What is a Frobenius-projective structure??

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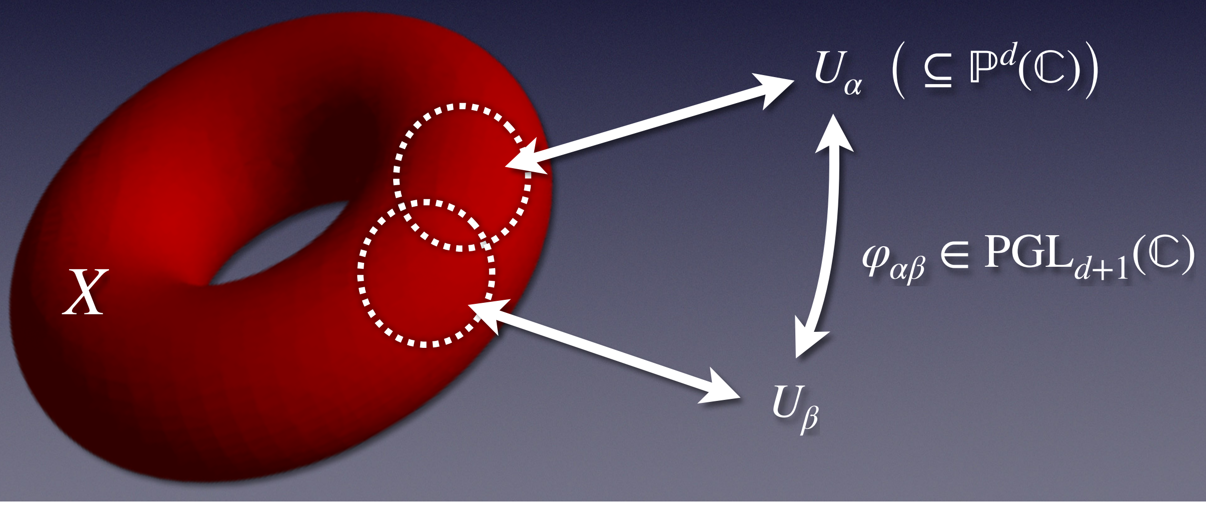
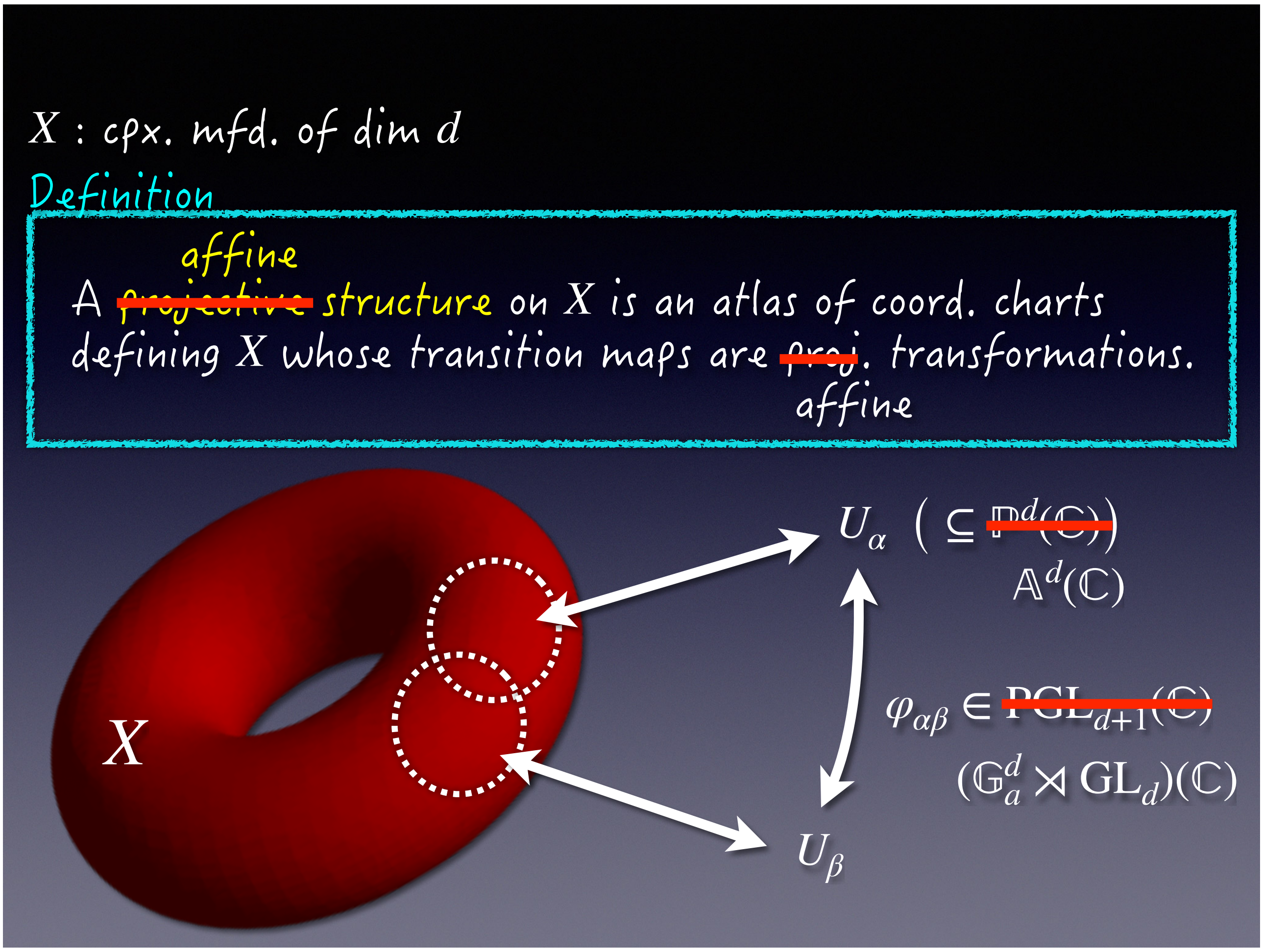
§1 Review of Complex projective structures

