Analysis of non-stationary Navier-Stokes equations approximated by the pressure stabilization method*

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1 Introduction

Navier-Stokes equations are given by

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla\pi = f, & t>0,\ x\in\Omega, \\ \nabla\cdot u = 0, & t>0,\ x\in\Omega, \\ u(0,x) = a, & x\in\Omega, \\ u(t,x) = 0, & t>0,\ x\in\partial\Omega, \end{cases} \tag{NS}$$

where the fluid vector fields $u = u(t,x) = (u_1(t,x),\dots,u_n(t,x))$ and the pressure $\pi = \pi(t,x)$ are unknown function, the external force $f = f(t,x)$ is a given vector function, the initial data a is a given solenoidal function and Ω is some bounded domain (see section 2 for detail). Equations (NS) are the partial differential equations which describe the motion of the viscous fluid flows and one of the most important equations in view of both mathematical analysis and engineering. However Equations (NS) are unsolved for a long time. One of the difficulty of analysis for (NS) is the pressure term $\nabla\pi$(and incompressible condition $\nabla\cdot u = 0$).

In order to overcome this difficulty, in real analysis, we often use the Helmholtz decomposition given by

$$L_p(\Omega)^n = L_{p,\sigma}(\Omega) \oplus G_p(\Omega)$$

for $1 < p < \infty$, where $L_{p,\sigma}(\Omega) = \overline{\{u \mid u_j \in C_0^\infty(\Omega), \nabla\cdot u = 0\}}^{\|\cdot\|_{L_p}}$ and $G_p(\Omega) = \{\nabla\pi \in L_p(\Omega)^n \mid \pi \in L_{p,\mathrm{loc}}(\Omega)\}$. We remark that whether the Helmholtz decomposition holds depends on the shape of the region in the case where $p \neq 2$ (see Galdi [6] for detail).

On the other hand, in numerical analysis, some penalty methods (quasi-compressibility methods) are employed as the method to overcome this difficulty. They are methods that eliminate the pressure by using approximated incompressible condition. For example, setting $\alpha > 0$ as a perturbation parameter, we use $\nabla\cdot u = -\pi/\alpha$ in the penalty method,

*This article is based on the paper [9] by the author and R. Matsui and the study with H. Kikuchi. More detail information and proofs can be found in [9](and [8]).

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$\nabla \cdot u = \Delta\pi/\alpha$ in the pressure stabilization method and $\nabla \cdot u = -\partial_t \pi/\alpha$ in the pseudo-compressible method (see [10] for other cases).

In this article, we consider the Navier-Stokes equations with incompressible condition approximated by pressure stabilization method. Namely we consider the approximated incompressible condition

$$(u_\alpha, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_\alpha, \nabla\varphi)_\Omega \qquad (\forall\varphi \in \widehat{W}^1_{q'}(\Omega)) \tag{1.1}$$

for $1 < q < \infty$ in stead of incompressible condition and the following Navier-Stokes equations under the approximated incompressible condition (1.1):

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + (u_\alpha \cdot \nabla)u_\alpha + \nabla\pi_\alpha = f, & t > 0,\ x \in \Omega, \\ u_\alpha(0, x) = a_\alpha, & x \in \Omega, \\ u_\alpha(t, x) = 0, & t > 0,\ x \in \partial\Omega. \end{cases} \tag{NSa}$$

Pressure stabilization method was first introduced by Brezzi and Pitkäranta [1]. They considered the approximated stationary Stokes equations which are linearized Navier-Stokes equations. They obtained the following error estimate by using the energy methods:

$$\|u_\alpha - u\|_{H^1(\Omega)} + \|\pi_\alpha - \pi\|_{L^2(\Omega)} \le C\alpha^{-1/2}\|f\|_{L^2(\Omega)}. \tag{1.2}$$

Nazarov and Specovius-Neugebauer [7] considered the same approximated Stokes problem and derived asymptotically precise estimates for solution to the approximated problem as $\alpha \to \infty$ by using the parameter-dependent Sobolev norms.

