Optimal execution under a generalized price impact model with Markovian exogenous orders in a continuous–time setting

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Abstract

In this paper, we analyze a continuous–time analog of the optimal trade execution problem with generalized price impacts, which was recently discussed in [11] for a discrete–time setting. The market model considers transient price impacts of random trade execution volumes posed by small traders as well as a large trader.

Our problem is formulated as a stochastic continuous control problem over a finite horizon of maximizing the expected utility from the final wealth of the large trader with Constant Absolute Risk Aversion (CARA) von Neumann–Morgenstern (vN-M) utility function. By examining the Hamilton–Jacobi–Bellman (HJB) equation, we characterize the optimal value function and optimal trade execution strategy, and conclude that the trade execution strategy is a time-dependent affine function of three state variables: the remained trade execution volume of the large trader and, so–called, the residual effects of past price impacts caused by both of the large trader and other small traders and the small traders' aggregate volume of orders itself. Further, the time-dependent coefficients could be derived from a solution of a system of ordinary differential equations (ODEs) with terminal conditions, which is numerically tractable.

1 Introduction

The optimal execution problem, which stems from the costs of trading a large volume of orders, has received much attention over a few decades. The so–called "high–frequency trading (HFT)" or "algorithmic trading (AT)" account for the problem as the trading systems are undergoing a revolution in terms of technological changes and the trading venues have diversified over a few decades. In a real market, institutional traders (or large traders) usually have a great influence on the market or the market price of traded (risky) assets they buy or sell through their own (large) order submission. Therefore, large traders have to take into account the "price impact," which can be seen as a kind of trading cost when constructing an execution strategy. On the other hand, small traders are generally assumed not to influence the market price as individual traders. The aggregate volume of orders submitted by them may have, however, have some impact on risky assets they trade through their submission. [39] statistically show that the small trades have relatively by far larger impacts on the price than that of large trades.

The last decades have witnessed a huge (and worldwide) change in the trading system on stock exchanges. For example, as stated in [33], the regulatory development of the HFT was accelerated over the 1990s for the financial market to be more competitive among market participants. The related regulation, Regulation ATS (alternative trading systems; Reg ATS) in 2000, enforced in the U. S., enabled sorts of non-exchange competitors to enter into the marketplace. However, the Regulation National Market System (Reg NMS) in 2007 and Market in Financial Instrument Directive (MiFID) in 2007, enforced in the U.S. and Europe, respectively, brought about a negative outcome: even though these regulations are designed for encouraging new competition and trading venues, equity markets in the U.S. and Europe fragmented since tradings spread out among sorts of exchanges and financial markets.

In the light of the emergence of MiFID in 2007, considerable concerns about the "dark pool" have arisen among practitioners and researchers. Since after the MiFID was enforced in Europe, there has been a rapid rise in the use of dark pools among institutional traders, such as a pension fund manager, where the execution of a large block of orders are not informed to the market participants and the trader of smaller orders are assured to be more protected. Then, the question "how traders act in the dark pool" and "to what extent the dark pool affects the market quality and market efficiency" has attracted both empirical and theoretical researchers in the last decade. The results obtained from [23] solve a cost minimization problem of trading at the traditional exchange and dark pool employing a multidimensional Poisson process to express the situation of dark pool and derive a relevant matrix differential equation. For more detail, see [28].

To address the difficulties large traders face in the paradigm mentioned above, this paper investigates a continuous–time execution problem associated with the interaction among a large trader and non–large traders (small traders hereafter) from a theoretical point of view. This research is an extension of [11]. The papers [27], [34], [35], [11], and [37] theoretically examine how the existence of small traders affects the execution strategy and trade performance of large traders through the following two models: a single–large–trader Markov decision model, and a two–large–trader Markov game model. These models then yield the optimal execution strategy and an equilibrium execution strategy at a Markov perfect equilibrium. These investigations reveal that both strategies are not necessarily deterministic, although a multitude of researches show that optimal and equilibrium execution strategies often become deterministic. Incorporating the price impact caused by small traders into the price impact modeling is the novelty of these researches. The formulation of an execution problem as a game model is also a significant factor in analyzing how the existence of other large traders affects the execution strategy of a large trader.

We conduct our research in line with previous researches such as the ones mentioned in the last paragraph: consider an execution problem where a risk–averse large trader maximizes his/her expected utility arising from the terminal wealth in a finite time interval. The price impact is supposed to be not only temporary or permanent but also transient. A multitude of researches prevail the importance of considering the transient price impact model from both theoretical and empirical points of view. Also, the aggregate order submission posed by small traders (which are assumed to be random), as well as the order submitted by a large trader, invoke the price impact in our assumption. This is the result of reflecting the research of [39]. This research includes similar results as [27], [35], and [11]: the optimal execution strategy for the large trader is obtained in explicit form and is not necessarily deterministic, and the aggregate orders posed by small traders have an indirect impact on the optimal execution strategy through the residual effect. What makes this research different from the existing researches is that the aggregate volume posed by small traders also have a direct impact on the optimal execution strategy. We will explain this point in more detail in Section 3.

This paper proceeds as follows. In Section 2, we construct a market model which characterizes the generalized price impact model through the definition of the price impact caused by the aggregate orders which small traders pose. Then we solve the maximization problem of the expected utility of a risk–averse large trader with Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility (or negative exponential utility) from the wealth at the maturity in section 3. This leads to an optimal execution strategy. Finally, Section 4 concludes.

1.1 Related Literature

Much of the researches have been conducted to search for the optimal trading performance with trading costs. The classical paper [38] argues the cost of trading (i.e., market impact or price impact) and the cost of not trading (i.e., adverse selection). [15] and [16] theoretically consider portfolio selection problems with transaction costs (which can be seen as price impact) by assuming the quadratic trading costs for the trading shares. They show that the optimal trading strategies, under the maximization problem of the sum of the all future expected return with the penalty for the risks and transaction cost, becomes a weighted average of the existing portfolio and the "aim portfolio," which is the weighted average of the current Markowitz portfolio and the expected Markowitz portfolio of the remaining infinite future horizon. Another research [31] further investigates these works when a CARA investor executes a large amount of orders in a finite time horizon and shows that the CARA investor is sensitive to the risk which the return–predicting factor causes while it is not the case in the above model. Moreover, [3] develop a model of a market where heterogeneous traders with mean–variance preference continuously trade with quadratic trading costs, which yields a unique equilibrium return using a system of coupled but linear forward–backward stochastic differential equations.

Another research derives the optimal (or best) execution strategies for institutional traders. The pioneering theoretical study for optimal execution strategy is done by [2] which address the optimization problem of minimizing the expected execution cost in a discrete–time framework via a dynamic programming approach and show that the optimal execution strategy is the one equally split over (finite) time horizon under the presence of temporary price impact. Subsequently, [1] derive an optimal execution strategy by considering both the execution cost and volatility risk, which entails the analysis with a mean–variance approach. [4] incorporates the price impact caused by other traders into the construction of the midprice process, showing that the optimal execution strategies are different from the one obtained in [1] when the price impacts caused by small traders and coincide with the one obtained in [1] when small traders are assumed to have no influence on the midprice. As an extension of this research, [6] demonstrates a multi–assets execution problem.

As in the analysis of [2], some researches apply a method of dynamic programming approach. For example, [5] study the optimal execution strategies considering the VWAP as well as the market order–flow and provide an optimal execution speed (and strategies) in explicit forms. On the other hand, [18] focuses on constructing a model which explains a guaranteed VWAP strategy with risk mitigating and finds that optimal trading speed for a guaranteed VWAP strategy is characterized with a Hamiltonian system (through Legendre transform). [6] consider a correlated multi–asset liquidation problem with the information of untraded assets incorporated into the price dynamics. [25] and [26] construct models for an investor to maximize an expected utility payoff from the final wealth at the maturity via a dynamic programming approach. In a discrete–time setting, these papers explicitly derive an optimal execution strategy in a deterministic and nonrandomized class under the assumption that there exists a residual effect of the transient price impact which dissipates over the trading time window. Our analysis is also involved in these kinds of works.

The modeling of the price impact plays an indispensable role in the research of optimal execution strategies. A number of empirical and theoretical researches investigate whether the "transient price impact" modeling is compatible with the real situation. [14], [21] and [42] consider the so–called "no–arbitrage" condition under the transient price impact model. The famous paper [32] shows that the resilience effect of the limit order book does affect the optimal execution strategies. For multiple large traders' optimal execution problems, [30] and [41] derive equilibrium execution strategies under a transient price impact model. These execution strategies are in a deterministic and static class, whereas the optimal execution strategies obtained from our model are random and dynamic. On the contrary, the temporary and permanent price impact are incorporated into consideration (from the classical works to the emerging papers). These researches, however, do not study the whole effect on the optimal (or equilibrium) execution strategies caused by the temporary, permanent and transient price impact.

