ESTIMATES OF THE REGULAR SET FOR NAVIER-STOKES FLOWS IN TERMS OF INITIAL DATA

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1. Introduction

The purpose of this note is to present a joint work with Kyungkeun Kang and Tai-Peng Tsai. We are concerned with the regularity of weak solutions for the incompressible Navier-Stokes equations

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \operatorname{div} v = 0 \tag{NS}$$

associated with the initial value $v|_{t=0} = v_0$ with $\operatorname{div} v_0 = 0$. Caffarelli, Kohn, and Nirenberg [5] established local regularity theory for suitable weak solutions. As an application of their celebrated ϵ-regularity criterion, they showed the following result:

Theorem D [5]. *There exists $\epsilon_0 > 0$ such that if $v_0 \in L^2(\mathbb{R}^3)$ satisfies*

$$\||x|^{-\frac{1}{2}} v_0\|_{L^2}^2 = \epsilon < \epsilon_0, \tag{1.1}$$

then there exists a suitable weak solution which is regular in the set $\Pi_{\epsilon_0 - \epsilon}$, where

$$\Pi_\delta := \left\{(x,t) : t > \frac{|x|^2}{\delta}\right\} \qquad \textit{for } \delta > 0.$$

This theorem asserts that the smallness of initial data in a weighted space implies regularity of the solution above a paraboloid with vertex at the origin. There are at least two interesting features in this result: No regularity condition (better than L^2) is assumed away from the origin and the regularity around the origin is propagated globally in time. We also note that if the size of v_0 tends to 0, $\Pi_{\epsilon_0 - \epsilon}$ increases and converges to a limit set Π_{ϵ_0}. This observation leads to the following questions:

(a) Can the size of regular set Π_δ be enlarged?
(b) Can the condition (1.1) of initial data be relaxed in terms of regularity and smallness?

One of the goals of this paper is trying to answer questions (a) and (b) by employing approach based on a framework of *scaled local energy* explained below.

It is known from works [6, 23, 12, 7] that for $v_0 \in L^q(\mathbb{R}^3)$ with $q \geq 3$, (NS) has a (unique) mild solution defined on some short time interval. Motivated by the problem for constructing large forward self-similar solutions to (NS), Jia and Šverák [9] asked under which condition this result can be localized in space. For $B_r(x) = \{y \in \mathbb{R}^3 : |x - y| < r\}$ and $B_r = B_r(0)$, their question can be stated as follows:

(c) If v_0 is a general initial data for which suitable weak solutions v is defined and $v_0|_{B_2} \in L^q(B_2)$, can we conclude that v is regular in $B_1 \times [0, t_1)$ for some time $t_1 > 0$?

Although non-local effect of the pressure might prevent the solution from having the same amount of the regularity as the one for the heat equation, such effect is expected to be handled at least for a short time and $q \geq 3$. Indeed this question is settled affirmatively for the scale subcritical case $q > 3$ in [9] and for the critical case $q = 3$ in [1, 11]; see also [21] for the condition on the initial enstrophy and [1] for further extension to the $L^{3,\infty}$ space and the critical Besov spaces. Notice that the results for the critical case have some similarities with Theorem D in [5]. Namely, the assumptions for the initial data ensure critical regularity at the origin and they lead to local-in-space regularity. As the first main result of this paper, we present a new type of local-in-space regularity estimate, which guarantees regularity in the set like Π_δ. In order to formulate it, define *the scaled local energy* of the initial data by

$$N_0 = N_0(v_0) := \sup_{r\in(0,1]} \frac{1}{r}\int_{B_r} |v_0(x)|^2\,dx,$$

which plays a central role in this paper.

Theorem 1.1. *Let (v,p) be a suitable weak solution in $B_2\times(0,4)$ with the initial data $v_0\in L^2(B_2)$ in the sense that $\lim_{t\to 0+}\|v(t)-v_0\|_{L^2(B_2)}=0$. Assume that*

$$M := \|v\|^2_{L^\infty_t(0,4;L^2_x(B_2))} + \|\nabla v\|^2_{L^2(B_2\times(0,4))} + \|p\|_{L^{\frac{3}{2}}(B_2\times(0,4))} < \infty \tag{1.2}$$

and that

$$N_0 \le \epsilon_*. \tag{1.3}$$

Then there exists $T=T(M)\ge \frac{c}{1+M^{18}}$ such that v is regular in the set

$$\Gamma = \left\{(x,t)\in B_1\times(0,1)\ : cN_0^2\,|x|^2 \le t < T\right\}$$

and satisfies

$$|v(x,t)| \le \frac{C}{t^{\frac{1}{2}}} \qquad \textit{for } (x,t)\in\Gamma,$$

where ϵ_*, c, and C are positive absolute constants.

Remark 1.2. (1) This theorem asserts that smallness of the scaled energy implies regularity above a paraboloid for a short time. It may be viewed as an ϵ-regularity criterion *in terms of the initial data.*
(2) One can relate Theorem 1.1 to results in [5, 1, 11] by noting that

$$N_0 \le C\min\{\|v_0\|^2_{L^3(B_1)}, \|v_0\|^2_{L^{2,-1}(B_1)}\},$$

where

$$\|v_0\|_{L^{2,\alpha}(\Omega)} := \||x|^{\frac{\alpha}{2}}v_0\|_{L^2(\Omega)}$$

for $\alpha\in\mathbb{R}$ and $\Omega\subset\mathbb{R}^3$. Thus (1.3) holds if either L^3 norm or $L^{2,-1}$ norm is small in B_1. Hence our theorem can be regarded as a local version of Theorem D in [5].

The proof of Theorem 1.1 is based on a local-in-space a priori estimate for the scaled energy of (v,p) defined by

$$E_r(t) := \operatorname*{ess\,sup}_{0<s<t}\frac{1}{r}\int_{B_r}|v(s)|^2 + \frac{1}{r}\int_0^t\int_{B_r}|\nabla v|^2 + \frac{1}{r^2}\int_0^t\int_{B_r}|p|^{\frac{3}{2}}. \tag{1.4}$$

Our strategy is partially inspired by a uniformly local L^2 estimate established in the fundamental work [15] of Lemarié-Rieusset. However, in contrast to his estimate, our a priori estimate guarantees scale-critical regularity at the origin so that the ϵ-regularity criterion of [5] can apply.

