The Cauchy problem for the Navier-Stokes type equations on the Heisenberg group

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Abstract

The aim of this article is to give a résumé of already published paper [28]. Thus we introduce an existence and uniqueness result of solutions for the Cauchy problem of the Navier-Stokes type equations associated with the sublaplacian provided by the left invariant vector fields on the Heisenberg group. To avoid the difficulty of the non-commutative which is intrinsic in 2-step stratified Lie groups, we construct the solenoidal space by using the right invariant vector fields on the Heisenberg group.

[1] [2]

1 Introduction and main results

In this article, we introduce without the proofs an existence and uniqueness result for the Cauchy problem of the Navier-Stokes type equations on the Heisenberg group in [28] as an analogy of them on the Euclidean space. For the Cauchy problem of the Navier-Stokes equations on the Euclidean space $\mathbb{R}^3$,

$$\begin{cases} \boldsymbol{u}_t - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = 0, \ x \in \mathbb{R}^3, \ t > 0, \\ \mathrm{div}\boldsymbol{u} = 0, \ x \in \mathbb{R}^3, \ t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{a}, \quad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

T. Kato [18] and Y. Giga [11] gave the following existence and uniqueness of local and global solutions of (1.1):

· If $a \in L^3_\sigma$, there exists a unique solution $u \in C([0,T), L^3_\sigma)$ for $T > 0$. (1.2)

· Moreover if there exists a constant δ such that $\|\boldsymbol{a}\|_{L^3} \leq \delta$, we can take $T = \infty$ in (1.2).

[1] *Keywords*: Heisenberg group, Navier-Stokes equations, Local solution, Global solution.
[2] **AMS Subject Classification 2010: 35A01, 35A02.**

After this result, this direction has been deeply investigated (for instance, [4], [20], [21], [22], [25] and so on). On the other hand, many studies of partial differential equations on 2-step stratified Lie groups have been done by many mathematicians (for instance, [1], [2], [3], [6], [7], [8], [9], [10], [12], [14], [15], [17], [24], [26], [27], [31], [32] and so on). Many of them, however, are linear partial differential equations. Recently it has started out even the results of nonlinear evolution equations (for instance, [5], [16], [26], [29], [30] and so on). The studies of partial differential equations on 2-step stratified Lie groups have become more difficult than the Euclidean cases by the non-commutative which is intrinsic in 2-step stratified Lie groups.

By the way, the following two results have led us to this investigation. One is Lewy equations. They strongly suggest that the solvability is characterized by the Lie group structure of the equation. The other is Hörmander's condition. It strongly suggests the existence of a differential operator on the group corresponding to the Laplacian in Euclidean space. Lewy equation was given from the investigation of differential equations without solutions in [23] and Hörmander's condition was given from the investigation of hypoellipticity in [13]. From these two results, we have the following question: What is the relationship between the structure of Lie groups and the solvability for nonlinear evolution equations? The motivation for our investigation is to (partially) clarify how groups affect solvability.

Here we prepare some notations needed to assert our results. Let $g = (x, y, s)$ and $g' = (x', y', s') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{2d+1}$. Then we consider $\mathbb{R}^{2d+1}$ with the group law defined by

$$g \cdot g' = (x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 1/2(x' \cdot y - x \cdot y')), \tag{1.3}$$

where $x \cdot y = \sum_{j=1}^{d} x_j y_j$. The group $\mathbb{R}^{2d+1}$ with respect to the group law defined by (1.3) is called the Heisenberg group and denoted by $\mathbb{H}^d$. For $\lambda > 0$, we define the dilation δ_λ by

$$\delta_\lambda(x, y, s) = (\lambda x, \lambda y, \lambda^2 s).$$

The left-invariant vector fields in the Heisenberg group $\mathbb{H}^d$ as $\mathbb{R}^{2d+1}$ are represented by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial s}, \; X_{d+j} = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial s} \text{ and } X_{2d+1} = \frac{\partial}{\partial s}$$

for $j = 1, 2, \cdots, d$ and these make a basis for the Lie algebra of $\mathbb{H}^d$. On the other hand, the right-invariant vector fields in $\mathbb{H}^d$ are represented by

$$\tilde{X}_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial s}, \; \tilde{X}_{d+j} = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial s} \text{ and } \tilde{X}_{2d+1} = \frac{\partial}{\partial s}$$

for $j = 1, 2, \cdots, d$ and these also make a basis for the Lie algebra $\mathfrak{h}$ of $\mathbb{H}^d$. Put

$$V_1 = \{X_1, X_2, \cdots, X_{2d}\} \;\; (\dim V_1 = 2d \; (= Q_1)) \;\; \text{and} \;\; V_2 = \{X_{2d+1}\} \;\; (\dim V_2 = 1 \; (= Q_2)).$$