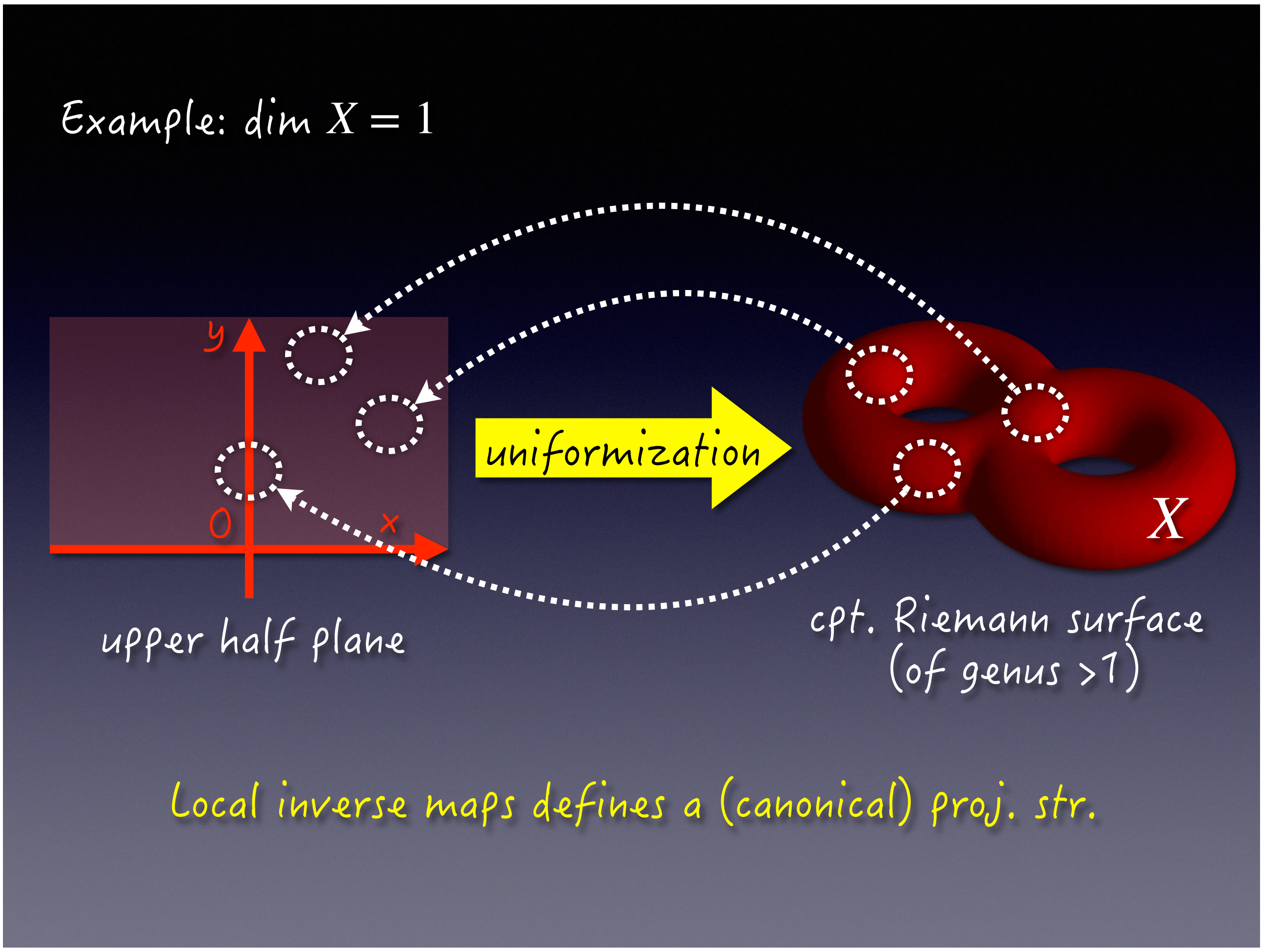
X : cpx. mfd. of dim d

Definition

> A *projective structure* on X is an atlas of coord. charts defining X whose transition maps are proj. transformations.




X : cpx. mfd. of dim d
Definition
affine
A ~~projective~~ structure on X is an atlas of coord. charts defining X whose transition maps are ~~proj~~. transformations.
affine
U_α ($\subseteq$ ~~$\mathbb{P}^d(\mathbb{C})$~~)
$\mathbb{A}^d(\mathbb{C})$
$\varphi_{\alpha\beta} \in$ ~~$\mathrm{PGL}_{d+1}(\mathbb{C})$~~
$(\mathbb{G}_a^d \rtimes \mathrm{GL}_d)(\mathbb{C})$
U_β
X


Example: dim $X = 1$
y
O
x
upper half plane
uniformization
X
cpt. Riemann surface
(of genus >1)
Local inverse maps defines a (canonical) proj. str.

Classification for proj. cpx. mfd.'s

Case of dim 1	proj. str.	aff. str.
genus 0	✓	✗
genus 1	✓	✓
genus >1	✓	✗

Case of dim 2: Kobayashi-Ochiai ('80, '81)