We now return to the questions (a) and (b) concerning the regular set. In order to state our result, it is natural and convenient to use the notion of *local energy solutions* introduced by Lemarié-Rieusset [15] and later slightly modified in [13, 9, 3]. The local energy solution is a suitable weak solution of (NS) defined in $\mathbb{R}^3$ which satisfies certain uniformly local energy bound and pressure representation; see Definition 2.1 for the details. In this context, let us recall the uniformly local L^q spaces for $1 \le q < \infty$. We say $f \in L^q_{\text{uloc}}$ if $f \in L^q_{\text{loc}}(\mathbb{R}^3)$ and

$$\|f\|_{L^q_{\text{uloc}}} = \sup_{x\in\mathbb{R}^3} \|f\|_{L^q(B_1(x))} < \infty. \tag{1.5}$$

Local-in-time existence of local energy solutions for initial data in L^2_{uloc} and also global existence for initial data in $E^2 := \overline{C_0^\infty}^{L^2_{\text{uloc}}}$ are established in [15]. One of the advantages of the local energy solution is it can be defined even for infinite energy data; see [14, 16, 10] and references therein for further developments and its applications, and [19] for the local energy solutions in the half space. We also define the global version of the scaled energy by

$$\dot{N}_0 := \sup_{r>0} \frac{1}{r} \int_{B_r} |v_0(x)|^2 \, dx$$

Here note that $\dot{N}_0$ is invariant under the Navier-Stokes scaling: $u(x,t) \mapsto u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$. The following result shows the estimates of the regular set for the local energy solution for initial data with small scaled energy and also about that for large data in $L^{2,-1}(\mathbb{R}^3)$:

Theorem 1.3. *Let (v,p) be a local energy solution in $\mathbb{R}^3 \times (0,\infty)$ for the initial data $v_0 \in L^2_{\text{uloc}}(\mathbb{R}^3)$.*
(i) *There exist absolute constants ϵ_* and c such that if v_0 satisfies*

$$\sup_{x_0\in\mathbb{R}^3} \sup_{r\ge 1} \frac{1}{r} \int_{B_r(x_0)} |v_0(x)|^2 \, dx < \infty, \tag{1.6}$$

and if

$$\dot{N}_0 \le \epsilon_*, \tag{1.7}$$

then v is regular in the set

$$\left\{ (x,t) \in \mathbb{R}^3 \times (0,\infty) \; : \; c\dot{N}_0^2 \, |x|^2 \le t \right\}.$$

(ii) *For any $v_0 \in L^{2,-1}(\mathbb{R}^3)$ there exist positive constants $T(v_0)$ and $c(v_0)$ such that v is regular in the set*

$$\left\{ (x,t) \in \mathbb{R}^3 \times (0,\infty) \; : \; c(v_0)|x|^2 \le t < T(v_0) \right\}.$$

Note that $\dot{N}_0 \le \|v_0\|^2_{L^{2,-1}(\mathbb{R}^3)}$ holds and that the condition (1.6) only assumes some mild decay of the data at infinity. We also note that [5, Theorem D] is an existence result for initial data satisfying the conditions, while Theorem 1.3 is a regularity result for *any* solution for such data.

The following corollary concerns estimates of regular set for the data in the weighted space $L^{2,\alpha}$ with $\alpha > -1$, which generalize the classical result [17] of Leray and [5, Theorem C] for the cases $\alpha = 0$ and $\alpha = 1$, respectively.

Corollary 1.4. *Let (v,p) be a local energy solution for the initial data in $L^2_{\text{uloc}}(\mathbb{R}^3)$.*
(i) *Assume that $v_0 \in L^{2,\alpha}(\mathbb{R}^3)$ for some $\alpha \ge 0$. Then $v_0 \in L^2(\mathbb{R}^3)$ and there is $K = K(\|v_0\|_{L^2}, \|v_0\|_{L^{2,\alpha}})$ such that v is regular in the set*

$$\left\{ (x,t) \in \mathbb{R}^3 \times (0,\infty) \; : \; t \ge \min\{K|x|^{-2\alpha}, C_0\|v_0\|^4_{L^2}\} \right\}.$$

(ii) *Assume that $v_0 \in L^{2,\alpha}(\mathbb{R}^3)$ for some $\alpha \in (-1,0)$ and that (1.6) holds. Then there exist $K = K(\|v_0\|_{L^{2,\alpha}})$ and $T = T(\|v_0\|_{L^{2,\alpha}})$ such that v is regular in the set*

$$\left\{ (x,t) \in \mathbb{R}^3 \times (0,\infty) \; : \; t \ge \max\{K|x|^{-2\alpha}, T\} \right\}.$$

2. Preliminaries

In this section, we recall some notions about the weak solution to (NS) and some results such as the ϵ-regularity theorems and a priori estimates for the solutions.

For any domain $\Omega \subset \mathbb{R}^3$ and open interval $I \subset (0,\infty)$, we say (v,p) is a suitable weak solution in $\Omega \times I$ if it satisfies (NS) in the sense of distributions in $\Omega \times I$,

$$v \in L^\infty(I;L^2(\Omega)) \cap L^2(I;\dot{H}^1(\Omega)), \quad p \in L^{3/2}(\Omega \times I),$$

and the local energy inequality:

$$\begin{aligned}&\int_\Omega |v(t)|^2\phi(t)\,dx + 2\int_0^t\int_\Omega |\nabla v|^2\phi\,dx\,dt \\ &\le \int_0^t\int_\Omega |v|^2(\partial_t\phi + \Delta\phi)\,dx\,dt + \int_0^t\int_\Omega (|v|^2+2p)(v\cdot\nabla\phi)\,dx\,dt\end{aligned} \tag{2.1}$$

for all non-negative $\phi \in C_c^\infty(\Omega \times I)$. Note that no boundary condition is required.

We next define the notion of local energy solutions, The following definition is formulated in [3], which is slightly revised from the notions of the local Leray solution defined in [15], the local energy solution in [13] and the Leray solution in [9]. We refer to [11, Section 2] for discussion of their relation.

Definition 2.1 (Local energy solutions [3])**.** *A vector field $v \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \times [0,T))$ is a local energy solution to* (NS) *with divergence free initial data $v_0 \in L^2_{\mathrm{uloc}}$ if*

(1) *for some $p \in L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3 \times [0,T))$, the pair (v,p) is a distributional solution to* (NS),
(2) *for any $R > 0$,*

$$\operatorname*{ess\,sup}_{0\le t<\min\{R^2,T\}} \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} |v|^2\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{\min\{R^2,T\}}\int_{B_R(x_0)} |\nabla v|^2\,dx\,dt < \infty, \tag{2.2}$$