Then $\mathfrak{h} = V_1 \oplus V_2$ and

$$\operatorname{rank}(\operatorname{Lie}\{X_1, X_2, \cdots X_{2d}\}(g)) = 2d + 1, \quad \text{(Hörmander's condition)}$$

for any $g \in \mathbb{H}^d$. The homogeneous dimension N of $\mathbb{H}^d$ is given by $N = 2d + 2$ (If $\mathbb{R}^{2d+1} = \mathbb{R}^{Q_1} \times \mathbb{R}^{Q_2} = \mathbb{R}^{2d} \times \mathbb{R}^1$, then $N = \sum_{i=1}^{2} iQ_i$). Define the vector ∇ of the left-invariant vector fields by $\nabla = (X_1, X_2, \cdots, X_{2d})$ and the vector $\tilde{\nabla}$ of the right-invariant vector fields by $\tilde{\nabla} = (\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_{2d})$. Moreover the divergence of the left invariant vector fields is denoted by div and the divergence of the right invariant vector fields is denoted by div_R, respectively. The sub-Laplacian $\mathcal{L}$ and $\mathcal{L}_R$ on $\mathbb{H}^d$ are defined by

$$\mathcal{L} = -\sum_{j=1}^{2d} X_j^2$$

and

$$\mathcal{L}_R = -\sum_{j=1}^{2d} \tilde{X}_j^2,$$

respectively.

We consider the following Cauchy problem of the Navier-Stokes type equations,

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \tilde{\nabla}\pi = 0, \\ \operatorname{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g, 0) = \boldsymbol{a}(g) \end{cases} \tag{1.4}$$

for $g \in \mathbb{H}^d$ and $t > 0$, limited to Lebesgue spaces on 2-step stratified Lie groups, especially Heisenberg group, instead of the Euclidean spaces.

Let $\boldsymbol{b} = (b_1, \cdots, b_{2d}) \in L^p(\mathbb{H}^d)^{2d}$, $1 < p < \infty$, and set

$$\boldsymbol{Pb} = (P_1\boldsymbol{b}, \cdots, P_{2d}\boldsymbol{b}), \ P_j\boldsymbol{b} = b_j + \sum_{k=1}^{2d} R_j \bar{R}_k b_k$$

and

$$Q\boldsymbol{b} = -\sum_{i=1}^{2d} \mathcal{L}_R^{-1} \tilde{X}_i b_i,$$

where $R_l = \tilde{X}_l \mathcal{L}_R^{-\frac{1}{2}}$ and $\bar{R}_l = \mathcal{L}_R^{-\frac{1}{2}} \tilde{X}_l$, $l = 1, \cdots, 2d$. Then we decompose $\boldsymbol{b}$ by

$$\boldsymbol{b} = \boldsymbol{Pb} + \tilde{\nabla} Q\boldsymbol{b}.$$

If $\pi = -Q(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, then (1.4) becomes

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} = -\boldsymbol{P}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \\ \mathrm{div}_R\boldsymbol{u} = 0, \\ \boldsymbol{u}(g,0) = \boldsymbol{a}(g). \end{cases} \tag{1.5}$$

If there exists a solution of (1.5), then by Proposition 5 in [28], a solution $\boldsymbol{u}$ can be expressed by

$$\boldsymbol{u}(t) = S(t)\boldsymbol{a} - \int_0^t S(t-\sigma)\boldsymbol{P}[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}](\sigma)d\sigma, \tag{1.6}$$

where $S(t)\boldsymbol{a} = (a_1 * h_t, a_2 * h_t, \cdots, a_{2d} * h_t)$ and a solution of the integral equations (1.6) satisfies (1.5) in the sence of distributions.

In this article, we introduce the following theorems for an existence and uniqueness of local solutions and global solutions. The basic idea of the proof is similar to [18] and [11]. We refer to [28] for detail proofs.