1.2 Notation

We use the notation $\mathbb{R}^n$ to denote the set of all n–dimensional real–valued column vectors and $\mathscr{M}_{n,m}(\mathbb{R})$ to denote the set of all $n \times m$ real–valued matrices. $\mathbf{I}_n$ stands for the $n \times n$ identity matrix. For an $n \times m$ real–valued matrix (or vector) $\mathbf{A}$, we denote by $\mathbf{A}^\top$ the transpose of the matrix (or vector). For any n–dimensional squared matrix $\mathbf{A} \in \mathscr{M}_{n,n}(\mathbb{R})$, $|\mathbf{A}|$ stands for the determinant of $\mathbf{A}$ and $\mathrm{tr}\,(\mathbf{A})$ is the trace of $\mathbf{A}$ defined as a map from $\mathscr{M}_{n,n}(\mathbb{R})$ to $\mathbb{R}$ such that

$$\mathrm{tr}\,(\mathbf{A}) = \mathrm{tr} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} := \sum_{i=1}^{n} a_{ii}. \tag{1.1}$$

2 Dynamics of Market Model

We assume a (risk–averse) large trader with the risk–aversion parameter $\gamma > 0$ in a financial market. He/she purchases $\mathfrak{Q} (\in \mathbb{R})$ volume of a risky asset in the time window $[0, T]$. Let $Q_t (\in \mathbb{R})$ be the cumulative purchase up to time $t \in [0, T]$ of a large trader. Then, the number of shares that remained to purchase at time $t \in [0, T]$ is described as

$$\overline{Q}_t = \mathfrak{Q} - Q_t, \tag{2.1}$$

with the initial and terminal conditions: $\overline{Q}_0 = \mathfrak{Q}$ and $\overline{Q}_T = 0$. We consider a continuous trading strategy:

$$\mathrm{d}Q_t = \dot{Q}_t \mathrm{d}t. \tag{2.2}$$

Here it is assumed that Q_t is continuously differentiable in time $t \in [0, T]$. We denote by the positive and negative $\dot{Q}_t$ the acquisition and liquidation of the risky asset, respectively. This leads to a similar setup for a selling problem. The execution price of an asset $\widehat{P}$ is assumed to follow a linear price impact model as follows:

$$\widehat{P}_t = P_t + \lambda_t \dot{Q}_t + \kappa_t v_t, \tag{2.3}$$

where $P_t (\in \mathbb{R})$ represents the market price of the asset at time $t \in [0, T]$ and $\lambda_t (\in \mathbb{R})$ is a price impact coefficient at time $t \in [0, T]$. Here v_t represents the aggregate volume of instantaneous order submitted by non–large traders (or small traders) and κ_t the price impact per unit at time $t \in [0, T]$ caused by the submission of small traders.

Remark 2.1 (Implication of κ_t)**.** We consider a different (or an extended) price impact coefficient for aggregate orders posed by small traders compared with the one for the large trader from the following perspective: the price impact coefficient for small traders is assumed to include the effect of history about trading order flow.

In the sequel of this paper, we assume that the buy– and sell–trade of the large trader induce the same (instantaneous) price impact, although it would be different in the real market. We can, however, justify this assumption from the statistical analysis of market data shown by [4] and [5]. These works estimate the permanent and temporary price impact by conducting a linear regression of price changes on net order–flow using trading data obtained from Nasdaq. This estimation and the relevant statistics reveal that the linear assumption of the price impact is compatible with the real stock market and that the price impact caused by both buy and sell trades are the same from the statistical point of view. The large trader's wealth process at time $t \in [0, T]$, denoted by W_t, evolves as

$$\mathrm{d}W_t = -\widehat{P}_t \mathrm{d}Q_t = -\widehat{P}_t \dot{Q}_t \mathrm{d}t = -\left(P_t + \lambda_t \dot{Q}_t + \kappa_t v_t\right) \dot{Q}_t \mathrm{d}t. \tag{2.4}$$

Besides the above temporary price impact, we assume permanent and transient parts of the market impact. The residual effect of the transient part is defined, with the deterministic linear temporary impact coefficient α_t, as

$$R_t = \mathrm{e}^{-\rho t} R_0 + \int_0^t \mathrm{e}^{-\rho(t-s)} \alpha_s \left(\lambda_t \dot{Q}_t + \kappa_t v_t \right) \mathrm{d}t, \tag{2.5}$$

or

$$\mathrm{d}R_t = -\rho R_t \mathrm{d}t + \alpha_t \left(\lambda_t \dot{Q}_t + \kappa_t v_t \right) \mathrm{d}t. \tag{2.6}$$

We define this transient price impact employing the exponential decay kernel G_t defined as:

$$G_t := \mathrm{e}^{-\rho t}, \tag{2.7}$$

where $\rho \in (0.\infty]$ stands for the deterministic resilience speed.[1] This residual effect indicates that the price impact decays gradually over the course of the trading epoch $[0, T]$. R_0 is assumed to be zero in the following analysis. It is quite plausible to assume this from the fact that the traders have no price impact on any risky asset before liquidating/acquiring the risky asset, and thereby there exist no residual effects caused by traders on the price before their execution. Note that $\alpha_t(\lambda_t \dot{Q}_t + \kappa_t v_t)$ represents the temporary price impact.

The market price is assumed to consist of the sum of two components:

$$P_t = P_t^f + R_t, \tag{2.8}$$

where P_t^f for $t \in [0, T]$ stands for the fundamental price defined by

$$\mathrm{d}P_t^f = \beta_t \left(\lambda_t \dot{Q}_t + \kappa_t v_t \right) \mathrm{d}t + \mathrm{d}Z_t. \tag{2.9}$$

Note that $\beta_t(\lambda_t \dot{Q}_t + \kappa_t v_t)$ represents the permanent price impact. Z_t represents the effect of some public news/information about the economic situation which may affect the market price (or quoted price).

The more mathematically formal setting of the above model is as follows. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space, where the processes of the aggregate volume v_t of the orders submitted by small traders and the effect Z_t caused by public news or information on the (quoted) price are defined as follows:

$$\mathrm{d}v_t = (a_t^v - b_t^v v_t)\,\mathrm{d}t + \sigma_t^v \mathrm{d}B_t^v; \tag{2.10}$$
$$\mathrm{d}Z_t = \mu_t^Z \mathrm{d}t + \sigma_t^Z \mathrm{d}B_t^Z, \tag{2.11}$$

where B_t^v and B_t^Z stand for standard Brownian motions with $B_0^v = 0$, $B_0^Z = 0$.[2] Then, the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is generated by (B_t^v, B_t^Z) and satisfies the usual conditions:

$$\mathcal{F}_t = \sigma \left\{ (B_s^v, B_s^Z), s \in [0, t] \right\}. \tag{2.12}$$

[1] Much of theoretical analysis, such as in [32], [44], deal with a (deterministic and) constant resilience speed. Many empirical kinds of research, however, demonstrate that the liquidity is variable over time, suggesting that the resilience speed is time–dependent. Our analysis allows the time–dependence for the resilience speed, i.e., ρ_t for all $t \in [0, T]$, as considered in [13]. Notwithstanding a meaningful extension from the viewpoint of real market analysis, we henceforth formulate the model without time–dependent parameter (i.e., with ρ) since the dependence will not offer additional intriguing results in the following analysis.

[2] Eq. (2.10) is an Ornstein–Uhlenbeck (OU) type process. This indicates that traders including both the large trader and small traders can access at time $t + \mathrm{d}t$ the information about the orders submitted by small traders at time t. This asumption is also compatible with a number of empirical literatures which emphasize the importance of taking the autocorrelattion of order flow into account. (see [19], [20], [7], [8].)

We assume that the quadratic co–variation of B_t^v and B_t^Z takes the following form:

$$\mathrm{d}\langle B^v, B^Z\rangle_t = \rho^{v,Z}\mathrm{d}t, \tag{2.13}$$

which implies that these two processes are correlated with each other. Note that $a_t^v, b_t^v, \mu_t^Z, \sigma_t^v, \sigma_t^Z$ in the above dynamicss of v_t and Z_t are all deterministic in time t.

If we assume that the information flow accessible for the large trader is carried by the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ then the executed volume Q_t of the large trader by time $t \in [0,T]$ is an $\mathcal{F}_t$–measurable (real–valued) random variable. Thus, the set of admissible execution strategies is defined as follows:

$$\mathcal{A} := \left\{\{Q_t\}_{t\in[0,T]} : \{\mathcal{F}_t\}_{t\in[0,T]}\text{–adapted process with differentiable path}, Q_0 = 0, Q_T = \mathfrak{Q}\right\}. \tag{2.14}$$

It turns out that, according to the dynamics of the market model, the wealth process, price dynamics, remaining execution volume, and residual effect depend on the process of the cumulative purchase (or liquidation) denoted by $Q = \{Q_s\}_{s\in[0,t]}$:

$$\begin{aligned}
\mathrm{d}W_t^Q &= -\widehat{P}_t^Q\mathrm{d}Q_t = -\widehat{P}_t^Q\dot{Q}_t\mathrm{d}t = -\left(P_t^Q + \lambda_t\dot{Q}_t + \kappa_t v_t\right)\dot{Q}_t\mathrm{d}t;\\
\mathrm{d}P_t^Q &= \beta_t\left(\lambda_t\dot{Q}_t\mathrm{d}t + \kappa_t v_t\mathrm{d}t\right) + \mathrm{d}Z_t + \mathrm{d}R_t^Q;\\
\mathrm{d}\overline{Q}_t^Q &= -\mathrm{d}Q_t = -\dot{Q}_t\mathrm{d}t;\\
\mathrm{d}R_t^Q &= -\rho R_t^Q\mathrm{d}t + \alpha_t\left(\lambda_t\dot{Q}_t\mathrm{d}t + \kappa_t v_t\mathrm{d}t\right).
\end{aligned}$$

However, to simplify the notations, we suppress the superscript Q in the above expressions representing the dependence on Q to each state variable, and simply use the ones $(W_t, P_t, \overline{Q}_t, R_t)$ defined in the previous description except the cases when we should emphasize the dependence explicitly.

Remark 2.2 (Markov Property of the Residual Effect)**.** Eq. (2.5) and Eq. (2.6) show the recursiveness of the residual effect. $\mathrm{d}R_t$ depends on only R_t and the price impact $\alpha_t(\lambda_t\mathrm{d}Q_t + \kappa_t\mathrm{d}v_t)$, which indicates that R_t has a Markov property in this settings. This Markov property of the residual effect arises thanks to the assumption of the exponantial decay kernel.

The Markov property of the residual effect also induces the Markov property of the price dynamics, which plays a fundamental role in constructing the value function and the related Hamilton–Jacobi–Bellman (HJB) equation.

3 Performance Criteria and HJB Equation

In this section, we formulate and solve an HJB equation, from which we obtain an optimal execution and a value function where the price impact caused by small traders exists.