(3) *for all compact subsets K of $\mathbb{R}^3$ we have $v(t) \to v_0$ in $L^2(K)$ as $t \to 0^+$,*
(4) *(v,p) satisfies the local energy inequality* (2.1) *for all non-negative functions $\phi \in C_c^\infty(Q)$ with all cylinder Q compactly supported in $\mathbb{R}^3 \times (0,T)$,*
(5) *for every $x_0 \in \mathbb{R}^3$ and $r > 0$, there exists $c_{x_0,r} \in L^{3/2}(0,T)$ such that*

$$\begin{aligned}p(x,t) - c_{x_0,r}(t) &= \frac{1}{3}|v(x,t)|^2 + \int_{B_{3r}(x_0)} K(x-y) : v(y,t)\otimes v(y,t)\,dy \\ &\quad + \int_{\mathbb{R}^3\setminus B_{3r}(x_0)} (K(x-y) - K(x_0-y)) : v(y,t)\otimes v(y,t)\,dy\end{aligned} \tag{2.3}$$

in $L^{3/2}(B_{2r}(x_0)\times(0,T))$, where $K(x) = \nabla^2(\frac{1}{4\pi|x|})$, and
(6) *for any compact supported functions $w \in L^2(\mathbb{R}^3)^3$,*

$$\text{the function} \quad t \mapsto \int_{\mathbb{R}^3} v(x,t)\cdot w(x)\,dx \quad \text{is continuous on } [0,T). \tag{2.4}$$

We now recall the scaled version of the ϵ-regularity theorem of Caffarelli-Kohn-Nirenberg [5, Proposition 1]. It is formulated in the present form in [20, 18].

Lemma 2.2. *There are absolute constants ϵ_{CKN} and $C_{CKN} > 0$ with the following property. Suppose (v,p) is a suitable weak solution of (NS) in $B_r(x_0) \times (t_0 - r^2, t_0))$, $r > 0$, with*

$$\frac{1}{r^2}\int_{t_0-r^2}^{t_0}\int_{B_r(x_0)} |v|^3 + |p|^{3/2} dx\,dt \le \epsilon_{CKN},$$

then $v \in L^\infty(B_{\frac{r}{2}(x_0)} \times (t_0 - \frac{r^2}{4}, t_0))$ *and*

$$\|v\|_{L^\infty(B_{\frac{r}{2}(x_0)} \times (t_0 - \frac{r^2}{4}, t_0))} \leq \frac{C_{CKN}}{r}. \tag{2.5}$$

We recall a useful Gronwall-type inequality from [3, Lemma 2.2].

Lemma 2.3. *Suppose* $f \in L^\infty_{\text{loc}}([0, T_0); [0, \infty))$ *(which may be discontinuous) satisfies, for some* a, $b > 0$, *and* $m \geq 1$,

$$f(t) \leq a + b \int_0^t (f(s) + f(s)^m) ds \qquad \text{for } t \in (0, T_0),$$

then we have $f(t) \leq 2a$ *for* $t \in (0, T)$ *with*

$$T = \min\left(T_0, \frac{C}{b(1 + a^{m-1})}\right),$$

where C *is a universal constant.*

Finally we also recall the following elementary bound for the scaled energy.

Lemma 2.4. *Assume that* $f \in L^2_{\text{loc}}(\mathbb{R}^3)$ *and let* $N_R = N_R(f) := \sup_{R<r\leq 1} \frac{1}{r} \int_{B_r} |f|^2$ *for some* $R \in [0, \frac{1}{2}]$. *If* $\delta \geq 2N_R$, *then for any* $x_0 \in B_{\frac{1}{2}}$ *we have*

$$\sup_{R(x_0)<r\leq 1-|x_0|} \frac{1}{r} \int_{B_r(x_0)} |f|^2 \leq \delta, \tag{2.6}$$

with $R(x_0) = \max\left(\frac{R}{2}, \frac{2N_R|x_0|}{\delta}\right)$.

3. Proof of Theorems

We first prove Theorem 1.1 regarding local regularity of suitable weak solutions. It is obtained as a consequence of the following theorem; see Remark 3.2 below.

Theorem 3.1. *Let* (v, p) *be a suitable weak solution in* $B_2 \times (0, T_0)$, $T_0 > 0$, *associated with the initial data* v_0 *in the sense that* $\lim_{t\to 0+} \|v(t) - v_0\|_{L^2(B_2)} = 0$. *There are absolute constants* c, $C \in (1, \infty)$ *such that the following holds true.*
(i) *Let* $N_R = \sup_{R<r\leq 1} \frac{1}{r} \int_{B_r} |v_0|^2 < \infty$, $R \geq 0$. *For any* $M \in (0, \infty)$ *and* $\delta \in [5N_R, \infty)$, *there exists* $T = T(M, \delta, T_0) \in (0, T_0]$ *such that if*

$$\|v\|^2_{L^\infty(0,T;L^2(B_2))} + \|\nabla v\|^2_{L^2(B_2\times(0,T))} + \|p\|_{L^{\frac{3}{2}}(B_2\times(0,T))} \leq M, \tag{3.1}$$

then $E_r(t)$ *defined by* (1.4) *and* (v, p) *satisfy*

$$E_r(t) \leq \delta \qquad \text{for } t \in (0, \min\{\lambda r^2, T\}] \qquad \text{for all } r \in (R, r_1] \tag{3.2}$$

if $R \leq r_1$, *and*

$$\frac{1}{r^2} \int_0^{\lambda r^2} \int_{B_r} |v|^3 + |p|^{\frac{3}{2}} dxdt \leq C(\delta + \delta^{\frac{3}{2}}) \qquad \text{for all } r \in (R, r_2] \tag{3.3}$$

if $R \leq r_2$, *where* T, λ, r_1, *and* r_2 *are given by*

$$T = \min\left(T_0, \frac{c\min\{1, \delta^{12}\}}{1 + M^{18}}\right), \qquad \lambda = \frac{c}{1 + \delta^2}, \qquad r_1 = \min\left(\frac{c\delta}{M^{\frac{3}{2}}}, \frac{1}{3}\right), \qquad r_2 := \min\left(\sqrt{\frac{T}{\lambda}}, r_1\right).$$

(ii) *There exists* $\epsilon_* > 0$ *such that if* $N_R \leq \epsilon_*$ *and* $R^2 \leq \widetilde{T}$, *then* v *is regular in the set*

$$\Pi = \left\{(x, t) \in B_{\frac{1}{2}} \times [R^2, \widetilde{T}] \,:\, t \geq cN_R^2 |x|^2\right\}$$

and satisfies

$$|v(x,t)| \le \frac{C}{\sqrt{t}} \qquad \text{for } (x,t) \in \Pi$$

with $\widetilde{T} = \min\{T_0, \frac{c}{1+M^{18}}\}$ *and absolute constants* c, $C > 0$.

Remark 3.2. Theorem 3.1 sharpens and generalizes Theorem 1.1 in the following senses: (a) The assumption (3.1) is weaker than (1.2). (b) It treats the case of general $R \ge 0$ so that the regular set Π is more specifically characterized. It should be emphasized that if $R > 0$ the assumption $N_R \le \epsilon_*$ in (ii) does not require scale critical regularity at the origin.