There are many results concerning the stationary Stokes equations and Navier-Stokes equations approximated by using the pressure stabilization method. However there are few results concerning the non-stationary Stokes equations and Navier-Stokes equations. As far as the authors know, only the result due to Prohl [10] is known as the results concerning the non-stationary problem. In [10], Prohl considered the sharp a priori estimate for the pressure stabilization method under some assumptions and showed the following error estimates:

$$\begin{aligned} \|u_\alpha - u\|_{L^\infty([0,T], L_2(\Omega))} + \|\tau(\pi_\alpha - \pi)\|_{L^\infty([0,T], W_2^{-1}(\Omega))} &\le C\alpha^{-1}, \\ \|u_\alpha - u\|_{L^\infty([0,T], W_2^1(\Omega))} + \|\sqrt{\tau}(\pi_\alpha - \pi)\|_{L^\infty([0,T], L_2(\Omega))} &\le C\alpha^{-1/2}, \end{aligned} \tag{1.3}$$

where $\tau = \tau(t) = \min(t, 1)$. He proved a priori error estimate by using energy method. In other words, he proved that if we can prove the existence of the local in time solution to (NSa), the solution to (NSa) satisfies (1.3). So the goal of this article is to show the existence theorem for (NSa) and the error estimates.

In our method, we shall use the maximal regularity theorem in order to prove the local in time existence theorem and the error estimates in the L_p in time and the L_q in space framework with $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$.

2 Local in time existence theorem and error estimates

Before we describe main theorem, we shall introduce some functional spaces and notations throughout this article. The letter C denotes generic constants and the constant $C_{a,b,\ldots}$

depends on $a, b, \dots$. The values of constants C and $C_{a,b,\dots}$ may change from line to line. For $1 < q < \infty$, let $q' = q/(q-1)$. As the complex domain where a resolvent parameter belongs, we use $\Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} \mid |\arg\lambda| < \pi - \varepsilon\}$ and $\Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \Sigma_\varepsilon \mid |\lambda| \geq \lambda_0\}$ for $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. For any domain D, Banach space X and $1 \leq q \leq \infty$, $L_q(D, X)$ denotes the usual Lebesgue space of X-valued functions defined on D and $\|\cdot\|_{L_q(D,X)}$ denotes its norm. We use the notation $L_q(D) = L_q(D, \mathbb{R})$, $\|\cdot\|_{L_q(D)} = \|\cdot\|_{L_q(D,\mathbb{R})}$ and for $a, b, \dots, c \in L_q(D)$, $\|(a, b, \dots, c)\|_{L_q(D)} = \|a\|_{L_q(D)} + \|b\|_{L_q(D)} + \dots + \|c\|_{L_q(D)}$. In a similar way, for $1 \leq q \leq \infty$ and a positive integer m, $W_q^m(D, X)$ denotes the Sobolev spaces of X-valued functions of defined on D. We often use the same symbols for denoting the vector and scalar function spaces if there is no confusion. For $1 \leq p, q \leq \infty$, $B_{q,p}^{2(1-1/p)}(D)$ denotes the real interpolation space defined by $B_{q,p}^{2(1-1/p)}(D) = (L_q(D), W_q^2(D))_{1-1/p,p}$. For a Banach space X and some $\gamma_0 \in \mathbb{R}$, we set

$$
\begin{aligned}
L_{p,\gamma_0}(\mathbb{R}, X) &= \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R}, X) \mid \|e^{-\gamma t} f\|_{L_p(\mathbb{R},X)} < \infty,\ (\gamma \geq \gamma_0)\},\\
L_{p,\gamma_0,(0)}(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_0}(\mathbb{R}, X) \mid f(t) = 0\ (t < 0)\},\\
W^1_{p,\gamma_0,(0)}(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_0,(0)}(\mathbb{R}, X) \mid f'(t) \in L_{p,\gamma_0}(\mathbb{R}, X)\}.
\end{aligned}
$$

In order to deal with the pressure term, we use the following functional spaces:

$$
\begin{aligned}
L_{q,\mathrm{loc}}(D) &= \{f \mid f|_K \in L_q(K),\ K \text{ is any compact set in } D\},\\
\widehat{W}_q^1(D) &= \{\theta \in L_{q,\mathrm{loc}}(D) \mid \nabla\theta \in L_q(D)^n\}.
\end{aligned}
$$

