $U_j(t) = (\phi_j(t), w_j(t), \nabla\psi_j(t)), \; j = 1, \infty$ satisfy the following integral equations;

$$\begin{cases} U_j(t) = e^{-tL} U_j(0) + \displaystyle\int_0^t e^{-(t-s)L} P_j N(U(s)) \mathrm{d}s, \\ \phi_j + \mathrm{div}\psi_j = 0, \\ U_j(0) = P_j U_0, \;\; \phi_0 + \mathrm{div}\psi_0 = 0. \end{cases} \tag{3.16}$$

Since U holds the linear constraint $\phi + \mathrm{div}\psi = 0$, the semigroup $e^{-tL}U_0$ decays as $t \to 0$ in L^p for $p > \frac{5}{4}$. Indeed, we obtain the following L^p estimates for U_1 and U_∞.

Lemma 3.4. *If $\phi_0 + \mathrm{div}\psi_0 = 0$, then the following estimates hold:*
(i) $\|e^{-tL}U_1(0)\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)} \|U_0\|_{L^1}, \; 1 \le p \le \infty.$
(ii) $\|e^{-tL}U_\infty(0)\|_{L^p} \le Ce^{-ct}\|U_0\|_{L^p}, \; 1 < p < \infty.$
(iii) $\|e^{-tL}U_\infty(0)\|_{L^\infty} \le Ce^{-ct}\|U_0\|_{H^2}.$

Proof To prove Lemma 3.4, we use the following expression of the semigroup $e^{-tL}U_0$:

$$e^{-tL}U_0 = \begin{pmatrix} \hat{\phi}(\xi,t) \\ \hat{w}(\xi,t) \\ \hat{\Psi}(\xi,t) \end{pmatrix} = \begin{pmatrix} \hat{K}^{11}(\xi,t) & \hat{K}^{12}(\xi,t) & \hat{K}^{13}(\xi,t) \\ \hat{K}^{21}(\xi,t) & \hat{K}^{22}(\xi,t) & \hat{K}^{23}(\xi,t) \\ \hat{K}^{31}(\xi,t) & \hat{K}^{32}(\xi,t) & \hat{K}^{33}(\xi,t) \end{pmatrix} \begin{pmatrix} \hat{\phi}_0(\xi) \\ \hat{w}_0(\xi) \\ \hat{\Psi}_0(\xi) \end{pmatrix}. \tag{3.17}$$

Here

$$\begin{aligned}
\hat{K}^{11}(\xi,t) =&\frac{\mu_3(\xi)e^{\mu_4(\xi)t}-\mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)},\\
\hat{K}^{12}(\xi,t) =&-i\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}{}^{\top}\xi,\\
\hat{K}^{13}(\xi,t) =&0,
\end{aligned}$$

$$\begin{aligned}
\hat{K}^{21}(\xi,t) =&-i\gamma^2\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\xi,\\
\hat{K}^{22}(\xi,t) =&\frac{\mu_1(\xi)e^{\mu_1(\xi)t}-\mu_2(\xi)e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^{\top}\xi}{|\xi|^2}\right)\\
&+\frac{\mu_3(\xi)e^{\mu_3(\xi)t}-\mu_4(\xi)e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^{\top}\xi}{|\xi|^2},\\
\hat{K}^{31}(\xi,t) =&0,\\
\hat{K}^{33}(\xi,t) =&\frac{\mu_1(\xi)e^{\mu_2(\xi)t}-\mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^{\top}\xi}{|\xi|^2}\right)\\
&+\frac{\mu_3(\xi)e^{\mu_4(\xi)t}-\mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^{\top}\xi}{|\xi|^2};
\end{aligned}$$