Proof. (i) For the convenience, we let

$$\mathcal{E}_{R,r_1}(t) := \sup_{R \le r \le r_1} E_r(t)$$

with the constant r_1 to be specified later. The local energy inequality (2.1) with a test function $\varphi \in C_0^\infty(B_{2r})$ such that $0 \le \varphi \le 1$ in B_{2r} with $\varphi = 1$ in B_r and $\|\nabla^k \varphi\|_{L^\infty} \le C_k r^{-k}$ leads to

$$\begin{aligned}\int |v(t)|^2\varphi^2 dx + 2\int_0^t\int |\nabla v|^2\varphi^2 &\le \int |v_0|^2\varphi^2 dx + \int_0^t\int |v|^2\Delta(\varphi^2) + (|v|^2+p)v\cdot\nabla\varphi^2 dxds\\ &\le \int_{B_{2r}} |v_0|^2 dx + \frac{C}{r^2}\int_0^t\int_{B_{2r}} |v|^2 + \frac{C}{r}\int_0^t\int_{B_{2r}} |v|^3 + |p|^{\frac{3}{2}} dxds.\end{aligned}$$

Note that we may take time-independent test functions in the local energy inequality provided the solution is continuous at $t=0$ in L^2_{loc}. See, e.g., [19, Remark 1.2]. This implies

$$\begin{aligned}E_r(t) \le& 2N_R + \frac{C}{r^3}\int_0^t\int_{B_{2r}} |v|^2 + \frac{C}{r^2}\int_0^t\int_{B_{2r}} |v|^3 + \frac{C}{r^2}\int_0^t\int_{B_{2r}} |p|^{\frac{3}{2}}\\ =:& 2N_R + I_{lin} + I_{nonlin} + I_{pr}.\end{aligned} \tag{3.4}$$

For any $\rho \in (3r, 1]$ we decompose the pressure as $p = \tilde{p} + p_h$ with

$$\tilde{p}(x) := \text{p.v.}\int K(x-y)\xi(y)(v\otimes v)(y)dy - \frac{1}{3}(\xi|v|^2)(x),$$

where ξ is a smooth cut-off function with $\xi = 1$ in B_ρ and supported in $B_{2\rho}$ and $K(x) = \nabla^2(\frac{1}{4\pi|x|})$. Since $\Delta p = \Delta\tilde{p}$ in B_ρ, p_h is harmonic in B_ρ. By the mean value property, we have

$$\begin{aligned}\int_{B_{2r}} |p|^{\frac{3}{2}} \le& C\int_{B_{2r}} |\tilde{p}|^{\frac{3}{2}} + C\int_{B_{2r}} |p_h|^{\frac{3}{2}}\\ \le& C\int_{B_{2r}} |\tilde{p}|^{\frac{3}{2}} + C(\frac{r}{\rho})^3\int_{B_\rho} |p_h|^{\frac{3}{2}}\\ \le& C\int_{B_{2r}} |\tilde{p}|^{\frac{3}{2}} + C(\frac{r}{\rho})^3\int_{B_\rho} |\tilde{p}|^{\frac{3}{2}} + C(\frac{r}{\rho})^3\int_{B_\rho} |p|^{\frac{3}{2}}\end{aligned}$$

By the Calderón-Zygmund estimate we see

$$\int_{B_{2r}} |p|^{\frac{3}{2}} \le C\int_{B_{2\rho}} |v|^3 + C(\frac{r}{\rho})^3\int_{B_\rho} |p|^{\frac{3}{2}}. \tag{3.5}$$

We now divide the proof into two cases. Let r_0 be a constant satisfying $0 \le r_0 \le r_1/3$ to be fixed later.

Case I: $R \le r \le r_0$. If $r_0 < R < r_1$, this case is empty, which is fine. Noting that $2r \in [R, r_1]$, we easily observe that

$$I_{lin} \le \frac{C}{r^2}\int_0^t \mathcal{E}_{R,r_1}(s)ds, \tag{3.6}$$

and also by the interpolation inequality,

$$
\begin{aligned}
I_{nonlin} &\leq \frac{C}{r^2}\int_0^t \left(\int_{B_{2r}}|\nabla v|^2\right)^{\frac{3}{4}}\left(\int_{B_{2r}}|v|^2\right)^{\frac{3}{4}} ds + \frac{C}{r^2}\int_0^t\left(\frac{1}{r}\int_{B_{2r}}|v|^2\right)^{\frac{3}{2}} ds\\
&\leq \frac{\epsilon}{r}\int_0^t\int_{B_{2r}}|\nabla v|^2 ds + \frac{C_\epsilon}{r^2}\int_0^t \left(\frac{1}{r}\int_{B_{2r}}|v|^2\right)^3 + \left(\frac{1}{r}\int_{B_{2r}}|v|^2\right)^{\frac{3}{2}} ds\\
&\leq \epsilon\mathcal{E}_{R,r_1}(t) + \frac{C_\epsilon}{r^2}\int_0^t \mathcal{E}^3_{R,r_1}(s) + \mathcal{E}^{\frac{3}{2}}_{R,r_1}(s)ds
\end{aligned}
\tag{3.7}
$$

with some constant $\epsilon \in (0,1)$. For the pressure term, integrating (3.5) in time yields

$$
I_{pr} \leq \frac{C}{r^2}\int_0^t\int_{B_{2\rho}}|v|^3 + \frac{C'r}{\rho}\frac{1}{\rho^2}\int_0^t\int_{B_\rho}|p|^{\frac{3}{2}}
$$

with an absolute constant $C' > 1$. Choose $\rho = 10C'r$ so that $\frac{C'r}{\rho} \leq \frac{1}{10}$ and also let $r_0 = \frac{r_1}{20C'}$. Noting that $2\rho = 20C'r \leq r_1$ and (3.7), we see that

$$
\begin{aligned}
I_{pr} &\leq \frac{C}{r^2}\int_0^t\int_{B_{20C'r}}|v|^3 + \frac{1}{10}\mathcal{E}_{R,r_1}(t)\\
&\leq \epsilon\mathcal{E}_{R,r_1}(t) + \frac{C_\epsilon}{r^2}\int_0^t \mathcal{E}^3_{R,r_1}(s) + \mathcal{E}^{\frac{3}{2}}_{R,r_1}(s)ds + \frac{1}{10}\mathcal{E}_{R,r_1}(t).
\end{aligned}
\tag{3.8}
$$

Hence applying (3.6)-(3.8) in (3.4) with $\epsilon = \frac{1}{20}$, we obtain

$$
\mathcal{E}_{R,r_0}(t) \leq \frac{2\delta}{5} + \frac{C}{R^2}\int_0^t \mathcal{E}^3_{R,r_1}(s) + \mathcal{E}_{R,r_1}(s)ds + \frac{1}{5}\mathcal{E}_{R,r_1}(t).
\tag{3.9}
$$