3.1 Performance Criteria of the Large Trader: A Hard Constraint

We first define the state of the process at time $t \in [0,T]$. The state, denoted by $\boldsymbol{s}_t$, is a 5–tuple and is defined as

$$\boldsymbol{s}_t := \left(W_t, P_t, \overline{Q}_t, R_t, v_t\right)^\top \in \mathbb{R}^5\,(= \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}) =: S. \tag{3.1}$$

Each component of the state is, as we have mentioned above, dependent on the process of the cumulative purchase/liquidation: Q_s, $s \in [0,t]$.

The utility function of the large trader is assumed to take a form of a Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility function. The utility payoff (or reward) arises only from the terminal wealth at the maturity:

$$g_T(\boldsymbol{s}_T) := \begin{cases} -\exp\{-\gamma W_T\} & \text{if } \overline{Q}_T = 0;\\ -\infty & \text{if } \overline{Q}_T \neq 0. \end{cases} \tag{3.2}$$

We define the (conditional) expected utility of the large trader at time $t \in [0,T]$ on an execution strartegy $Q = \{Q_t\}_{0 \le t \le T}$ as

$$V_t^Q := \mathbb{E}\left[g_T(\boldsymbol{s}_T^Q) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[-\exp\left\{ -\gamma W_T^Q \right\} \cdot \mathbb{1}_{\{\overline{Q}_T^Q = 0\}} + (-\infty) \cdot \mathbb{1}_{\{\overline{Q}_T^Q \neq 0\}} \middle| \mathcal{F}_t \right], \quad t \in [0,T], \quad (3.3)$$

where $\mathbb{1}_A$ $(A \in \mathcal{F})$ represents the indicator function of the event A.

Let the optimal (expected utility) value from time $t \in [0,T]$ by

$$V_t := \operatorname*{ess\,sup}_{Q \in \mathcal{A}} V_t^Q, \quad t \in [0,T]. \quad (3.4)$$

Then V_t depends on the history or information $\mathcal{F}_t$ only through the (controlled) state: $\boldsymbol{s}_t = \left(W_t, P_t, \overline{Q}_t, R_t, v_t\right)^\top \in S = \mathbb{R}^5$ and we denote this functional dependence by the optimal value function as

$$V\left[t, W_t, P_t, \overline{Q}_t, R_t, v_t\right] := V_t, \quad t \in [0,T]. \quad (3.5)$$

3.2 In the Case with Target Close Order: A Soft Constraint

Here we consider a model with a closing price. The time framework $t \in [0,T]$ is the same in the model mentioned above. However, we add an assumption that a large trader can execute his/her remaining execution volume at the terminal, $\overline{Q}_T$, with closing price P_T. We further assume that the trading at time T imposes the large trader to pay the additive cost χ_T per unit of the remaining volume.

According to the above settings, the value function at maturity becomes

$$V[T, \boldsymbol{s}_T] = -\exp\Big\{ -\gamma\big[W_T - (P_T + \chi_T \overline{Q}_T)\overline{Q}_T\big]\Big\}, \quad (3.6)$$

and the conditional expectational utility at time $t \in [0,T]$ is defined by

$$V_t^Q := \mathbb{E}\left[-\exp\Big\{ -\gamma\big[W_T - (P_T + \chi_T \overline{Q}_T)\overline{Q}_T\big]\Big\} \middle| \mathcal{F}_t \right]. \quad (3.7)$$

This formulation is similar to [4] and [5], which incorporates the (quadratic) cost of trading the remaining execution at maturity into the performance criteria. The term $\chi_T \overline{Q}_T^2$ corresponds to a quadratic transaction cost emerging at the terminal. A multitude of researches formulate a model with quadratic transaction costs and give it some economic meaning. For example, [15] and [16] solve a portfolio selection problem with quadratic transaction costs, meaning that the buying or selling a unit of risky asset incurs an additional cost proportional to the number of buying/sell assets (which we can regard as price impact). A plausible interpretation for the formulation of Eq. (3.6) would seem to be that the large trader can execute the orders remaining at the terminal T in a dark pool. Consider the following case that a brokerage has to buy $\mathfrak{Q} (\in \mathbb{R})$ volume which a client asked him/her to manage in a daily lit market. These insights infer that the analysis of the optimal execution strategies in the soft constraint case lays the foundation of how a large trader should allocate the execution volumes (or speeds) in the lit and dark pool.

Formulating the terminal condition as Eq. (3.6) also plays an indispensable role from a mathematical viewpoint. Changing the terminal condition from Eq. (3.2) to Eq. (3.6) makes a system of (Riccati type) ordinary differential equations (ODE) (accompanied by the derivation of optimal execution strategies) analytically tractable. The relationship between the two formulations explained above concludes this subsection.

Remark 3.1 (Relationship between Hard and Soft Constraints)**.** We can obtain the optimal execution strategies for the primary problem (the optimal execution problem with a hard constraint) by $\chi_T \to \infty$ in the following problem (the optimal execution problem with a soft constraint). If the additional trading cost χ_T goes to infinity, then the large trader will not use the dark pool and execute only in the lit pool, as shown in [26]. The statement indicates that the large trader does not rely on a dark pool with too much high trading costs (commitment fee), even if the dark pool protects the leakage of the information about his/her trading in the venue.

3.3 HJB Equation

The optimal value function denoted by $V[t, \boldsymbol{s}_t] := V\big[t, W_t, P_t, \overline{Q}_t, R_t, v_t\big]$ with the terminal condition:

$$V\big[T, W_T, P_T, \overline{Q}_T, R_T, v_T\big] = -\exp\Big\{-\gamma\big[W_T - (P_T + \chi_T\overline{Q}_T)\overline{Q}_T\big]\Big\}, \tag{3.8}$$

satisfies, from the dynamic programming principle, the following Hamilton–Jacobi–Bellman (HJB) equation (or dynamic programming equation) for the optimal (policy) function $\dot{Q}$:

$$\begin{aligned}\sup_{\dot{Q}_t \in \mathbb{R}} \Big[&\partial_t V - (P_t + \lambda_t\dot{Q}_t + \kappa v_t)\dot{Q}_t\partial_W V + \big\{-\rho R_t + (\alpha_t + \beta_t)(\lambda_t\dot{Q}_t + \kappa_t v_t) + \mu_t^Z\big\}\partial_P V - \dot{Q}_t\partial_{\overline{Q}} V \\ &+ \big\{-\rho R_t + \alpha_t(\lambda_t\dot{Q}_t + \kappa_t v_t)\big\}\partial_R V + (a_t^v - b_t^v v_t)\partial_v V \\ &+ \frac{1}{2}\big\{(\sigma_t^Z)^2\partial_{PP}V + 2\sigma_t^v\sigma_t^Z\rho^{v,Z}\partial_{Pv}V + (\sigma_t^v)^2\partial_{vv}V\big\}\Big] = 0, \quad 0 \le t < T\end{aligned} \tag{3.9}$$

if we assume that the function $V\colon [0,T] \times S \to \mathbb{R}$ is in $C^{1,2}$, that is, V is continuously differentiable with respect to time t and continuously twice differentiable with respect to each state variable. The above HJB equation stems from the following (stochastic) differential equations:

$$\begin{aligned}\mathrm{d}W_t &= -(P_t + \lambda_t\dot{Q}_t + \kappa_t v_t)\dot{Q}_t\mathrm{d}t; \\ \mathrm{d}P_t &= \mathrm{d}P_t^f + \mathrm{d}R_t \\ &= -\rho R_t\mathrm{d}t + (\alpha_t + \beta_t)(\lambda_t\dot{Q}_t + \kappa_t v_t)\mathrm{d}t + \mu_t^Z\mathrm{d}t + \sigma_t^Z\mathrm{d}B_t^Z; \\ \mathrm{d}\overline{Q}_t &= -\dot{Q}_t\mathrm{d}t; \\ \mathrm{d}R_t &= \{-\rho R_t + \alpha_t(\lambda_t\dot{Q}_t + \kappa_t v_t)\}\mathrm{d}t; \\ \mathrm{d}v_t &= (a_t^v - b_t^v v_t)\mathrm{d}t + \sigma_t^v\mathrm{d}B_t^v.\end{aligned} \tag{3.10}$$

Thus, rewriting this results in

$$\begin{aligned}\sup_{\dot{Q}_t \in \mathbb{R}} &\Big[-(P_t + \lambda_t\dot{Q}_t + \kappa_t v_t)\dot{Q}_t\partial_W V + (\alpha_t + \beta_t)\lambda_t\dot{Q}_t\partial_P V - \dot{Q}_t\partial_{\overline{Q}}V + \alpha_t\lambda_t\dot{Q}_t\partial_R V\Big] \\ &+ \partial_t V + \big\{-\rho R_t + (\alpha_t + \beta_t)\kappa_t v_t + \mu_t^Z\big\}\partial_P V + (-\rho R_t + \alpha_t\kappa_t v_t)\partial_R V + (a_t^v - b_t^v v_t)\partial_v V \\ &+ \frac{1}{2}\big\{(\sigma_t^Z)^2\partial_{PP}V + 2\sigma_t^v\sigma_t^Z\rho^{v,Z}\partial_{Pv}V + (\sigma_t^v)^2\partial_{vv}V\big\} = 0, \quad 0 \le t < T.\end{aligned} \tag{3.11}$$

Then we can derive the optimal execution strategy and its associated value function of Eq. (3.5) explicitly by appropriately guessing the form of the value function and verifying the obtained solution.

3.4 Optimal Execution Strategy and Optimal Value Function

The optimal dynamic execution strategy is acquired by solving the above equation (3.11). We obtain the following theorem.