Since our proof is based on Fourier analysis, we next introduce the Fourier transform and the Laplace transform. We define the Fourier transform, its inverse Fourier transform, the Laplace transform and its inverse Laplace transform by

$$
\begin{aligned}
\hat{f}(\xi) &= \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, & \mathcal{F}_\xi^{-1}[f](x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi)d\xi,\\
\mathcal{L}_t[f](\lambda) &= \mathcal{F}_t[e^{-\gamma t} f(t)](\tau), & \mathcal{L}_\tau^{-1}[f](t) &= e^{\gamma t}\mathcal{F}_\tau^{-1}[f](t),
\end{aligned}
$$

respectively, where $x, \xi \in \mathbb{R}^n$, $\lambda = \gamma + i\tau \in \mathbb{C}$ and $x \cdot \xi$ is usual inner product: $x \cdot \xi = \sum_{j=1}^n x_j\xi_j$. Furthermore, we define the Fourier-Laplace transform by

$$
\mathcal{L}_t[\mathcal{F}_x[v(t,x)]](\lambda, \xi) = \mathcal{F}_{t,x}[e^{-\gamma t} v(t,x)](\lambda, \xi) = \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n} e^{-(\lambda t + ix\cdot\xi)} v(t,x)dx\right)dt.
$$

By using Fourier transform and Laplace transform, we define the operator Λ_γ^s as

$$
(\Lambda_\gamma^s f)(t) = \mathcal{L}_\tau^{-1}[|\lambda|^s \mathcal{L}_t[f](\lambda)](t) = e^{\gamma t}\mathcal{F}_\tau^{-1}[(\tau^2 + \gamma^2)^{s/2}\mathcal{F}_t[e^{-\gamma t} f(t)](\tau)](t)
$$

for $\lambda = \gamma + i\tau$.

In this paper, we assume next assumption for our domain Ω.

Assumption 2.1. *Let $n/2 < q < \infty$ and $n < r < \infty$. Let Ω be a uniform $W_r^{2-1/r}$ domain introduced in [5] and $L_q(\Omega)$ has the Helmholtz decomposition.*

Here the assumption on a uniformly $W_r^{2-1/r}$ domain is used when we reduce the problem on the bounded domain to one on the bent half-space and on the whole space. The possibility of Helmholtz decomposition is used in the processing of pressure terms. According to Galdi [6], that "$L_q(\Omega)$ has the Helmholtz decomposition" is equivalent that the following weak Neumann problem is uniquely solvable: for $f \in L_q(\Omega)$,

$$(\nabla\theta, \nabla\varphi) = (f, \nabla\varphi) \qquad (\forall \varphi \in \widehat{W}^1_{q'}(\Omega)).$$

($L_2(\Omega)$ has the Helmholtz decomposition for any Ω.) Let θ be the solution to the above weak Neumann problem for $f \in L_q(\Omega)$ and the maps P_Ω and Q_Ω be defined by $Q_\Omega f = \theta$ and $P_\Omega f = f - \nabla Q_\Omega f$. P_Ω is called the Helmholtz projection. By (1.1) and the map Q_Ω, we see that $\nabla\pi_\alpha = \alpha\nabla Q_\Omega u_\alpha$.

First main result is concerned with the local in time existence theorem for (NSa) under (1.1).

Theorem 2.1. *Let $n \geq 2$, $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$. Let $\alpha > 0$ and $T_0 \in (0,\infty)$. For any $M > 0$, assume that the initial data $a_\alpha \in B^{2(1-1/p)}_{q,p}(\Omega)$ and the external force $f \in L_p((0,T_0), L_q(\Omega))$ satisfy*

$$\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|f\|_{L_p((0,T_0),L_q(\Omega))} \leq M. \tag{2.1}$$

Then, there exists $T^* \in (0, T_0)$ depending on only M such that (NSa) *under* (1.1) *has a unique solution (u_α, π_α) of the following class:*

$$u_\alpha \in W^1_p((0,T^*), L_q(\Omega)) \cap L_p((0,T^*), W^2_q(\Omega)), \qquad \pi_\alpha \in L_p((0,T^*), \widehat{W}^1_q(\Omega)).$$

Moreover the following estimate holds:

$$\|u_\alpha\|_{L_\infty((0,T^*),L_q(\Omega))} + \|(\partial_t u_\alpha, \nabla^2 u_\alpha, \nabla\pi_\alpha)\|_{L_p((0,T^*),L_q(\Omega))} + \|\nabla u_\alpha\|_{L_r((0,T^*),L_q(\Omega))} \leq C_{n,p,q,T^*}$$

for $1/p - 1/r \leq 1/2$.