Case II: $r_0 \leq r \leq r_1$. For $t \leq T$ with T specified later, we estimate the right hand side of (3.4) with the aid of M. A straightforward estimate yields

$$
I_{lin} \leq \frac{C}{r_0^3}\int_0^t\int_{B_2}|v|^2 \leq \frac{Ct}{r_0^3}M \leq \frac{\delta}{30},
$$

provided $t \leq \frac{c\delta r_0^3}{M}$ with a small absolute constant $c \in (0,1)$. In the similar way to (3.7) we have

$$
\begin{aligned}
I_{nonlin} \leq& \frac{C}{r_0^2}\int_0^t\int_{B_2}|v|^3\\
\leq& \frac{C}{r_0^2}\int_0^t\left(\int_{B_2}|\nabla v|^2\right)^{\frac{3}{4}}\left(\int_{B_2}|v|^2\right)^{\frac{3}{4}} ds + \frac{C}{r_0^2}\int_0^t\left(\int_{B_2}|v|^2\right)^{\frac{3}{2}} ds\\
\leq& \frac{Ct^{\frac{1}{4}}M^{\frac{3}{2}}}{r_0^2} + \frac{CtM^{\frac{3}{2}}}{r_0^2}.
\end{aligned}
\tag{3.10}
$$

Thus I_{nonlin} is bounded by $\frac{\delta}{30}$ if $t \le \min\left(\frac{c\delta^4 r_0^8}{M^6}, \frac{c\delta r_0^2}{M^{\frac{3}{2}}}\right)$ with a suitable absolute constant $c \in (0,1)$. Concerning the pressure, we choose $\rho = 1$ (using $3r_1 \le 1 = \rho$) in (3.5) and apply (3.10) to get

$$\begin{aligned} I_{pr} \le & \frac{C}{r^2}\int_0^t \int_{B_{2r}} |\tilde{p}|^{\frac{3}{2}} + Cr\int_0^t \int_{B_1} |p|^{\frac{3}{2}} \\ \le & \frac{C}{r_0^2}\int_0^t \int_{B_1} |\tilde{p}|^{\frac{3}{2}} + Cr_1 M^{\frac{3}{2}} \\ \le & \frac{C}{r_0^2}\int_0^t \int_{B_2} |v|^3 + Cr_1 M^{\frac{3}{2}} \\ \le & \frac{Ct^{\frac{1}{4}}M^{\frac{3}{2}}}{r_0^2} + \frac{CtM^{\frac{3}{2}}}{r_0^2} + Cr_1 M^{\frac{3}{2}} \le \frac{\delta}{30}, \end{aligned} \tag{3.11}$$

provided $t \le \min\left(\frac{c\delta^4 r_0^8}{M^6}, \frac{c\delta r_0^2}{M^{\frac{3}{2}}}\right)$ and $r_1 \le \min\left(\frac{c\delta}{M^{\frac{3}{2}}}, \frac{1}{3}\right)$ with an absolute constant $c > 0$. Making use of these estimates in (3.4), we obtain that

$$\mathcal{E}_{r_0,r_1}(t) \le \frac{\delta}{2} \quad \text{if } \; t \le \min\left(\frac{c\delta r_0^3}{M}, \frac{c\delta^4 r_0^8}{M^6}, \frac{c\delta r_0^2}{M^{\frac{3}{2}}}\right) \;\text{ and }\; r_1 \le \min\left(\frac{c\delta}{M^{\frac{3}{2}}}, \frac{1}{3}\right). \tag{3.12}$$

Combining (3.9) and (3.12), we see

$$\mathcal{E}_{R,r_1}(t) \le \frac{\delta}{2} + \frac{C}{R^2}\int_0^t \mathcal{E}^3_{R,r_1}(s) + \mathcal{E}_{R,r_1}(s)ds, \quad 0 < t < T, \tag{3.13}$$

where

$$T = \min\left(T_0, \frac{c\min\{1,\delta^{12}\}}{1+M^{18}}\right). \tag{3.14}$$

Note that $T \le \min\left(\frac{c\delta r_0^3}{M}, \frac{c\delta^4 r_0^8}{M^6}, \frac{c\delta r_0^2}{M^{\frac{3}{2}}}\right)$ with a suitable small constant $c > 0$ upon the choice of $r_0 = \frac{r_1}{20C'}$ with $r_1 = \min\left(\frac{c\delta}{M^{\frac{3}{2}}}, \frac{1}{3}\right)$. The inequality (3.13) is also true if R is replaced by any $r \in (R, r_1]$ with the same δ. Therefore we may invoke Gronwall Lemma 2.3 with $f(t) = \mathcal{E}_{r,r_1}(t)$ with $a = \frac{\delta}{2}$, $b = \frac{C}{r^2}$, and $m = 3$ to see that

$$\mathcal{E}_{r,r_1}(t) \le \delta \quad \text{for } t \in \left(0, \min\left\{T, \lambda r^2\right\}\right] \;\text{ and }\; r \in (R, r_1], \quad \lambda = \frac{c}{1+\delta^2}. \tag{3.15}$$

This proves (3.2).

In the similar way to (3.7), we see that if $\lambda r^2 \in (0, T]$ and $r \in (R, r_1]$,

$$\begin{aligned} \frac{1}{r^2}\int_0^{\lambda r^2}\int_{B_r} |v|^3 \le & \frac{C}{r^2}\int_0^{\lambda r^2}\left(\int_{B_r}|\nabla v|^2\right)^{\frac{3}{4}}\left(\int_{B_r}|v|^2\right)^{\frac{3}{4}} ds + \frac{C}{r^2}\int_0^{\lambda r^2}\left(\frac{1}{r}\int_{B_r}|v|^2\right)^{\frac{3}{2}} ds \\ \le & C(\lambda^{\frac{1}{4}} + \lambda)E_r(\lambda r^2)^{\frac{3}{2}}. \end{aligned}$$

Hence taking $r_2 := \min\left(\sqrt{\frac{T}{\lambda}}, r_1\right)$, we have from (3.2) and $\lambda \le 1$ that

$$\frac{1}{r^2}\int_0^{\lambda r^2}\int_{B_r} |v|^3 \le C\delta^{\frac{3}{2}} \qquad \text{for } r \in (R, r_2].$$

This together with the pressure bound

$$\frac{1}{r^2}\int_0^{\lambda r^2}\int_{B_r} |p|^{\frac{3}{2}} \le E_r(\lambda r^2) \le \delta$$

leads to (3.3) as desired.