Theorem 3.1 (Optimal Execution Strategy and Optimal Value Function)**.** Under a set of regularity conditions:

1. The optimal execution speed at time $t \in [0,T]$, denoted as $\dot{Q}_t^*$, becomes an affine function of the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect R_t and the orders posed by small traders v_t at time t:

$$\dot{Q}_t^* = f_t(\boldsymbol{s}_t) = a_t + b_t\overline{Q}_t + c_t R_t + d_t v_t, \quad 0 \le t \le T. \tag{3.12}$$

2. The optimal value function $V[t, \mathbf{s_t}] := V[t, W, P, \overline{Q}, R, v_t]$ at time $t \in [0, T]$ is represented as follows:

$$V[t, W_t, P_t, \overline{Q}_t, R_t, v_t] = -\exp\Big\{-\gamma\big[W_t - P_t\overline{Q}_t + G_t\overline{Q}_t^2 + H_t\overline{Q}_t + I_t\overline{Q}_tR_t + J_tR_t^2 + L_tR_t + M_t\overline{Q}_tv_t + N_tR_tv_t + X_tv_t^2 + Y_tv_t + K_t\big]\Big\}, \quad (3.13)$$

where a_t, b_t, c_t, d_t; $G_t, H_t, I_t, J_t, L_t, M_t, N_t, X_t, Y_t, K_t$ for $t \in [0, T]$ are deterministic functions of time t which are dependent on the model parameters, and these are assumed to exist as a unique solution of a simultaneous system of ordinary differential equations derived in the proof.

Proof. See Appendix A □

As for the system of ODEs, we have the following concise representation.

Proposition 3.1 (Concise Representation of ODEs)**.** These conditions are represented via the following three Riccati-type (matrix) differential equations:

$$\dot{\mathbf{\Omega}}_t = -\mathbf{\Omega}_t^\top\mathbf{A}_t\mathbf{\Omega}_t + \mathbf{\Omega}_t^\top\mathbf{B}_t + \mathbf{B}_t^\top\mathbf{\Omega}_t + \mathbf{C}_t; \quad (3.14)$$

$$\dot{\mathbf{\Gamma}}_t = \mathbf{D}_t\mathbf{\Gamma}_t + \mathbf{F}_t; \quad (3.15)$$

$$\dot{K}_t = -\frac{1}{4\lambda_t}\left(\mathbf{\Phi}_t\mathbf{\Gamma}_t\right)^2 - \mathbf{\Psi}_t^\top\mathbf{\Gamma}_t + \frac{1}{2}\mathbf{\Gamma}^\top\mathbf{\Sigma}\mathbf{\Sigma}^\top\mathbf{\Gamma} + \mathrm{Tr}\left(\mathbf{\Sigma}_t^\top\mathbf{\Omega}_t\mathbf{\Sigma}_t\right). \quad (3.16)$$

with the terminal conditions:

$$\mathbf{\Omega}_T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 0 & -1/2 & -\chi_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{\Gamma}_T := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad K_T = 0. \quad (3.17)$$

Proof. See Appendix C □

Eq. (3.14), (3.15) and (3.16) are Eq. (B.13), (B.14) and (B.15), respectively, which we derive in Appendix B. In the appendix, we show how to obtain the matrix representation of the optimal execution strategy (or Eq. (3.12) and (3.13).) Here we assume that the solution of the Riccati-type (matrix) differencial equations uniquely exist under a set of regularity consitions.

It remains to show that the above system of ODEs exists uniquely.

From the theorem above, the optimal execution volume $\dot{Q}_t^*$ depends on the state $\boldsymbol{s}_t = (W_t, P_t, \overline{Q}_t, R_t, v_t)$ of the controlled process only through the remaining execution volume $\overline{Q}_t$, the cumulative residual effect R_t and the aggregate volume submitted by small traders v_t, not through the wealth W_t or market price P_t. In addition, by the definition of residual effect R_t and the aggregate volume submitted by small traders, the optimal execution volume $\dot{Q}_t$ includes a nondeterministic term (random variable) thorough the (aggregate) volume v_t submitted by small traders at time $t \in [0, T]$ in the residual effect R_t and itself. This fact indicates that v_t affects the optimal execution strategies both directly and indirectly, which makes the optimal execution strategy a stochastic one. Therefore, we can obtain a deterministic execution strategy if the total execution volume of small traders are deterministic for all trading time window $[0, T]$.

Corollary 3.1. If the orders posed by small traders v_t for $t \in [0, T]$ are deterministic, the optimal execution volumes $\dot{Q}_t^*$ at time $t \in [0, T]$ also become deterministic functions of time in a class of the static (and non–randomized) execution strategy.

This is our contribution to the field of the optimal execution problem. A large number of researches that focus on execution problems of a single large trader derive an optimal execution strategy in a deterministic class. For example, [40] considers an execution problem of a large trader in a continuous–time setting and derives an optimal execution strategy in a deterministic class. However, we can confirm from our analysis that the optimal execution strategy does not always become a deterministic one when focusing on a single large trader's execution problem.

4 Conclusion

We constructed, in a (finite) continuous–time framework, a model focusing on a single large trader. The large trader maximizes the expected CARA utility arising from his/her wealth at the end of the trading epoch in a market with small traders. By formulating a generalized price impact model, the backward induction method of dynamic programming based on the dynamic programming principle permitted us to derive the optimal execution strategy. The most important result which emerged from this research is as follows. The aggregate orders of small traders have an impact on the execution of the large trader. This kind of work concerned with an execution problem through the backward induction procedure of dynamic programming will be explored from a more in–depth and extensive perspective, which we can expect will also give us a more illuminating insight into all the other problems left in this field of research as follows.

In the above models, we have assumed that the price reversion rate and the resilience speed are deterministic. This assumption makes the fundamental price of the risky asset observable for large traders before the trading time. The fundamental value of a risky asset is, however, unobservable and uncertain in a real market. Therefore, we can evolve the model built in this paper as an incomplete state information model, which leads to an analysis in a more realistic situation of the marketplace. Developing an incomplete state information model of either single– or multiple–large traders will contribute to some developments of a study involved in a trading market.

Adding to the possibility of extending our research to an incomplete state information model, there would be room for formulating an execution problem as a stochastic differential game of multiple large traders. In our current research, we assume that there is a single large trader in a security market, although multiple large traders participate in a real market. The assumption may be relaxed by assuming that multiple large traders influence the execution price with each other. This makes us capable of formulating the problem as a stochastic differential game played by multiple large traders. The formulation is, however, intractable in terms of obtaining an analytical solution. Thus, there might be room for searching for a more tractable model of an execution problem concerned with multiple large traders which yields a semi-analytical or, if possible, an analytical solution.

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Appendix

A Proof of Theorem 3.1

From the discrete time analogue [11], we guess the objective (or value) function as follows:

$$V\left[t, W_t, P_t, \overline{Q}_t, R_t, v_t\right] = -\exp\Big\{-\gamma\big[W_t - P\overline{Q}_t + G_t\overline{Q}_t^2 + H_t\overline{Q}_t + I_t\overline{Q}_tR_t + J_tR_t^2 + L_tR_t + M_t\overline{Q}_tv_t + N_tR_tv_t + X_tv_t^2 + Y_tv_t + K_t\big]\Big\}, \tag{A.1}$$

with the terminal condition:

$$V\left[T, W_T, P_T, \overline{Q}_T, R_T, v_T\right] = -\exp\Big\{-\gamma\big[W_T - (P_T + \chi_T\overline{Q}_T)\overline{Q}_T\big]\Big\}. \tag{A.2}$$

The partial differentiation of $V\left[t, W_t, P_t, \overline{Q}_t, R_t, v_t\right]$ with respect to time and each state variable is calculated as follows:

$$\begin{aligned}
&\partial_t V = -\gamma\left\{\dot{G}_t(\overline{Q}_t)^2 + \dot{H}_t\overline{Q}_t + \dot{I}_t\overline{Q}_tR_t + \dot{J}_tR_t^2 + \dot{L}_tR_t + \dot{M}_t\overline{Q}_tv_t + \dot{N}_tR_tv_t + \dot{X}_tv_t^2 + \dot{Y}_tv_t + \dot{K}_t\right\}V;\\
&\partial_W V = -\gamma V;\\
&\partial_P V = \gamma\overline{Q}_tV;\\
&\partial_{\overline{Q}} V = -\gamma\left(-P_t + 2G_t\overline{Q}_t + H_t + I_tR_t + M_tv_t\right)V;\\
&\partial_R V = -\gamma\left(I_t\overline{Q}_t + 2J_tR_t + L_t + N_tv_t\right)V;\\
&\partial_v V = -\gamma(M_t\overline{Q}_t + N_tR_t + 2X_tv_t + Y_t)V;\\
&\partial_{PP} V = \gamma^2\overline{Q}_t^2V;\\
&\partial_{Pv} V = -\gamma\overline{Q}_t(M_t\overline{Q}_t + N_tR_t + 2X_tv_t + Y_t)V;\\
&\partial_{vv} V = -2X_tV + \gamma^2\left(-P_t + 2G_t\overline{Q}_t + H_t + I_tR_t + M_tv_t\right)^2V.
\end{aligned}$$