Here we state the outline of the proof of main theorem (Theorem 2.1). We can prove Theorem 2.1 by using the contraction mapping principle with two type maximal regularity theorems (Theorem 2.2 and Theorem 2.5). In order to prove Theorem 2.2, we use the Weis' operator valued Fourier multiplier theorem. For this purpose, we have to show the existence of $\mathcal{R}$-bouned solution operator to the generalized resolvent problem of (NSa). In order to prove Theorem 2.5, we need the some estimate of semigroup $T_\alpha(t)$ for linearlized problem of (NSa) (See [8] and [9] for the detail proof).

From here we shall introduce the two type maximal L_p-L_q regularity theorems for the following linearized problems corresponding to (NSa):

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + \nabla\pi_\alpha = f, & t > 0, x \in \Omega, \\ u_\alpha(t,x) = 0, & x \in \partial\Omega, \\ u_\alpha(0,x) = a_\alpha, & x \in \Omega \end{cases} \tag{2.2}$$

under the approximated incompressible condition

$$(u_\alpha, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_\alpha, \nabla\varphi)_\Omega + (g, \nabla\varphi)_\Omega \qquad \varphi \in \widehat{W}^1_{q'}(\Omega). \tag{2.3}$$

First result is concerned with the maximal L_p-L_q regularity theorem for (2.2) with $a_\alpha = 0$ under (2.3) .

Theorem 2.2. *Let $1 < p, q < \infty$ and $\alpha > 0$. Then there exists a positive number γ_0 such that the following assertion holds: for any $f, g \in L_{p,\gamma_0,(0)}(\mathbb{R}, L_q(\Omega))$, (2.2) under (2.3) with $a_\alpha = 0$ has a unique solution:*

$$u_\alpha \in L_{p,\gamma_0,(0)}(\mathbb{R}, W_q^2(\Omega)) \cap W^1_{p,\gamma_0,(0)}(\mathbb{R}, L_q(\Omega)), \qquad \pi_\alpha \in L_{p,\gamma_0,(0)}(\mathbb{R}, \widehat{W}_q^1(\Omega)).$$

Moreover, the following estimate holds:

$$\begin{aligned}&\|e^{-\gamma t}(\partial_t u_\alpha, \gamma u_\alpha, \Lambda_\gamma^{\frac{1}{2}} \nabla u_\alpha, \Lambda_{\gamma+\alpha}^{1/2}(\nabla \cdot u_\alpha), \nabla^2 u_\alpha, \nabla \pi_\alpha)\|_{L_p(\mathbb{R}, L_q(\Omega))}\\ &\leq C_{n,p,q}\|e^{-\gamma t}(f, \alpha g)\|_{L_p(\mathbb{R}, L_q(\Omega))}\end{aligned}$$

for any $\gamma \geq \gamma_0$.

Remark 2.3. *By the property of Helmholtz decomposition, we can solve (2.3) for $u_\alpha, g \in L_q(\Omega)$ and we see $\pi_\alpha = \alpha Q_\Omega(u_\alpha - g)$.*

In order to prove the maximal L_p-L_q regularity theorem for (2.2) with $f = 0$ under (2.3), we consider the following generalized resolvent problem concerning with (2.2):

$$\begin{cases} \lambda v_\alpha - \Delta v_\alpha + \nabla \rho_\alpha = f, & x \in \Omega, \\ v_\alpha = 0, & x \in \partial\Omega \end{cases} \tag{2.4}$$

under

$$(v_\alpha, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\rho_\alpha, \nabla\varphi)_\Omega + (g, \nabla\varphi)_\Omega \qquad \varphi \in \widehat{W}^1_{q'}(\Omega), \tag{2.5}$$

where the resolvent parameter λ varies in $\Sigma_{\varepsilon,\lambda_0}$ $(0 < \varepsilon < \pi/2)$. Then the following resolvent estimate holds:

Proposition 2.4. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. There exists a positive constant λ_0 such that for $f, g \in L_q(\Omega)$ and $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, there exists a unique solution (v_α, ρ_α) to (2.4) under (2.5) which satisfies the following inequality:*

$$\|(\lambda v_\alpha, \lambda^{1/2}\nabla v_\alpha, \nabla^2 v_\alpha, (\lambda + \alpha)^{1/2}(\nabla \cdot v_\alpha), \nabla\rho_\alpha)\|_{L_q(\Omega)} \leq C\|(f, \alpha g)\|_{L_q(\Omega)}.$$