(ii) In order to show the statement (ii), we claim that there exist ϵ_* and $c > 0$ such that if $N_R < \epsilon_*$, then for any $x_0 \in B_{\frac{1}{2}}$ and $r \in (\max(R, cN_R|x_0|), c(1-|x_0|)r_2]$,

$$\frac{1}{r^2}\int_0^{r^2}\int_{B_r(x_0)} |v|^3 + |p|^{\frac{3}{2}}\, dxdt \le \epsilon_{\text{CKN}} \tag{3.16}$$

holds. Here ϵ_{CKN} is the small constant in Lemma 2.2. To this end, we first note that for each $\eta \ge 2N_R$ and $x_0 \in B_{1/2}$, Lemma 2.4 implies

$$\sup_{R(x_0)<r\le\rho} \frac{1}{r}\int_{B_r(x_0)} |v_0|^2 dx \le \eta, \quad \rho = 1-|x_0|,$$

with $R(x_0) = \max\left(\frac{R}{2}, \frac{2N_R}{\eta}|x_0|\right)$. Let $v_{x_0}(x,t) = \rho v(x_0+\rho x, \rho^2 t)$, $p_{x_0}(x,t) = \rho^2 p(x_0+\rho x, \rho^2 t)$ and $\delta = 5\eta$. Since (v_{x_0}, p_{x_0}) solves (NS) in $B_2(0)\times(0,\rho^{-2}T_0)$, corresponding to (v,p) in $B_{2\rho}(x_0)\times(0,T_0)$, and $1/2 \le \rho \le 1$,

$$\|v_{x_0}\|^2_{L^\infty(0,\rho^{-2}T_0;L^2(B_2))} + \|\nabla v_{x_0}\|^2_{L^2(B_2\times(0,\rho^{-2}T_0))} + \|p_{x_0}\|_{L^{\frac{3}{2}}(B_2\times(0,\rho^{-2}T_0))} \le CM,$$

$$\sup_{\rho^{-1}R(x_0)<r\le 1} \frac{1}{r}\int_{B_r(0)} |v_{x_0}(x,0)|^2 dx = \sup_{R(x_0)<r\le\rho} \frac{1}{r}\int_{B_r(x_0)} |v_0|^2 dx \le \frac{\delta}{5},$$

by (3.3) and (3.14) we get

$$\sup_{\rho^{-1}R(x_0)<r\le r_2} \frac{1}{r^2}\int_0^{\lambda r^2}\int_{B_r(0)} |v_{x_0}|^3 + |p_{x_0}|^{\frac{3}{2}} \le C(\delta+\delta^{\frac{3}{2}}).$$

Here $r_2 = \min\left(\sqrt{\frac{T'}{\lambda}}, r_1\right)$ with

$$T' = \min\left(\rho^{-2}T_0, \frac{c\min\{1,\delta^{12}\}}{1+M^{18}}\right), \quad r_1 = \min\left(\frac{c\delta}{M^{\frac{3}{2}}}, \frac{1}{3}\right),$$

with a smaller constant c. Note T' differs from T in (3.14) by the factor ρ^{-2} for T_0. This implies

$$\sup_{R(x_0)\le r\le\rho r_2} \frac{1}{\lambda r^2}\int_0^{\lambda r^2}\int_{B_{\sqrt{\lambda}r}(x_0)} |v|^3 + |p|^{\frac{3}{2}} \le \frac{C(\delta+\delta^{\frac{3}{2}})}{\lambda} \le C(1+\delta^2)(\delta+\delta^{\frac{3}{2}}). \tag{3.17}$$

Take a constant $\delta_0 > 0$ so small that $C(1+\delta_0^2)(\delta_0+\delta_0^{\frac{3}{2}}) \le \epsilon_{\text{CKN}}$. We now assume that v_0 satisfies $N_R \le \delta_0/10$. Then we may choose $\delta = \delta_0$ since $\delta_0 \ge 10N_R$. With this choice and with $\lambda_0 = \lambda(\delta_0)$, (3.17) shows (3.16) holds for $\lambda_0^{\frac{1}{2}}R(x_0) < r \le \lambda_0^{\frac{1}{2}}\rho r_2$. This enables us to apply Lemma 2.2 for $x_0 \in B_{\frac{1}{2}}$ and $t_0 = r^2 \in (\lambda_0 \max\left(R^2, \frac{CN_R^2|x_0|^2}{\delta_0^2}\right), \lambda_0\rho^2 r_2^2]$ to see

$$|v(x_0,t_0)| \le \frac{C_{\text{CKN}}}{r} = \frac{C_{\text{CKN}}}{\sqrt{t_0}},$$

and hence v is regular at (x_0,t_0). Since $r_2 = \min\left(\sqrt{\frac{T'}{\lambda_0}}, r_1\right)$ and $1/2 \le \rho \le 1$,

$$\rho^2\lambda_0 r_2^2 = \rho^2\min(T', \lambda_0 r_1^2) = \rho^2\min\left(\rho^{-2}T_0, \frac{c\min\{1,\delta_0^{12}\}}{1+M^{18}}, \lambda_0 r_1^2\right) \ge \min\left(T_0, \frac{c}{1+M^{18}}\right).$$

Thus $|v(x_0,t_0)| \le C_{\text{CKN}} t_0^{-1/2}$ for $x_0 \in B_{1/2}$ and

$$\max\left(R^2, cN_R^2|x_0|^2\right) \le t_0 \le \min\left(T_0, \frac{c}{1+M^{18}}\right).$$

This shows Part (ii) of Theorem 3.1. $\square$

One implication of Theorem 3.1 is that for L^2_{uloc} initial data in $\mathbb{R}^3$, similar conclusions hold for the local energy solutions:

Theorem 3.3. *Let v be a local energy solution of (NS) in $\mathbb{R}^3 \times (0, T_0)$ associated with initial data $v_0 \in L^2_{\mathrm{uloc}}$, $N_R = \sup_{R<r\le 1} \frac{1}{r}\int_{B_r(0)} |v_0|^2 < \infty$. The following holds true.*
(i) *For $R \in [0, \frac{1}{3}]$, let $\delta \ge 5N_R$. Then*

$$E_r(t) \le \delta \quad \textit{for } t \le \min\{\lambda r^2, T_1\} \qquad \textit{for all } r \in (R, \frac{1}{3}], \tag{3.18}$$

holds with constants given by

$$\lambda = \frac{c}{1+\delta^2} \le 1, \qquad T_1 = \min\left(T_0, \frac{c\min(1,\delta^4)}{1+\|v_0\|^{12}_{L^2_{\mathrm{uloc}}}}\right).$$