Therefore, by substituting these into Eq. (3.11), we have

$$\begin{aligned}
&\sup_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[\lambda_t\dot{Q}_t^2 + \big[\{(\alpha_t+\beta_t)\lambda_t - \alpha_t\lambda_tI_t + 2G_t\}\overline{Q}_t + \{(\alpha_t+\beta_t)\lambda_t + L_t - 2\alpha_t\lambda_tJ_t\}R\\
&+ (\kappa_t + M_t - \alpha_t\lambda_tN_t)v_t + (H_t - \alpha_t\lambda_tL_t)\big]\dot{Q}_t\Big]V\\
&+ \gamma\Big\{-\dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}M_t + \frac{1}{2}\gamma(\sigma_t^v)^2M_t^2\Big\}\overline{Q}_t^2V\\
&+ \gamma\Big\{-\dot{H}_t + \mu^Z - a_t^vM_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}Y_t + \gamma(\sigma_t^v)^2M_tY_t\Big\}\overline{Q}_tV\\
&+ \gamma\Big\{-\dot{I}_t - \rho + \rho I_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}N_t + \gamma(\sigma_t^v)^2M_tN_t\Big\}\overline{Q}_tR_tV\\
&+ \gamma\Big\{-\dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^v)^2N_t^2\Big\}R_t^2V\\
&+ \gamma\Big\{-\dot{L}_t + \rho L_t - a_t^vN_t + \gamma(\sigma_t^v)^2N_tY_t\Big\}R_tV\\
&+ \gamma\Big\{-\dot{M}_t + (\alpha_t+\beta_t)\kappa_t - \alpha_t\kappa_tI_t + b_t^vM_t - 2\gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}X_t + 2\gamma(\sigma_t^v)^2M_tX_t\Big\}\overline{Q}_tv_tV\\
&+ \gamma\Big\{-\dot{N}_t + \rho N_t - 2\alpha_t\kappa_tJ_t + b_t^vN_t + 2\gamma(\sigma_t^v)^2N_tX_t\Big\}R_tv_tV\\
&+ \gamma\Big\{-\dot{X}_t - \alpha_t\kappa_tN_t + 2b_t^vX_t + \gamma(\sigma_t^v)^2X_t^2\Big\}v_t^2V\\
&+ \gamma\Big\{-\dot{Y}_t - \alpha_t\kappa_tL_t - 2a_t^vX_t + b_t^vY_t + 2\gamma(\sigma_t^v)^2X_tY_t\Big\}v_tV\\
&+ \gamma\Big\{-\dot{K}_t - a_t^vY_t - (\sigma_t^v)^2X_t + \frac{1}{2}\gamma(\sigma_t^v)^2Y_t^2\Big\}V\\
&= 0.
\end{aligned} \tag{A.3}$$

Since we assume the negative exponential utility function above,

$$\begin{aligned}
&\sup_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[\lambda_t\dot{Q}_t^2 + \big[\big\{(\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t\big\}\overline{Q}_t + \big\{I_t - 2\alpha_t\lambda_t J_t\big\}R_t \\
&+ (\kappa_t + M_t - \alpha_t\lambda_t N_t)v_t + (H_t - \alpha_t\lambda_t L_t)\big]\dot{Q}_t\Big]V \\
={}& V\inf_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[\lambda_t\dot{Q}_t^2 + \big[\big\{(\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t\big\}\overline{Q}_t + \big\{I_t - 2\alpha_t\lambda_t J_t\big\}R_t \\
&+ (\kappa_t + M_t - \alpha_t\lambda_t N_t)v_t + (H_t - \alpha_t\lambda_t L_t)\big]\dot{Q}\Big] \\
={}& V\inf_{\dot{Q}_t} \gamma\Big[a_t\dot{Q}_t^2 + \big[b_t\overline{Q}_t + c_t R_t + d_t v_t + e_t\big]\,\dot{Q}_t\Big],
\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}
a_t &:= \lambda_t; \\
b_t &:= (\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t; \\
c_t &:= I_t - 2\alpha_t\lambda_t J_t; \\
d_t &:= \kappa_t + M_t - \alpha_t\lambda_t N_t, \\
e_t &:= H_t - \alpha_t\lambda_t L_t.
\end{aligned} \tag{A.5}$$

Therefore, Eq. (A.4) attains the infimum at the optimal execution speed:

$$\dot{Q}_t^* = -\frac{b_t\overline{Q}_t + c_t R_t + d_t v_t + e_t}{2a_t} := a_t^* + b_t^*\overline{Q}_t + c_t^* R_t + d_t^* v_t, \tag{A.6}$$

where

$$a_t^* := -\frac{e_t}{2a_t};\quad b_t^* := -\frac{b_t}{2a_t};\quad c_t^* := -\frac{c_t}{2a_t};\quad \text{and}\quad d_t^* := -\frac{d_t}{2a_t}, \tag{A.7}$$

and the value of Eq. (A.4) at the infimum becomes

$$V\inf_{\dot{Q}_t} \gamma\Big[a_t\dot{Q}_t^2 + \big[b_t\overline{Q}_t + c_t R_t + d_t\big]\,\dot{Q}_t\Big] = -\frac{(b_t\overline{Q}_t + c_t R_t + d_t v_t + e_t)^2}{4a_t}V. \tag{A.8}$$

Substituting this into Eq. (A.3) yields

$$
\begin{aligned}
&\sup_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[\lambda_t\dot{Q}_t^2 + \big[\big\{(\alpha_t+\beta_t)\lambda_t - \alpha_t\lambda_t I_t + 2G_t\big\}\overline{Q}_t + \big\{(\alpha_t+\beta_t)\lambda_t + L_t - 2\alpha_t\lambda_t J_t\big\}R \\
&+ (\kappa_t + M_t - \alpha_t\lambda_t N_t)v_t + (H_t - \alpha_t\lambda_t L_t)\big]\dot{Q}_t\Big]V \\
&+ \gamma\Big\{-\dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}M_t + \frac{1}{2}\gamma(\sigma_t^v)^2M_t^2\Big\}\overline{Q}_t^2V \\
&+ \gamma\Big\{-\dot{H}_t + \mu^Z - a_t^vM_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}Y_t + \gamma(\sigma_t^v)^2M_tY_t\Big\}\overline{Q}_tV \\
&+ \gamma\Big\{-\dot{I}_t - \rho + \rho I_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}N_t + \gamma(\sigma_t^v)^2M_tN_t\Big\}\overline{Q}_tR_tV \\
&+ \gamma\Big\{-\dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^v)^2N_t^2\Big\}R_t^2V \\
&+ \gamma\Big\{-\dot{L}_t + \rho L_t - a_t^vN_t + \gamma(\sigma_t^v)^2N_tY_t\Big\}R_tV \\
&+ \gamma\Big\{-\dot{M}_t + (\alpha_t+\beta_t)\kappa_t - \alpha_t\kappa_tI_t + b_t^vM_t - 2\gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}X_t + 2\gamma(\sigma_t^v)^2M_tX_t\Big\}\overline{Q}_tv_tV \\
&+ \gamma\Big\{-\dot{N}_t + \rho N_t - 2\alpha_t\kappa_tJ_t + b_t^vN_t + 2\gamma(\sigma_t^v)^2N_tX_t\Big\}R_tv_tV \\
&+ \gamma\Big\{-\dot{X}_t - \alpha_t\kappa_tN_t + 2b_t^vX_t + \gamma(\sigma_t^v)^2X_t^2\Big\}v_t^2V \\
&+ \gamma\Big\{-\dot{Y}_t - \alpha_t\kappa_tL_t - 2a_t^vX_t + b_t^vY_t + 2\gamma(\sigma_t^v)^2X_tY_t\Big\}v_tV \\
&+ \gamma\Big\{-\dot{K}_t - a_t^vY_t - (\sigma_t^v)^2X_t + \frac{1}{2}\gamma(\sigma_t^v)^2Y_t^2\Big\}V \\
=\ &\gamma\left\{-\dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}M_t + \frac{1}{2}\gamma(\sigma_t^v)^2M_t^2 - \frac{b_t^2}{4a_t}\right\}\overline{Q}_t^2V \\
&+ \gamma\left\{-\dot{H}_t + \mu^Z - a_t^vM_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}Y_t + \gamma(\sigma_t^v)^2M_tY_t - \frac{b_te_t}{2a_t}\right\}\overline{Q}_tV \\
&+ \gamma\left\{-\dot{I}_t - \rho + \rho I_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}N_t + \gamma(\sigma_t^v)^2M_tN_t - \frac{b_tc_t}{2a_t}\right\}\overline{Q}_tR_tV \\
&+ \gamma\left\{-\dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^v)^2N_t^2 - \frac{c_t^2}{4a_t}\right\}R_t^2V \\
&+ \gamma\left\{-\dot{L}_t + \rho L_t - a_t^vN_t + \gamma(\sigma_t^v)^2N_tY_t - \frac{c_te_t}{2a_t}\right\}R_tV \\
&+ \gamma\left\{-\dot{M}_t + (\alpha_t+\beta_t)\kappa_t - \alpha_t\kappa_tI_t + b_t^vM_t - 2\gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}X_t + 2\gamma(\sigma_t^v)^2M_tX_t - \frac{b_td_t}{2a_t}\right\}\overline{Q}_tv_tV \\
&+ \gamma\left\{-\dot{N}_t + \rho N_t - 2\alpha_t\kappa_tJ_t + b_t^vN_t + 2\gamma(\sigma_t^v)^2N_tX_t - \frac{c_td_t}{2a_t}\right\}R_tv_tV \\
&+ \gamma\left\{-\dot{X}_t - \alpha_t\kappa_tN_t + 2b_t^vX_t + \gamma(\sigma_t^v)^2X_t^2 - \frac{d_t^2}{4a_t}\right\}v_t^2V \\
&+ \gamma\left\{-\dot{Y}_t - \alpha_t\kappa_tL_t - 2a_t^vX_t + b_t^vY_t + 2\gamma(\sigma_t^v)^2X_tY_t - \frac{d_te_t}{2a_t}\right\}v_tV \\
&+ \gamma\left\{-\dot{K}_t - a_t^vY_t - (\sigma_t^v)^2X_t + \frac{1}{2}\gamma(\sigma_t^v)^2Y_t^2 - \frac{e_t^2}{4a_t}\right\}V \\
=\ &0. \qquad \text{(A.9)}
\end{aligned}
$$