Let $\mathcal{A}_\alpha$ be the linear operator defined by $\mathcal{A}_\alpha u_\alpha = \Delta u_\alpha - \alpha\nabla Q_\Omega u_\alpha$ and $\mathcal{D}(\mathcal{A}_\alpha) = \{u \in W_q^2(\Omega) \mid u|_{\partial\Omega} = 0\}$. By Proposition 2.4 with $g = 0$, we see that $\mathcal{A}_\alpha$ generates the semigroup $\{T_\alpha(t)\}_{t\geq 0}$ on $L_q(\Omega)$. Moreover there exists a positive constant $C > 0$ such that for any $a_\alpha \in L_q(\Omega)$, $u_\alpha(t) = T_\alpha(t)a_\alpha$ satisfies

$$\begin{aligned}\|(u_\alpha, t^{1/2}\nabla u_\alpha, t\nabla^2 u_\alpha, t\partial_t u_\alpha)\|_{L_q(\Omega)} &\leq Ce^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)} \qquad (t > 0),\\ \|\nabla Q_\Omega u_\alpha\|_{L_q(\Omega)} &\leq Ce^{-\alpha t}\|a_\alpha\|_{L_q(\Omega)}.\end{aligned}$$

By the equations (2.2), we have

$$\|\nabla\pi_\alpha\|_{L_q(\Omega)} \leq \|\partial_t u_\alpha\|_{L_q(\Omega)} + \|-\Delta u_\alpha\|_{L_q(\Omega)} \leq Ct^{-1}e^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)}, \tag{2.6}$$

which means that the pressure term $\nabla\pi_\alpha$ has the singularity at $t=0$. On the other hands, since $\nabla\pi_\alpha = \alpha\nabla Q_\Omega u_\alpha$, (u_α, π_α) enjoys (2.2) under (2.3) and $\nabla\pi_\alpha$ satisfies the following estimate:

$$\|\nabla\pi_\alpha\|_{L_q(\Omega)} = \alpha\|\nabla Q_\Omega u_\alpha\|_{L_q(\Omega)} \le C\alpha e^{-\alpha t}\|a_\alpha\|_{L_q(\Omega)},$$

which implies that $\nabla\pi_\alpha$ does not have the singularity at $t=0$. We think that this advantage may be the reason for using the pressure stabilization method in numerical simulations.

By real interpolation, we can see the following maximal L_p-L_q regularity theorem for (2.2) with $f=g=0$.

Theorem 2.5. *Let $\alpha>0$ and $1<p,q<\infty$. Let λ_0 be a number obtained in Proposition 2.4. For $a_\alpha \in B^{2(1-1/p)}_{q,p}(\Omega)$, $u_\alpha = T_\alpha(t)a_\alpha$ satisfy*

$$\begin{aligned}
\|e^{-\lambda_0 t}(\partial_t u_\alpha, \nabla^2 u_\alpha)\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)},\\
(\gamma-\lambda_0)^{1/p}\|e^{-\gamma t}u_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{L_q(\Omega)},\\
(\gamma-\lambda_0)^{1/(2p)}\|e^{-\gamma t}\nabla u_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)}
\end{aligned}$$

for any $\gamma>\lambda_0$.

Next we consider the error estimate between the solution (u,π) to (NS) under the weak incompressible condition

$$(u, \nabla\varphi)_\Omega = 0 \qquad (\varphi \in \widehat{W}^1_{q'}(\Omega)) \tag{2.7}$$

and solution (u_α, π_α) to (NSa) under (1.1). To this end, setting $u_E = u - u_\alpha$ and $\pi_E = \pi - \pi_\alpha$, we see that (u_E, π_E) enjoys that

$$\begin{cases}
\partial_t u_E - \Delta u_E + \nabla\pi_E + N(u_E, u_\alpha) = 0, & t>0,\ x\in\Omega,\\
u_E(0,x) = a_E, & x\in\Omega,\\
u_E(t,x) = 0, & t>0,\ x\in\partial\Omega,
\end{cases} \tag{2.8}$$

where $N(u_E, u_\alpha) = (u_E\cdot\nabla)u_E + (u_E\cdot\nabla)u_\alpha + (u_\alpha\cdot\nabla)u_E$ and $a_E = a - a_\alpha$ under the approximated weak incompressible condition

$$(u_E, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_E, \nabla\varphi)_\Omega + \alpha^{-1}(\nabla\pi, \nabla\varphi)_\Omega \qquad \varphi \in \widehat{W}^1_{q'}(\Omega) \tag{2.9}$$

for $1<q<\infty$. In a similar way to Theorem 2.1, we consider (2.8) under (2.9) for $a_\alpha = a_E$.