$\hat{K}^{23}(\xi,t)\hat{\Psi}_0(\xi)$ and $\hat{K}^{32}(\xi,t)\hat{w}_0(\xi)$ are defined by

$$\begin{aligned}
\hat{K}^{23}(\xi,t)\hat{\Psi}_0(\xi) =&i\beta^2\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^{\top}\xi}{|\xi|^2}\right)\hat{\Psi}_0(\xi)\xi\\
&+i\beta^2\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^{\top}\xi}{|\xi|^2}\hat{\Psi}_0(\xi)\xi,
\end{aligned}$$

$$\begin{aligned}
\hat{K}^{32}(\xi,t)\hat{w}_0(\xi) =&i\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^{\top}\xi}{|\xi|^2}\right)\hat{w}_0(\xi)^{\top}\xi\\
&+i\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^{\top}\xi}{|\xi|^2}\hat{w}_0(\xi)^{\top}\xi,
\end{aligned}$$

where $\mu_j(\xi),\ j=1,2,3,4$, are given by

$$
\begin{aligned}
\mu_1(\xi) &= \frac{-\nu|\xi|^2+\sqrt{\nu^2|\xi|^4-4\beta^2|\xi|^2}}{2},\\
\mu_2(\xi) &= \frac{-\nu|\xi|^2-\sqrt{\nu^2|\xi|^4-4\beta^2|\xi|^2}}{2},\\
\mu_3(\xi) &= \frac{-(\nu+\tilde{\nu})|\xi|^2+\sqrt{(\nu+\tilde{\nu})^2|\xi|^4-4(\beta^2+\gamma^2)|\xi|^2}}{2},\\
\mu_4(\xi) &= \frac{-(\nu+\tilde{\nu})|\xi|^2-\sqrt{(\nu+\tilde{\nu})^2|\xi|^4-4(\beta^2+\gamma^2)|\xi|^2}}{2}.
\end{aligned}
$$

The derivation of the above expression can be confirmed in [9, Appendix A.].

Since $\mu_j,\ j=1,2,3,4$ have the following properties

$$
\begin{gathered}
\mu_j(\xi)\sim-\frac{\nu}{2}|\xi|^2+i(-1)^{j+1}\beta|\xi|,\ \text{for } |\xi|\ll 1,\ j=1,2,\\
\mu_1(\xi)\sim-\frac{\beta}{\nu},\ \mu_2(\xi)\sim-\nu|\xi|^2,\ \text{for } |\xi|\gg 1,\\
\mu_j(\xi)\sim-\frac{\nu+\tilde{\nu}}{2}|\xi|^2+i(-1)^{j+1}\sqrt{\beta^2+\gamma^2}|\xi|,\ \text{for } |\xi|\ll 1,\ j=3,4,\\
\mu_3(\xi)\sim-\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}},\ \mu_4(\xi)\sim-(\nu+\tilde{\nu})|\xi|^2,\ \text{for } |\xi|\gg 1,
\end{gathered}
$$

the estimates (i) and (ii) follow from the results in [11, 20]. The estimate (iii) can be proved by the energy method. This completes the proof. ■

To prove Theorem 2.2 (i), we apply Lemma 3.4 (i) to the equation (3.16) for $j=1$. For the high frequency part U_∞, we use the energy method. Theorem 2.2 (ii) can be shown by applying Lemma 3.4 (ii) to (3.16) for $j=\infty$ and using Lemma 3.1 and Lemma 3.2. We note that since $N(U)$ also satisfies the linear constraints $N_1(U)+\mathrm{tr}N_3(U)=0$, one can see from Lemma 3.4 that the Duammel terms $\int_0^t e^{-(t-s)L}P_jN(U(s))\mathrm{d}s, j=1,\infty$ are estimated as

$$
\begin{aligned}
&\left\|\int_0^t e^{-(t-s)L} P_1 N(U(s))\mathrm{d}s\right\|_{L^p}\\
&\le C\int_0^t (1+t-s)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}\|N(U(s))\|_{L^1}\mathrm{d}s\\
&\le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}\|u_0\|_{L^1\cap H^3},\\
&\left\|\int_0^t e^{-(t-s)L} P_\infty N(U(s))\mathrm{d}s\right\|_{L^p}\\
&\le C\int_0^t e^{-c(t-s)}\|N(U(s))\|_{L^p}\mathrm{d}s\\
&\le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}\|u_0\|_{L^1\cap H^3},\ 1<p\le 2.
\end{aligned}
$$

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