This equation holds for all states, and hence we can derive the following conditions:

$$-\dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}M_t + \frac{1}{2}\gamma(\sigma_t^v)^2M_t^2 - \frac{b_t^2}{4a_t} = 0; \tag{A.10}$$

$$-\dot{H}_t + \mu^Z - a_t^vM_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}Y_t + \gamma(\sigma_t^v)^2M_tY_t - \frac{b_te_t}{2a_t} = 0; \tag{A.11}$$

$$-\dot{I}_t - \rho + \rho I_t - \gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}N_t + \gamma(\sigma_t^v)^2M_tN_t - \frac{b_tc_t}{2a_t} = 0; \tag{A.12}$$

$$-\dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^v)^2N_t^2 - \frac{c_t^2}{4a_t} = 0; \tag{A.13}$$

$$-\dot{L}_t + \rho L_t - a_t^vN_t + \gamma(\sigma_t^v)^2N_tY_t - \frac{c_te_t}{2a_t} = 0; \tag{A.14}$$

$$-\dot{M}_t + (\alpha_t + \beta_t)\kappa_t - \alpha_t\kappa_tI_t + b_t^vM_t - 2\gamma\sigma_t^v\sigma_t^Z\rho^{v,Z}X_t + 2\gamma(\sigma_t^v)^2M_tX_t - \frac{b_td_t}{2a_t} = 0; \tag{A.15}$$

$$-\dot{N}_t + \rho N_t - 2\alpha_t\kappa_tJ_t + b_t^vN_t + 2\gamma(\sigma_t^v)^2N_tX_t - \frac{c_td_t}{2a_t} = 0; \tag{A.16}$$

$$-\dot{X}_t - \alpha_t\kappa_tN_t + 2b_t^vX_t + \gamma(\sigma_t^v)^2X_t^2 - \frac{d_t^2}{4a_t} = 0; \tag{A.17}$$

$$-\dot{Y}_t - \alpha_t\kappa_tL_t - 2a_t^vX_t + b_t^vY_t + 2\gamma(\sigma_t^v)^2X_tY_t - \frac{d_te_t}{2a_t} = 0; \tag{A.18}$$

$$-\dot{K}_t - a_t^vY_t - (\sigma_t^v)^2X_t + \frac{1}{2}\gamma(\sigma_t^v)^2Y_t^2 - \frac{e_t^2}{4a_t} = 0. \tag{A.19}$$

with the terminal conditions:

$$G_T = -\chi_T; \quad H_T = I_T = J_T = L_T = M_T = N_T = X_T = Y_T = K_T = 0. \tag{A.20}$$

Then, by substituting the dynamics of a_t, b_t, c_t, d_t, e_t into the condition derived above and rearranging, we obtain a system of ordinary differential equations consisting of $G_t, H_t, I_t, J_t, L_t, M_t, N_t, X_t, Y_t, Z_t$. □

B Matrix Representation of the Optimal Execution Strategy

We here show that the optimal execution problem considered above can be rewritten with a matrix representation. The matrix representation allows us to clearly understand the relationships between state variables and also prevails the interaction of the deterministic functions of time (expressed the equations from (A.10) to (A.19)) explicitly. Then we can examine a set of weak sufficient conditions satisfied by the system of ODEs with more ease than directly carrying out the analysis of equations from (A.10) to (A.19). To this end, let us first redefine the state process and the ansatz of the corresponding value function.

B.1 State Process: Revisited

The dynamics of the state process is, based on the argument in Section 2, reformed in the following stacked form:

$$\mathrm{d}\boldsymbol{s}_t = \begin{pmatrix} \mathrm{d}W_t \\ \mathrm{d}P_t \\ \mathrm{d}\overline{Q}_t \\ \mathrm{d}R_t \\ \mathrm{d}v_t \end{pmatrix} = \begin{pmatrix} -\left(P_t + \lambda_t\dot{Q}_t + \kappa_tv_t\right)\dot{Q}_t \\ -\rho R_t + (\alpha_t + \beta_t)(\lambda_t\dot{Q}_t + \kappa_tv_t) + \mu_t^Z \\ -\dot{Q}_t \\ -\rho R_t + \alpha_t(\lambda_t\dot{Q}_t + \kappa_tv_t) \\ (a_t^v - b_t^vv_t) \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 & 0 \\ \sigma_t^Z & 0 \\ 0 & 0 \\ 0 & \sigma_t^v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{d}B_t^Z \\ \mathrm{d}B_t^v \end{pmatrix}$$

$$=: \boldsymbol{\mu}_t(\boldsymbol{s}_t, \dot{Q}_t)\mathrm{d}t + \widetilde{\boldsymbol{\Sigma}}_t\mathrm{d}\widetilde{\boldsymbol{B}}_t, \tag{B.1}$$

where

$$\boldsymbol{\mu}_t(\boldsymbol{s}_t, \dot{Q}_t) := \begin{pmatrix} -\Big(P_t + \lambda_t \dot{Q}_t + \kappa_t v_t\Big)\dot{Q}_t \\ -\rho R_t + (\alpha_t + \beta_t)(\lambda_t \dot{Q}_t + \kappa_t v_t) + \mu_t^Z \\ -\dot{Q}_t \\ -\rho R_t + \alpha_t(\lambda_t \dot{Q}_t + \kappa_t v_t) \\ (a_t^v - b_t^v v_t) \end{pmatrix}; \quad \widetilde{\boldsymbol{\Sigma}}_t := \begin{pmatrix} 0 & 0 \\ \sigma_t^Z & 0 \\ 0 & 0 \\ 0 & \sigma_t^v \\ 0 & 0 \end{pmatrix}; \quad \mathrm{d}\widetilde{\boldsymbol{B}}_t := \begin{pmatrix} \mathrm{d}B_t^Z \\ \mathrm{d}B_t^v \end{pmatrix}.$$

We can rewrite $\mathrm{d}\widetilde{\boldsymbol{B}}_t$ with a Brownian motion orthogonal to B_t^Z, denoted by $B_t^{Z\perp}$, as

$$\mathrm{d}\widetilde{\boldsymbol{B}}_t := \begin{pmatrix} \mathrm{d}B_t^Z \\ \mathrm{d}B_t^v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \mathrm{d}B_t^Z \\ \mathrm{d}B_t^{Z\perp} \end{pmatrix} =: \boldsymbol{\rho}\mathrm{d}\mathbf{B}_t. \tag{B.2}$$

Plugging Eq. (B.2) into Eq. (B.1), we have

$$\mathrm{d}\boldsymbol{s}_t = \boldsymbol{\mu}_t(\boldsymbol{s}_t, \dot{Q}_t)\mathrm{d}t + \boldsymbol{\Sigma}_t' \boldsymbol{\rho} \mathrm{d}\mathbf{B}_t' = \boldsymbol{\mu}_t(\boldsymbol{s}_t, \dot{Q}_t)\mathrm{d}t + \boldsymbol{\Sigma}_t \mathrm{d}\mathbf{B}_t, \tag{B.3}$$

where $\boldsymbol{\Sigma}_t := \boldsymbol{\Sigma}_t' \boldsymbol{\rho} \in \mathscr{M}_{5,2}(\mathbb{R})$.

B.2 HJB Equation: Revisited

From the dynamic programming principle, the optimal value function, $V[t, \boldsymbol{s}_t] := V\left[t, W_t, P_t, \overline{Q}_t, R_t, v_t\right]$, with the terminal condition:

$$V[t, \boldsymbol{s}_T] = V\left[T, W_T, P_T, \overline{Q}_T, R_T, v_T\right] = -\exp\Big\{ -\gamma\big[W_T - (P_T + \chi_T \overline{Q}_T)\overline{Q}_T\big]\Big\}, \tag{B.4}$$

satisfies the following HJB equation (or dynamic programming equation) for the optimal (policy) function $\dot{Q}$:

$$\sup_{\dot{Q}\in\mathbb{R}} \left[\partial_t V + \boldsymbol{\mu}_t(\boldsymbol{s}_t, \dot{Q}_t)^\top \partial_{\boldsymbol{s}} V + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^\top \partial_{\boldsymbol{s}\boldsymbol{s}^\top} V \boldsymbol{\Sigma}\right)\right] = 0, \tag{B.5}$$

Here $\boldsymbol{\mu}_t(\boldsymbol{s}_t, \dot{Q}_t)$ also has another description associated with $\dot{Q}_t$:

$$\boldsymbol{\mu}_t(\boldsymbol{s}_t, \dot{Q}_t) := \left(-\boldsymbol{\Lambda}_t \dot{Q}_t^2 + (\boldsymbol{\Theta}_t \boldsymbol{s}_t + \boldsymbol{\Phi}_t)\dot{Q}_t + (\boldsymbol{\Pi}_t \boldsymbol{s}_t + \boldsymbol{\Psi})\right), \tag{B.6}$$

where

$$\boldsymbol{\Lambda}_t := \begin{pmatrix} \lambda_t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{\Theta}_t := \begin{pmatrix} 0 & -1 & 0 & 0 & -\kappa_t \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boldsymbol{\Phi}_t := \begin{pmatrix} 0 \\ (\alpha_t + \beta_t)\lambda_t \\ -1 \\ \alpha_t \lambda_t \\ 0 \end{pmatrix};$$

$$\boldsymbol{\Pi}_t := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho & (\alpha_t + \beta_t)\kappa_t \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho & \alpha_t \kappa_t \\ 0 & 0 & 0 & 0 & -b_t^v \end{pmatrix}; \quad \boldsymbol{\Psi} := \begin{pmatrix} 0 \\ \mu_t^Z \\ 0 \\ 0 \\ a_t^v \end{pmatrix}.$$

Putting Eq. (B.2) and (B.6) together with Eq. (B.5) yields

$$\sup_{\dot{Q}\in\mathbb{R}} \left[\left(-\boldsymbol{\Lambda}\dot{Q}^2 + (\boldsymbol{\Theta}\boldsymbol{s} + \boldsymbol{\Phi})\dot{Q}\right)^\top \partial_{\boldsymbol{s}} V\right] + \partial_t V + (\boldsymbol{\Pi}\boldsymbol{s} + \boldsymbol{\Psi})^\top \partial_{\boldsymbol{s}} V + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^\top \partial_{\boldsymbol{s}\boldsymbol{s}^\top} V \boldsymbol{\Sigma}\right) = 0. \tag{B.7}$$

We can then derive the matrix representation of the optimal execution strategy and its associated value function explicitly by appropriately guessing the form of the value function and verifying the obtained solution.