Theorem 2.6. *Let $n\ge 2$, $n/2<q<\infty$, $\max\{1, n/q\}<p<\infty$ and $\alpha>0$. Let T^* be a positive constant obtained in Theorem 2.1 and (u_α, π_α) be a solution obtained in Theorem 2.1. For any $M>0$, assume that $a_E \in B^{2(1-1/p)}_{q,p}(\Omega)$ satisfies*

$$\|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le M\alpha^{-1}. \tag{2.10}$$

Then there exists $T^\flat \in (0, T^*)$ such that (2.8) *has a unique solution (u_E, π_E) which satisfies*

$$\begin{aligned}\|u_E\|_{L_\infty((0,T^\flat),L_q(\Omega))} &+ \|\nabla u_E\|_{L_r((0,T^\flat),L_q(\Omega))} \\ &+\|(\nabla^2 u_E, \partial_t u_E, \nabla \pi_E)\|_{L_p((0,T^\flat),L_q(\Omega))} \le C_{n,p,q,T^\flat}\alpha^{-1}\end{aligned} \tag{2.11}$$

for $1/p - 1/r \le 1/2$.

Remark 2.7. (2.11) *means the following error estimates for the Navier-Stokes equations:*

$$\begin{aligned}&\|u - u_\alpha\|_{L_\infty((0,T^\flat),L_q(\Omega))} \le C\alpha^{-1}, \\ &\|(\nabla^2(u - u_\alpha), \partial_t(u - u_\alpha), \nabla(\pi - \pi_\alpha))\|_{L_p((0,T^\flat),L_q(\Omega))} \le C\alpha^{-1},\end{aligned}$$

In comparison with the result due to Prohl [10], we can extend L_2 framework to L_q framework with respect to the error estimate.

3 Analysis of the resolvent Stokes problem

In order to show the global in time solution, it is necessary to analyze when the resolve parameter is near the origin. In this section, we introduce the recent results obtained when the resolve parameter is near the origin.

The resolvent problem for the approximated Stokes equation which is linearized problem for (NSa) is given by

$$\begin{cases} \lambda v_\alpha - \Delta v_\alpha + \nabla \rho_\alpha = f, & x \in \Omega \\ v_\alpha = 0, & x \in \partial\Omega \end{cases} \tag{3.1}$$

under the approximated incompressible condition:

$$(v_\alpha, \nabla\varphi)_\Omega = \frac{1}{\alpha}(\nabla\rho_\alpha, \nabla\varphi)_\Omega, \qquad (\varphi \in \widehat{W}^1_{q'}). \tag{3.2}$$

Lemma 3.1. *Let $1 < q < \infty$ and $\alpha > 1/C_0^2$, where C_0 is the optimal constant in the Poincaré inequality. Then for any $f \in L_q(\Omega)$ and $\lambda \in \mathbb{C} \setminus (-\infty, -C_0^{-2}]$, there exists the unique solution (v_α, ρ_α) to* (3.1) *under* (3.2)*. Moreover (v_α, ρ_α) satisfies*

$$\|(\lambda v_\alpha, |\lambda|^{1/2}\nabla v_\alpha)\|_{L_q(\Omega)} + \|v_\alpha\|_{W^2_q(\Omega)} + \|\nabla\rho_\alpha\|_{L_q(\Omega)} \le C\|f\|_{L_q(\Omega)}.$$

Remark 3.2. (i) *It is well-known that for a smooth bounded domain Ω, C_0 is the minimal eigenvalue of the Laplace operator in the space H^1_0.*

(ii) *For usual resolvent Stokes problem, the same result holds as Lemma 3.1. From this lemma, when using the pressure stabilization method in numerical simulations, we suggest that α should satisfy $\alpha > 1/C_0^2$ at least .*

The key to show Lemma 3.1 is the following lemma concerning uniqueness:

Proposition 3.3. *Let $1 < q < \infty$, $\alpha > 1/C_0^2$ and $\lambda \in \mathbb{C}\setminus(-\infty, -1/C_0^2]$. If $(v_\alpha, \rho_\alpha) \in W^2_q(\Omega) \times \widehat{W}^1_q(\Omega)$ is the solution to* (3.1) *with $f = 0$ under* (3.2)*, then $v_\alpha = 0$ and $\rho_\alpha = 0$.*