Case of dim 3: Jahnke-Radloff ('04)

§ 2 Projective structures in char. $p > 0$

k : field of char. $p > 0$ s.t. $k = \overline{k}$

X : sm. proj. k-variety of dim d s.t. $p > (d+1)$

$\mathscr{P}$: sheaf on X given by $U \mapsto \{$étale mor.'s $U \to \mathbb{P}^d\}$

$G^{(N)}$: sheaf on X given by $U \mapsto \mathrm{PGL}_{d+1}(U^{(N)})$ ← N-th Frob. twist of U

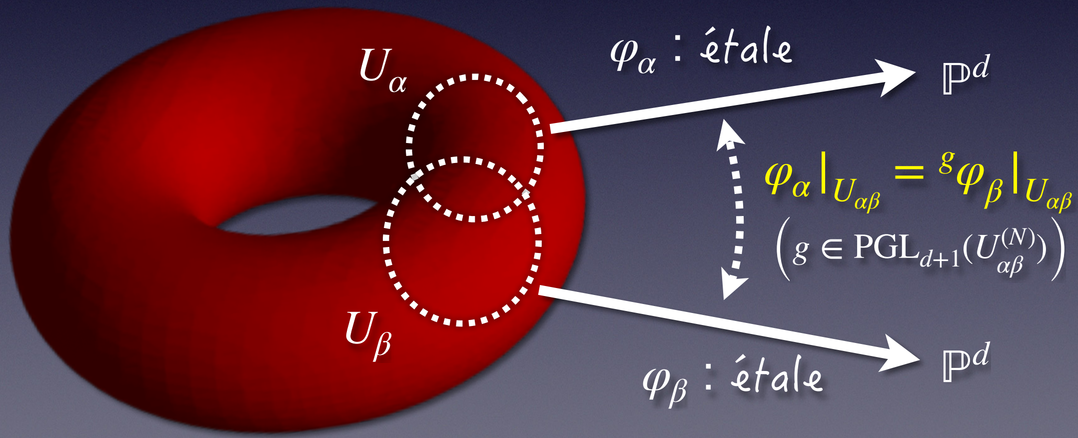
$(N > 0)$

⟶
- $G^{(1)} \supseteq G^{(2)} \supseteq \cdots \supseteq G^{(N)} \supseteq \cdots$
- $\mathscr{P}$ has a $G^{(N)}$-action

Definition

An F^N-projective structure on X
is a subsheaf of $\mathscr{P}$ forming a $G^{(N)}$-torsor

(An F^N-affine structure is defined in a similar manner.)



Definition

An F^∞-projective structure on X is a "compatible" collection of F^N-proj. str.'s for various $N > 0$

(An F^∞-affine structure is defined in a similar manner.)