B.3 Matrix Representation of Optimal Execution Strategy and Optimal Value Function

Theorem B.1 (Matrix Representation of Optimal Execution Strategy and Optimal Value Function)**.** Under a set of regularity conditions:

1. The optimal execution volume at time $t \in [0,T]$, denoted as $\dot{Q}^*_t$, becomes an affine function of the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect R_t and the orders posed by small traders v_t at time t:
$$\dot{Q}^*_t = f_t(\boldsymbol{s}_t) = \frac{1}{2\lambda_t}\left\{\left(2\boldsymbol{\Phi}_t\boldsymbol{\Omega}_t + \boldsymbol{\Gamma}_t^\top\boldsymbol{\Theta}_t\right)\boldsymbol{s}_t + \boldsymbol{\Phi}_t^\top\boldsymbol{\Gamma}_t\right\} (= a_t + b_t\overline{Q}_t + c_tR_t + d_tv_t). \tag{B.8}$$
where a_t, b_t, c_t, d_t are the ones defined in Section 3.

2. The optimal value function $V[t,\boldsymbol{s}_t]$ at time $t \in [0,T]$ is represented as follows:
$$V\left[t,\boldsymbol{s}_t\right] = -\exp\Big\{-\gamma\big[\boldsymbol{s}_t^\top\boldsymbol{\Omega}_t\boldsymbol{s}_t + \boldsymbol{\Gamma}_t^\top\boldsymbol{s}_t + K_t\big]\Big\}, \tag{B.9}$$
where
$$\boldsymbol{\Omega}_t := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 0 & -1/2 & G_t & 1/2I_t & 1/2M_t \\ 0 & 0 & 1/2I_t & J_t & 1/2N_t \\ 0 & 0 & 1/2M_t & 1/2N_t & X_t \end{pmatrix}; \quad \boldsymbol{\Gamma}_t := \begin{pmatrix} 1 \\ 0 \\ H_t \\ L_t \\ Y_t \end{pmatrix},$$
with the terminal conditions:
$$\boldsymbol{\Omega}_T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 0 & -1/2 & -\chi_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathscr{M}_{5,5}(\mathbb{R}); \quad \boldsymbol{\Gamma}_T := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^5; \quad K_T = 0,$$
and $G_t, H_t, I_t, J_t, L_t, M_t, N_t, X_t, Y_t, K_t$ for $t \in [0,T]$ are the ones defined in Section 3.

Proof. Writing the ansatz (A.1) in matrix form yields Eq. (B.9). Here we have
$$\begin{aligned} \partial_t V\left[t,\boldsymbol{s}\right] &= V\left[t,\boldsymbol{s}\right]\{-\gamma[\boldsymbol{s}^\top\dot{\boldsymbol{\Omega}}\boldsymbol{s} + \dot{\boldsymbol{\Gamma}}\boldsymbol{s} + \dot{K}\}; \\ \partial_{\boldsymbol{s}} V\left[t,\boldsymbol{s}\right] &= V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s} + \boldsymbol{\Gamma}]\}; \\ \partial_{\boldsymbol{s}\boldsymbol{s}^\top} V\left[t,\boldsymbol{s}\right] &= V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s} + \boldsymbol{\Gamma}]\}\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s} + \boldsymbol{\Gamma}]\}^\top + V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}]\} \end{aligned}$$
Thus, by substituting the above equations into the HJB equation (B.7), we obtain
$$\begin{aligned} &\sup_{\dot{Q}\in\mathbb{R}}\left[\left(-\boldsymbol{\Lambda}\dot{Q}^2 + (\boldsymbol{\Theta}\boldsymbol{s} + \boldsymbol{\Phi})\dot{Q}\right)^\top V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s} + \boldsymbol{\Gamma}]\}\right] \\ &\quad + V\left[t,\boldsymbol{s}\right]\left\{-\gamma[\boldsymbol{s}^\top\dot{\boldsymbol{\Omega}}\boldsymbol{s} + \dot{\boldsymbol{\Gamma}}\boldsymbol{s} + \dot{K}\right\} + (\boldsymbol{\Pi}\boldsymbol{s} + \boldsymbol{\Psi})^\top V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s} + \boldsymbol{\Gamma}]\} \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^\top\left[V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s} + \boldsymbol{\Gamma}]\}\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s} + \boldsymbol{\Gamma}]\}^\top + V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}]\}\right]\boldsymbol{\Sigma}\right) = 0. \end{aligned}$$

Rearranging the equation results in

$$
\begin{aligned}
&\sup_{\dot{Q}\in\mathbb{R}} V\left[t,\boldsymbol{s}\right](-\gamma)\left[\left(-\boldsymbol{\Lambda}\dot{Q}^2+(\boldsymbol{\Theta}\boldsymbol{s}+\boldsymbol{\Phi})\dot{Q}\right)^\top\{[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]\}\right]\\
&\quad+V\left[t,\boldsymbol{s}\right]\left\{-\gamma[\boldsymbol{s}^\top\dot{\boldsymbol{\Omega}}\boldsymbol{s}+\dot{\boldsymbol{\Gamma}}\boldsymbol{s}+\dot{K}\}+(\boldsymbol{\Pi}\boldsymbol{s}+\boldsymbol{\Psi})^\top V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]\right\}\\
&\quad+\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^\top\left[V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]\}\{-\gamma[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]\}^\top+V\left[t,\boldsymbol{s}\right]\{-\gamma[2\boldsymbol{\Omega}]\}\right]\boldsymbol{\Sigma}\right)\\
=&\sup_{\dot{Q}\in\mathbb{R}} V\left[t,\boldsymbol{s}\right](-\gamma)\left[-\left(2\boldsymbol{\Lambda}^\top\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Lambda}^\top\boldsymbol{\Gamma}\right)\dot{Q}^2+(\boldsymbol{\Theta}\boldsymbol{s}+\boldsymbol{\Phi})^\top[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]\dot{Q}\right]\\
&\quad+V\left[t,\boldsymbol{s}\right](-\gamma)\left[\boldsymbol{s}^\top\dot{\boldsymbol{\Omega}}\boldsymbol{s}+\dot{\boldsymbol{\Gamma}}^\top\boldsymbol{s}+\dot{K}+(\boldsymbol{\Pi}\boldsymbol{s}+\boldsymbol{\Psi})^\top\left[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}\right]\right.\\
&\quad\left.+\frac{1}{2}(-\gamma)\mathrm{tr}\left(\boldsymbol{\Sigma}^\top\left[-\gamma[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}][2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]^\top\right]\boldsymbol{\Sigma}+\boldsymbol{\Sigma}^\top\left[2\boldsymbol{\Omega}\right]\boldsymbol{\Sigma}\right)\right]=0. \qquad \text{(B.10)}
\end{aligned}
$$

The first–order condition becomes

$$
-2\left(2\boldsymbol{\Lambda}^\top\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Lambda}^\top\boldsymbol{\Gamma}\right)\dot{Q}+(\boldsymbol{\Theta}\boldsymbol{s}+\boldsymbol{\Phi})^\top[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]=0.
$$

Therefore, the optimal execution speed is

$$
\dot{Q}^*=\frac{1}{2\lambda_t}\left(\boldsymbol{s}^\top\boldsymbol{\Theta}^\top+\boldsymbol{\Phi}^\top\right)[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]=\frac{1}{2\lambda_t}\left(\left(2\boldsymbol{\Phi}^\top\boldsymbol{\Omega}+\boldsymbol{\Gamma}^\top\boldsymbol{\Theta}\right)\boldsymbol{s}+\boldsymbol{\Phi}^\top\boldsymbol{\Gamma}\right). \qquad \text{(B.11)}
$$

In the derivation of the optimal trading speed, we have used the following facts:

1. $\boldsymbol{\Lambda}_t^\top\boldsymbol{\Omega}_t=\boldsymbol{0}\in\mathscr{M}_{1,5}(\mathbb{R})$:

$$
\boldsymbol{\Lambda}_t^\top\boldsymbol{\Omega}_t=\begin{pmatrix}\lambda_t & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1/2 & 0 & 0\\ 0 & -1/2 & G_t & 1/2I_t & 1/2M_t\\ 0 & 0 & 1/2I_t & J_t & 1/2N_t\\ 0 & 0 & 1/2M_t & 1/2N_t & X_t\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0 & 0\end{pmatrix};
$$

2. $\boldsymbol{\Theta}_t^\top\boldsymbol{\Omega}_t=\boldsymbol{0}\in\mathscr{M}_{5,5}(\mathbb{R})$:

$$
\boldsymbol{\Theta}_t^\top\boldsymbol{\Omega}_t=\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -\kappa_t & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1/2 & 0 & 0\\ 0 & -1/2 & G_t & 1/2I_t & 1/2M_t\\ 0 & 0 & 1/2I_t & J_t & 1/2N_t\\ 0 & 0 & 1/2M_t & 1/2N_t & X_t\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.
$$

These relationships make the optimal execution speed an affine function of the state process. Then, Eq. (B.10) becomes

$$
\begin{aligned}
&V\left[t,\boldsymbol{s}\right](-\gamma)\frac{1}{4\lambda}\left(\left(2\boldsymbol{\Phi}^\top\boldsymbol{\Omega}+\boldsymbol{\Gamma}^\top\boldsymbol{\Theta}\right)\boldsymbol{s}+\boldsymbol{\Phi}^\top\boldsymbol{\Gamma}\right)^2\\
&+V\left[t,\boldsymbol{s}\right](-\gamma)\left[\boldsymbol{s}^\top\dot{\boldsymbol{\Omega}}\boldsymbol{s}+\dot{\boldsymbol{\Gamma}}^\top\boldsymbol{s}+\dot{K}+(\boldsymbol{\Pi}\boldsymbol{s}+\boldsymbol{\Psi})^\top\left[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}\right]\right.\\
&\left.+\frac{1}{2}V\left[t,\boldsymbol{s}\right](-\gamma)\ \mathrm{tr}\left(\boldsymbol{\Sigma}^\top\left[-\gamma[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}][2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]^\top\right]\boldsymbol{\Sigma}+\boldsymbol{\Sigma}^\top\left[2\boldsymbol{\Omega}\right]\boldsymbol{\Sigma}\right)\right]=0. \qquad \text{(B.12)}
\end{aligned}
$$