Theorem 1 ([28])**.** *Let $N = 2d+2 < p < \infty$. Then for an initial value $\boldsymbol{a} \in J^N(\mathbb{H}^d) = \{\boldsymbol{a} = (a_1, \cdots, a_{2d}) \in L^N(\mathbb{H}^d)^{2d},\ \mathrm{div}_R\boldsymbol{a} = 0\}$, there exists a positive constant T and the following holds:*

(i) (Existence) The integral equation (1.6) has a solution $\boldsymbol{u}(t)$ satisfying the following conditions:

$$\cdot\, \boldsymbol{u}(t) \in C([0,T), L^N(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \|\boldsymbol{u}(t) - \boldsymbol{a}\|_N = 0,$$

$$\cdot\, \mathrm{div}_R\boldsymbol{u}(t) = 0, \tag{1.7}$$

$$\cdot\, \boldsymbol{u}(t) \in C((0,T), L^p(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}-\frac{N}{2p}} \|\boldsymbol{u}(\sigma)\|_p = 0 \text{ and} \tag{1.8}$$

$$\cdot\, \nabla\boldsymbol{u}(t) \in C((0,T), L^N(\mathbb{H}^d)^{2d\times 2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}} \|\nabla\boldsymbol{u}\|_N = 0. \tag{1.9}$$

(ii) (Uniqueness) If $\boldsymbol{v}(t)$ satisfies (1.6), (1.7), (1.8) and (1.9), then $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for $0 \le t < T$.

Theorem 2 ([28])**.** *Let $N = 2d+2 < p < \infty$.*

(i) (Existence) If an initial value $\boldsymbol{a} \in J^N(\mathbb{H}^d)$ satisfies

$$\|\boldsymbol{a}\|_N \le C_{N,p}$$

for a positive constant $C_{N,p}$, the integral equation (1.6) has a solution $\boldsymbol{u}(t)$ satisfying the following conditions.

$$\cdot\, \boldsymbol{u}(t) \in C([0,\infty), L^N(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \|\boldsymbol{u}(t) - \boldsymbol{a}\|_N = 0,$$

$$\cdot\, \mathrm{div}_R \boldsymbol{u}(t) = 0, \tag{1.10}$$

$$\cdot\, \boldsymbol{u}(t) \in C((0,\infty), L^p(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}-\frac{N}{2p}} \|\boldsymbol{u}(\sigma)\|_p = 0, \tag{1.11}$$

$$\cdot\, \nabla\boldsymbol{u}(t) \in C((0,\infty), L^N(\mathbb{H}^d)^{2d\times 2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}} \|\nabla\boldsymbol{u}\|_N = 0 \ \textit{and} \tag{1.12}$$

$\cdot$ *there exists a positive constant C such that*

$$\|\boldsymbol{u}(t)\|_p \le Ct^{-\frac{1}{2}-\frac{N}{2p}},\ \|\nabla\boldsymbol{u}(t)\|_N \le Ct^{-\frac{1}{2}}.$$

(ii) (Uniqueness) If $\boldsymbol{v}(t)$ satisfies the integral equation (1.6), (1.10), (1.11) and (1.12). Then $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for $0 \le t < \infty$.

We find out the following from the main results:

- The integrability of the space of solutions depends on the dilation δ_λ and the homogeneous dimension N on the Heisenberg group, not the topological dimension.
- The dimension of the space of solutions depends on dimV_1, that is, Hörmander's condition on the Heisenberg group.
- Our idea to avoid the difficulty of the non-commutative coming from the Heisenberg group is to use the right-invariant fields to construct the solenoidal space. Hence we use the left-invariant vector fields to construct the equations and use the right-invariant vector fields to construct the space of the initial value and solutions.

2 Future work

In this article, we have introduced an existence and uniqueness result of solutions for the Cauchy problem of the Navier-Stokes type equations associated with the sublaplacian on the Heisenberg group. As we have seen, they depend on the dilation, homogeneous dimension, Hörmander's condition and non-commutative on the Heisenberg group. What happens if we change groups? It is known that the Heisenberg group has a close relationship with the Gabor transform. On the other hand, it is also known that NA group (or $ax + b$ group) has a close relationship with the wavelet transform (for instance, we refer to [19]). So it is natural to consider the NA group after the consideration on the Heisenberg group from the point of view of time-frequency analysis. Since the Heisenberg group is a nilpotent Lie group, that is, it is a unimodular Lie group, but NA group is a solvable Lie group, that is, it is a non-unimodular Lie group, the situation on NA group is considerably more complicated than on the Heisenberg group. However, we consider that it is an interesting subject to clarify the relationship between the properties of non-unimodular Lie groups and the solvability of nonlinear evolution equations, which are not encountered as far as nilpotent Lie groups (for example the Heisenberg group, the diamond group, stratified Lie groups and graded Lie groups and so on) are considered.

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