Remark

F^∞-proj. str. on X ⟶ $\pi_1^{\mathrm{str}}(X, x) \to \mathrm{PGL}_{d+1}$

Chern class formula ('20 W.)

$\exists$ F^N-proj. (resp., F^N-aff.) str. on X

$$\Longrightarrow \forall i, \quad c_i \equiv \frac{1}{(d+1)^i} \cdot \binom{d+1}{i} c_1^i \quad (\text{resp., } c_i \equiv 0\,)$$

in $H^{2i}_{\mathrm{crys}}(X/W)$ mod p^N.

Characterization of $\mathbb{P}^d$ ('20 W.)

$\exists$ F^∞-proj. str. on X, and either

- X contains a rational curve, or
- $\pi_1^{\text{ét}}(X) = \{e\}$

$\Longrightarrow X \cong \mathbb{P}^d$

Classification for dim1 ('99 Mochizuki, '20 W., '20 Hoshi)

	F^N-proj. str.		F^N-aff. str.	
genus 0	✓		✗	
genus 1	$N \geq 2$ ✓ (⇔ ordinary)	$N = 1$ ✓	$N \geq 2$ ✓ (⇔ ordinary)	$N = 1$ ✓
genus >1	✓		$N = \infty$ ✗	$N < \infty$ ✓ & ✗

Classification for dim2 ('20 W.)

	F^∞-proj. str.	F^∞-aff. str.
rational surface	✓ ($\Leftrightarrow \cong \mathbb{P}^2$)	✗
ruled surface	✗	✗
Enriques surface	✗	✗
K3 surface	✗	✗
Abelian surface	✓ ($\Leftrightarrow$ ordinary)	✓ ($\Leftrightarrow$ ordinary)
bielliptic surface	✓ ($\Leftarrow$ general)	✓ ($\Leftarrow$ general)
properly ell. surface	✗ ($\Leftarrow$ general)	✗ ($\Leftarrow$ general)
surface of gen. type	✓ ($\Rightarrow c_1^2 = 3c_2$)	✓ ($\Rightarrow c_1^2 = c_2 = 0$)

Classification for Abelian var.'s ('20 W.)

X : Abelian var. of dim d.

(i) $\exists$ F^∞-affine str. on $X \Longleftrightarrow X$ is ordinary

(ii) If X is ordinary, then

$$\{F^\infty\text{-aff. str.'s}/X\} \cong \left\{(b_i) \in (T_p\hat{X})^d \;\middle|\; \wedge_i b_i \in \det(T_p\hat{X})^\times\right\}/\mathfrak{S}_n$$

(The set of F^∞-proj. str.'s is obtained as a quotient of this set.)

Theorem ('14-, W.)

X : curve of genus g

$$\sharp\{F^1\text{-proj. str.'s}/X\} = \begin{cases} 1 & (g=0) \\ \left(\frac{p-1}{2}\right)^{p\text{-rank}} & (g=1) \\ \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \sin^{2-2g}\left(\frac{\pi\cdot\theta}{p}\right) & (g>1) \end{cases}$$

(counted with multiplicity)

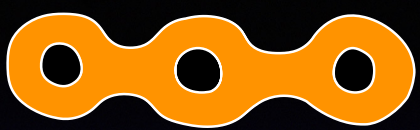
How to calculate :

(1) Apply a computation of GW inv. of rel. Grass.

(2) Compare with $\mathfrak{sl}_2(\mathbb{C})$ CFT

(3) Count via a graph-theoretic description

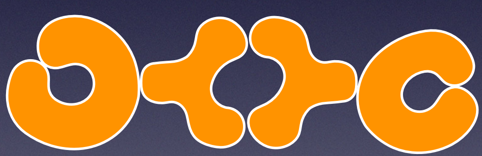
(smooth) curve



Degeneration

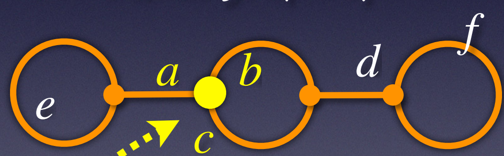
F^1-proj. str.'s on

X : totally degenerate



Edge-numberings on

Γ : dual graph of X



$$\begin{cases} 0 \leq a \leq b+c \\ 0 \leq b \leq c+a \\ 0 \leq c \leq a+b \\ a+b+c \leq p-2 \end{cases}$$

Theorem ('20 W., '21 Hoshi)

$\mathcal{N}_g$: locus of $\mathcal{M}_g$ classifying curves having an F^1-aff. str.

- $\mathcal{N}_g \neq \varnothing \iff p \mid g-1$
- $p \mid g-1 \implies \dim \mathcal{N}_g = \dfrac{2(p+1)(g-1)}{p}$