Proof. We remark $\nabla\rho_\alpha = \alpha\nabla Q v_\alpha$ by Helmholtz decomposition and that (3.1) is equivalent to the following equations:

$$\begin{cases} \lambda v_\alpha - \Delta v_\alpha + \alpha\nabla Q v_\alpha = 0, & x \in \Omega, \\ v_\alpha = 0, & x \in \partial\Omega. \end{cases}$$

We first consider the case where $2 \le q < \infty$. By Hölder inequality, $(v_\alpha, \rho_\alpha) \in W_2^2(\Omega) \times \widehat{W}_2^1(\Omega)$. By divergence theorem, we have

$$\begin{aligned} 0 &= (\lambda v_\alpha - \Delta v_\alpha + \alpha\nabla Q_\Omega v_\alpha, v_\alpha)_\Omega \\ &= \lambda\|v_\alpha\|_{L^2}^2 + \|\nabla v_\alpha\|_{L^2}^2 + \alpha(\nabla Q_\Omega v_\alpha, v_\alpha)_\Omega. \end{aligned}$$

By $v_\alpha = P_\Omega v_\alpha + \nabla Q_\Omega v_\alpha$ and $(P_\Omega v_\alpha, \nabla Q_\Omega v_\alpha)_\Omega = 0$, we see

$$0 = \lambda\|v_\alpha\|_{L^2}^2 + \|\nabla v_\alpha\|_{L^2}^2 + \alpha\|\nabla Q_\Omega v_\alpha\|_{L^2}^2.$$

Taking the real part and the imaginary part, we obtain

$$\begin{cases} (\mathrm{Re}\lambda)\|P_\Omega v_\alpha\|_{L^2}^2 + (\mathrm{Re}\lambda + \alpha)\|\nabla Q_\Omega v_\alpha\|_{L^2}^2 + \|\nabla v_\alpha\|_{L^2}^2 = 0, \\ (\mathrm{Im}\lambda)\|v_\alpha\|_{L^2}^2 = 0. \end{cases}$$

In the case where $\mathrm{Im}\lambda \neq 0$, we obtain $\|v_\alpha\|_{L^2} = 0$. Therefore by $\nabla\rho_\alpha = \alpha\nabla Q_\Omega v_\alpha$, we have $v_\alpha = 0$ and $\nabla\rho_\alpha = 0$.

In the case where $\mathrm{Im}\lambda = 0$ and $\lambda \ge 0$, we obtain $v_\alpha = 0$ and $\nabla\rho_\alpha = 0$ by $\mathrm{Re}\lambda > 0, \mathrm{Re}\lambda + \alpha > 0$.

In the case where $\mathrm{Im}\lambda = 0$ and $-C_0^{-2} < \mathrm{Re}\lambda < 0$, set $\tilde{\lambda} = -\lambda$ $(0 \le \tilde{\lambda} < C_0^{-2})$. Since $\alpha > C_0^{-2}$, we have $\mathrm{Re}\lambda + \alpha = -\tilde{\lambda} + \alpha > 0$.

By $\|P_\Omega v_\alpha\|_{L^2} \le \|v_\alpha\|_{L^2}$ and $\|v_\alpha\|_{L^2} \le C_0\|\nabla v_\alpha\|_{L^2}$, we have

$$\begin{aligned} 0 &\ge -\tilde{\lambda}\|P_\Omega v_\alpha\|_{L^2}^2 + \|\nabla v_\alpha\|_{L^2}^2 \\ &\ge -\tilde{\lambda}\|v_\alpha\|_{L^2}^2 + \|\nabla v_\alpha\|_{L^2}^2 \\ &\ge -\tilde{\lambda}C_0^2\|\nabla v_\alpha\|_{L^2}^2 + \|\nabla v_\alpha\|_{L^2}^2 = (1 - \tilde{\lambda}C_0^2)\|\nabla v_\alpha\|_{L^2}^2. \end{aligned}$$

Therefore when $1 - \tilde{\lambda}C_0^2 > 0$, namely $\tilde{\lambda} < C_0^{-2}$, we have $\|\nabla v_\alpha\|_{L^2} = 0$. By $v_\alpha|_{\partial\Omega} = 0$, we obtain $v_\alpha = 0$ and $\nabla\rho_\alpha = 0$.

In the case where $1 < q < 2$, by using duality argument, we can obtain the uniqueness. □

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