Moreover for any $r \in (R, R_1]$ with $R_1 := \min\{\sqrt{\frac{T_1}{\lambda}}, \frac{1}{3}\}$, there exists $c_2(t)$ such that

$$\frac{1}{r^2}\int_0^{\lambda r^2}\int_{B_r} |v|^3 + |p - c_2(t)|^{\frac{3}{2}} dxdt \le C(\delta + \delta^{\frac{3}{2}}). \tag{3.19}$$

Here c and $C > 0$ are absolute constants.
(ii) *There are absolute constants ϵ_*, c_0, c_1, and $C_2 > 0$ such that the following holds. If $N_R \le \epsilon_*$ for some $R \le \min\{\sqrt{T_2}, \frac{1}{3}\}$ with $T_2 = \min(T_0, \frac{c_1}{1+\|v_0\|^{12}_{L^2_{\mathrm{uloc}}}})$, then v is regular in the set*

$$\Pi := \left\{(x,t) \in B_{\frac{1}{2}} \times [R^2, T_2];\ c_0 N_R^2 |x|^2 \le t\right\}$$

and satisfies

$$|v(x,t)| \le \frac{C_2}{\sqrt{t}} \qquad \textit{for } (x,t) \in \Pi. \tag{3.20}$$

We now show Theorem 3.4, which contains Theorem 1.3 as a special case and is useful for further applications. To this end, define $\dot{N}_R$ by

$$\dot{N}_R := \sup_{r>R} \frac{1}{r}\int_{B_r(0)} |v_0|^2 \qquad (R \ge 0).$$

Theorem 3.4. *Let (v,p) be a local energy solution in $\mathbb{R}^3 \times (0,\infty)$ for the initial data $v_0 \in L^2_{\mathrm{uloc}}(\mathbb{R}^3)$. Let ϵ_* and c_0 be the absolute constants in Theorem 3.3 (ii). The following statements hold:*
(i) *If v_0 satisfies (1.6), i.e.,*

$$M_1 := \sup_{x_0\in\mathbb{R}^3}\sup_{r\ge 1} \frac{1}{r}\int_{B_r(x_0)} |v_0|^2\, dx < \infty,$$

and if

$$\dot{N}_R \le \epsilon_* \qquad \textit{for some } R \ge 0, \tag{3.21}$$

then v is regular in the set $\left\{(x,t) \in \mathbb{R}^3 \times (0,\infty)\ :\ \max\{R^2, c_0 \dot{N}_R^2 |x|^2\} \le t\right\}$.
(ii) *Suppose $v_0 \in L^{2,-1}(\mathbb{R}^3)$. For any $0 < \delta \le \epsilon_*$, there exist positive constant $T(v_0,\delta)$ such that v is regular in the set*

$$\left\{(x,t) \in \mathbb{R}^3 \times (0,\infty):\ c_0\delta^2|x|^2 \le t \le T(v_0,\delta)\right\}. \tag{3.22}$$

If $v_0 \in L^{2,-1}(\mathbb{R}^3)$ also satisfies (1.6), then for any $\delta \in (0,\epsilon_*]$, there is $T'(v_0,\delta)$ such that v is regular in

$$\left\{(x,t) \in \mathbb{R}^3 \times (0,\infty):\ \max(c_0\delta^2|x|^2, T'(v_0,\delta)) \le t\right\}. \tag{3.23}$$

It is also regular in

$$\{(x,t) \in \mathbb{R}^3 \times (0,\infty) : \; c_0\epsilon_*^2|x|^2 \le t < r_*^2 T_2(M_2)\}, \tag{3.24}$$

with $r_* := \sup\{r > 0 \mid \|v_0\|^2_{L^{2,-1}(B_r)} \le \epsilon_*\}$, $0 < r_* \le \infty$, *and* $M_2 = [(\max(1, \frac{1}{r_*})M_1]^{1/2}$. *When* $r_* = \infty$, *i.e.,* $\|v_0\|^2_{L^{2,-1}(\mathbb{R}^3)} \le \epsilon_*$, *the regular set* (3.24) *has no time upper bound.*

Proof. (i) We first consider the case $R > 0$. Let $u_0(x) = \lambda v_0(\lambda x)$, $u(x,t) = \lambda v(\lambda x, \lambda^2 t)$ for $\lambda > 2R$. By the assumption, we have

$$\sup_{\frac{R}{\lambda} \le r < 1} \frac{1}{r}\int_{B_r} |u_0|^2 \le \dot{N}_R \le \epsilon_*,$$

and

$$\|u_0\|_{L^2_{\text{uloc}}} = \sup_{x_0 \in \mathbb{R}^3} \left(\frac{1}{\lambda}\int_{|x-x_0|<\lambda} |v_0(x)|^2 dx\right)^{1/2} \le \max\left(\frac{1}{R}\|v_0\|^2_{L^2_{\text{uloc}}}, M_1\right)^{1/2} =: C_R, \tag{3.25}$$

with $C_R > 0$ independent of λ. By Theorem 3.3 (ii), there exists $T_2' = T_2'(C_R)$ independent of λ such that u is regular if $\max\{R^2/\lambda^2, c_0\dot{N}_R^2\,|x|^2\} \le t \le T_2'$ for $\lambda \ge \frac{R}{\sqrt{T_2'}}$. Scaling back we see that v is regular if $\max\{R^2, c_0\dot{N}_R^2\,|x|^2\} \le t \le \lambda^2 T_2'$. Since $\lambda > \max\{2R, \frac{R}{\sqrt{T_2'}}\}$ is arbitrary, v is regular in the set $\{(x,t) : \max\{R^2, c_0\dot{N}_R^2\,|x|^2\} \le t\}$. This proves the case $R > 0$. For the case $R = 0$, by $\dot{N}_r \le \dot{N}_0$, the above argument shows v is regular in $\{(x,t) : \max\{r^2, c_0\dot{N}_r^2\,|x|^2\} \le t\}$. Since $r > 0$ is arbitrary, we conclude the proof.

(ii) Suppose now $v_0 \in L^{2,-1}(\mathbb{R}^3)$. For any $\delta \in (0, \epsilon_*]$, there exists $R_0 > 0$ such that

$$\sup_{0<r\le R_0} \frac{1}{r}\int_{B_r} |v_0|^2\,dx \le \int_{B_{R_0}} \frac{|v_0|^2}{|x|} dx \le \delta.$$

Let $u_0(x) = \lambda v_0(\lambda x)$ and $u(x,t) = \lambda v(\lambda x, \lambda^2 t)$ with $\lambda = R_0$. We easily see $u_0 \in L^2_{\text{uloc}}$ and $\sup_{0<r\le 1} \frac{1}{r}\int_{B_r} |u_0|^2\,dx \le \delta \le \epsilon_*$. By Theorem 3.3 (ii), u is regular if $c_0\delta^2|x|^2 \le t \le T_2(u_0)$. Hence v is regular in the set $\{(x,t) \mid c_0\delta^2|x|^2 \le t \le R_0^2 T_2(u_0)\}$. Note $T_2(u_0)$ depends on both R_0 and $\|v_0\|_{L^2_{\text{uloc}}}$ and goes to zero rapidly as $R_0 \to 0$.