Here we have

$$\begin{aligned}
&\operatorname{tr}\left(\boldsymbol{\Sigma}^\top\left[-\gamma[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}][2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]^\top\right]\boldsymbol{\Sigma}+2\boldsymbol{\Sigma}^\top\boldsymbol{\Omega}\boldsymbol{\Sigma}\right)\\
=&-\gamma\operatorname{tr}\left(\boldsymbol{\Sigma}^\top[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}][2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]^\top\boldsymbol{\Sigma}\right)+2\operatorname{tr}\left(\boldsymbol{\Sigma}^\top\boldsymbol{\Omega}\boldsymbol{\Sigma}\right)\\
=&-\gamma\operatorname{tr}\underbrace{\left([2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]^\top\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]\right)}_{\in\mathbb{R}}+2\operatorname{tr}\left(\boldsymbol{\Sigma}^\top\boldsymbol{\Omega}\boldsymbol{\Sigma}\right)\\
=&-\gamma[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]^\top\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top[2\boldsymbol{\Omega}\boldsymbol{s}+\boldsymbol{\Gamma}]+2\operatorname{tr}\left(\boldsymbol{\Sigma}^\top\boldsymbol{\Omega}\boldsymbol{\Sigma}\right)\\
=&-4\gamma\boldsymbol{s}^\top\boldsymbol{\Omega}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\boldsymbol{\Omega}\boldsymbol{s}-2\gamma\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\boldsymbol{\Omega}\boldsymbol{s}-2\gamma\boldsymbol{s}^\top\boldsymbol{\Omega}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\boldsymbol{\Gamma}-\gamma\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\boldsymbol{\Gamma}+2\operatorname{tr}\left(\boldsymbol{\Sigma}^\top\boldsymbol{\Omega}\boldsymbol{\Sigma}\right).
\end{aligned}$$

Substituting this equation into Eq. (B.12) and rearranging yields

$$\begin{aligned}
&\boldsymbol{s}_t^\top\left[\dot{\boldsymbol{\Omega}}_t-\boldsymbol{\Omega}_t^\top\left(2\gamma\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t^\top-\frac{1}{\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Phi}_t^\top\right)\boldsymbol{\Omega}_t+\boldsymbol{\Omega}_t^\top\left(\boldsymbol{\Pi}_t+\frac{1}{\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Gamma}_t^\top\boldsymbol{\Theta}_t\right)+\left(\boldsymbol{\Pi}_t+\frac{1}{\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Gamma}_t^\top\boldsymbol{\Theta}_t\right)^\top\boldsymbol{\Omega}_t+\frac{1}{\lambda_t}\boldsymbol{\Theta}_t^\top\boldsymbol{\Gamma}_t\boldsymbol{\Gamma}_t^\top\boldsymbol{\Theta}_t\right]\boldsymbol{s}_t\\
&+\left[\dot{\boldsymbol{\Gamma}}_t+\boldsymbol{\Gamma}_t^\top\left(\frac{1}{2\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Theta}_t^\top\right)\boldsymbol{\Gamma}_t-\left(\boldsymbol{\Pi}_t-2\gamma\boldsymbol{\Omega}_t\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t^\top-\frac{1}{\lambda_t}\boldsymbol{\Omega}_t\boldsymbol{\Phi}_t\boldsymbol{\Phi}_t^\top\right)\boldsymbol{\Gamma}_t-2\boldsymbol{\Omega}_t\boldsymbol{\Psi}_t\right]^\top\boldsymbol{s}_t\\
&+\left[\dot{K}_t+\frac{1}{4\lambda_t}\left(\boldsymbol{\Phi}_t\boldsymbol{\Gamma}_t\right)^2+\boldsymbol{\Psi}_t^\top\boldsymbol{\Gamma}_t-\frac{1}{2}\gamma\boldsymbol{\Gamma}_t^\top\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t^\top\boldsymbol{\Gamma}_t+\operatorname{tr}\left(\boldsymbol{\Sigma}_t^\top\boldsymbol{\Omega}_t\boldsymbol{\Sigma}_t\right)\right]=0.
\end{aligned}$$

Therefore, for the above equation to be satisfied for all $\boldsymbol{s}_t \in S$, the following three equations must hold:

1. with respect to $\boldsymbol{\Omega}_t \in \mathscr{M}_{5,5}(\mathbb{R})$:

$$\begin{aligned}
\dot{\boldsymbol{\Omega}}_t=&\boldsymbol{\Omega}_t^\top\left(2\gamma\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t^\top-\frac{1}{\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Phi}_t^\top\right)\boldsymbol{\Omega}_t-\boldsymbol{\Omega}_t^\top\left(\boldsymbol{\Pi}_t+\frac{1}{\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Gamma}_t^\top\boldsymbol{\Theta}_t\right)-\left(\boldsymbol{\Pi}_t+\frac{1}{\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Gamma}_t^\top\boldsymbol{\Theta}_t\right)^\top\boldsymbol{\Omega}_t\\
&-\frac{1}{\lambda_t}\boldsymbol{\Theta}_t^\top\boldsymbol{\Gamma}_t\boldsymbol{\Gamma}_t^\top\boldsymbol{\Theta}_t; \qquad \text{(B.13)}
\end{aligned}$$

2. with respect to $\boldsymbol{\Gamma}_t \in \mathbb{R}^5$:

$$\dot{\boldsymbol{\Gamma}}_t=\boldsymbol{\Gamma}_t^\top\left(-\frac{1}{2\lambda_t}\boldsymbol{\Phi}_t\boldsymbol{\Theta}_t^\top\right)\boldsymbol{\Gamma}_t+\left(\boldsymbol{\Pi}_t-2\gamma\boldsymbol{\Omega}_t\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t^\top-\frac{1}{\lambda_t}\boldsymbol{\Omega}_t\boldsymbol{\Phi}_t\boldsymbol{\Phi}_t^\top\right)\boldsymbol{\Gamma}_t+2\boldsymbol{\Omega}_t\boldsymbol{\Psi}_t; \qquad \text{(B.14)}$$

3. with respect to $K_t \in \mathbb{R}$:

$$\dot{K}_t=-\frac{1}{4\lambda_t}\left(\boldsymbol{\Phi}_t\boldsymbol{\Gamma}_t\right)^2-\boldsymbol{\Psi}_t^\top\boldsymbol{\Gamma}_t+\frac{1}{2}\gamma\boldsymbol{\Gamma}_t^\top\boldsymbol{\Sigma}_t\boldsymbol{\Sigma}_t^\top\boldsymbol{\Gamma}_t+\operatorname{tr}\left(\boldsymbol{\Sigma}_t^\top\boldsymbol{\Omega}_t\boldsymbol{\Sigma}_t\right). \qquad \text{(B.15)}$$

As these equations show, the relationships which $\boldsymbol{\Omega}_t, \boldsymbol{\Gamma}_t$, and K_t satisfy are represented by a system of matrix Riccati equations. If $\boldsymbol{\Omega}_t$ and $\boldsymbol{\Gamma}_t$ uniquely exist, then K_t are also determined uniquely by integrating $\dot{K}_t$:

$$K_t=\int_T^t \dot{K}_s \mathrm{d}s,$$

due to the terminal condition: $K_T \equiv 0$. □

C Proof of Proposition 3.1

Calculating $\boldsymbol{\Theta}_t^\top \boldsymbol{\Gamma}_t$ yields

$$\boldsymbol{\Theta}_t^\top \boldsymbol{\Gamma}_t = \begin{pmatrix} 0 & -1 & 0 & 0 & -\kappa_t \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^\top \begin{pmatrix} 1 \\ 0 \\ H_t \\ L_t \\ Y_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\kappa_t & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ H_t \\ L_t \\ Y_t \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -\kappa_t \end{pmatrix} \in \mathbb{R}^5. \tag{C.1}$$

Therefore, the vector $\boldsymbol{\Theta}_t^\top \boldsymbol{\Gamma}_t$ becomes a one whose elements are all constant (in time t) and so does the matrix $\boldsymbol{\Theta}_t^\top \boldsymbol{\Gamma}_t \boldsymbol{\Gamma}_t^\top \boldsymbol{\Theta}_t$. Therefore, Eq. (B.13) satisfies the following ordinary matrix Riccati differential equation:

$$\dot{\boldsymbol{\Omega}}_t = -\boldsymbol{\Omega}_t^\top \mathbf{A}_t \boldsymbol{\Omega}_t + \boldsymbol{\Omega}_t^\top \mathbf{B}_t + \mathbf{B}_t^\top \boldsymbol{\Omega}_t + \mathbf{C}_t \tag{C.2}$$

with the terminal condition:

$$\boldsymbol{\Omega}_T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 0 & -1/2 & -\chi_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathscr{M}_{5,5}(\mathbb{R}),$$

where

$$\begin{aligned} \mathbf{A}_t &:= -2\gamma \boldsymbol{\Sigma}_t \boldsymbol{\Sigma}_t^\top + \frac{1}{\lambda_t} \boldsymbol{\Phi}_t \boldsymbol{\Phi}_t^\top \in \mathscr{M}_{5,5}(\mathbb{R}); \\ \mathbf{B}_t &:= -\boldsymbol{\Pi}_t - \frac{1}{2\lambda_t} \boldsymbol{\Phi}_t \boldsymbol{\Gamma}_t^\top \boldsymbol{\Theta}_t \in \mathscr{M}_{5,5}(\mathbb{R}); \\ \mathbf{C}_t &:= -\frac{1}{\lambda_t} \boldsymbol{\Theta}_t^\top \boldsymbol{\Gamma}_t \boldsymbol{\Gamma}_t^\top \boldsymbol{\Theta}_t \in \mathscr{M}_{5,5}(\mathbb{R}), \end{aligned} \tag{C.3}$$

and $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{C}_t$ are all constant matrices (in time t). □