Suppose now $v_0 \in L^{2,-1}$ also satisfies (1.6). There is $\rho > 0$ such that $\int_{\mathbb{R}^3 \setminus B_\rho} \frac{1}{|x|}|v_0|^2 \le \delta/2$. Let $R_1 = \max(\rho, \frac{2}{\delta}\int_{B_\rho} |v_0|^2)$. For any $r \ge R_1$, we have

$$\frac{1}{r}\int_{B_r} |v_0|^2 \le \frac{1}{r}\int_{B_\rho} |v_0|^2 + \int_{B_r \setminus B_\rho} \frac{1}{|x|}|v_0|^2 \le \frac{\delta}{2} + \frac{\delta}{2}.$$

Thus $\dot{N}_{R_1} \le \delta$. By Part (i), v is regular in the set $\{\max(R_1^2, c_0\delta^2|x|^2) \le t\}$.

The remaining statement follows by choosing $\delta = \epsilon_*$, $\lambda = R_0 \to r_*$, and noting $\|u_0\|_{L^2_{\text{uloc}}} \le M_2$ using (3.25). □

We next apply Theorem 3.4 to prove Corollary 1.4.

Proof of Corollary 1.4. (i) By the assumptions, we have $v_0 \in L^2(\mathbb{R}^3)$, and hence it satisfies (1.6) since

$$\sup_{x_0 \in \mathbb{R}^3} \frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \frac{1}{r}\|v_0\|^2_{L^2}$$

This estimate also implies

$$\sup_{x_0 \in \mathbb{R}^3, r \ge R_*} \frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \epsilon_* \qquad \text{with } R_* := \frac{\|v_0\|^2_{L^2}}{\epsilon_*}. \tag{3.26}$$

Therefore applying Theorem 3.4 (i) to $v_{x_0}(x,t) := v(x+x_0,t)$ for each $x_0 \in \mathbb{R}^3$, we see

$$\Big\{(x,t) \in \mathbb{R}^3 \times (0,\infty) \; : \; \max\{R_*^2, c_0 \dot{N}_{R_*}^2 \, |x-x_0|^2\} \le t\Big\}$$

is the regular set of v. Since $x_0 \in \mathbb{R}^3$ is arbitrary, this shows that v is regular for $t \ge R_*^2 = \|v_0\|_{L^2}^4/\epsilon_*^2$.

We next show that there exists $M = M(\|v_0\|_{L^{2,\alpha}})$ such that if $|x_0| \ge 2R_*$,

$$\sup_{M|x_0|^{-\alpha} \le r \le R_*} \frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \epsilon_*. \tag{3.27}$$

Indeed since $r \le R_* \le |x_0|/2$, we see

$$\frac{1}{r}\int_{B_r(x_0)} |v_0|^2 = \frac{1}{r}\int_{B_r(x_0)} \frac{|x|^\alpha |v_0|^2}{|x|^\alpha} \le \frac{2^\alpha}{r|x_0|^\alpha}\|v_0\|_{L^{2,\alpha}}^2,$$

from which (3.27) follows with $M = \frac{2^\alpha \|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*}$. Combining this with (3.26) implies

$$\sup_{M|x_0|^{-\alpha} \le r} \frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \epsilon_* \qquad \text{provided } |x_0| \ge 2R_*.$$

Then if $|x_0| \ge 2R_*$, we may apply Theorem 3.4 (i) for v_{x_0} with $R = M|x_0|^{-\alpha}$, to see that v is regular at (x_0,t) for $t \ge M^2|x_0|^{-2\alpha}$. Since v is also regular for $t \ge R_*^2 = \|v_0\|_{L^2}^4/\epsilon_*^2$, this finishes the proof of (i) of Corollary 1.4, with $K = \max(M^2, 4^\alpha R_*^{2+2\alpha})$. Note that $K|x|^{-2\alpha} \ge R_*^2$ when $|x| \le 2R_*$.

(ii) Now $\alpha \in (-1,0)$. We use similar approach as above: If $r \ge |x_0|/2$,

$$\frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \frac{1}{r}\int_{B_{3r}} |v_0|^2 = \frac{1}{r}\int_{B_{3r}} \frac{|x|^\alpha |v_0|^2}{|x|^\alpha} \le \frac{C_1}{r^{1+\alpha}}\|v_0\|_{L^{2,\alpha}}^2.$$

Hence we have

$$\frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \epsilon_* \qquad \text{if } \; r \ge \max\left\{\frac{|x_0|}{2}, \left(\frac{C_1\|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*}\right)^{\frac{1}{1+\alpha}}\right\}. \tag{3.28}$$

Therefore, by virtue of (1.6), we may use Theorem 3.4 (i) to see that

$$v \text{ is regular at } (x_0,t) \; \text{ if } \; t \ge \max\left\{\frac{|x_0|^2}{4}, \left(\frac{C_1\|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*}\right)^{\frac{2}{1+\alpha}}\right\}. \tag{3.29}$$

For $r \le |x_0|/2$, we have

$$\frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \frac{C_2}{r|x_0|^\alpha}\int_{B_r(x_0)} |x|^\alpha \, |v_0|^2 \le \frac{C_2}{r|x_0|^\alpha}\|v_0\|_{L^{2,\alpha}}^2,$$

and hence

$$\frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \epsilon_* \qquad \text{if } \; \frac{C_2\|v_0\|_{L^{2,\alpha}}^2}{|x_0|^\alpha \epsilon_*} \le r \le \frac{|x_0|}{2}. \tag{3.30}$$

We may increase C_1 depending only on α and C_2 so that when $\frac{|x_0|}{2} \ge \left(\frac{C_1\|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*}\right)^{\frac{1}{1+\alpha}}$, then $\frac{|x_0|}{2} \ge \frac{C_2\|v_0\|_{L^{2,\alpha}}^2}{|x_0|^\alpha \epsilon_*}$. For such x_0, (3.28) and (3.30) imply

$$\frac{1}{r}\int_{B_r(x_0)} |v_0|^2 \le \epsilon_* \qquad \text{if } \; r \ge \frac{C_2\|v_0\|_{L^{2,\alpha}}^2}{|x_0|^\alpha \epsilon_*}.$$

Thus Theorem 3.4 (i) shows v is regular for $t \ge \frac{C_2^2\|v_0\|_{L^{2,\alpha}}^4}{|x_0|^{2\alpha}\epsilon_*^2}$. This and (3.29) show Part (ii). □

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