ジェットスキームの有理２重点上の ファイバーの既約成分の配置

Yoshimune Koreeda

Hiroshima university

October 27th 2021

Overview

Setting

S : surface with rational double point singularity at 0
$m > 0$
S_m : "m-th jet scheme"
$\pi_m : S_m \to S$: "truncation map"

The following one-to-one correspondence is known (Mourtada) :

Irred. comps. of $\pi_m^{-1}(0) \leftrightarrow$ Exceptional curves in a minimal resolution

Problem

Can one reconstruct the resolution graph from the informations of the jet scheme?

Overview

Main Theorem

For A_n- or D_4-type singular surfaces, the following conditions are equivalent:

- The intersection of two distinct irreducible components of $\pi_m^{-1}(0)$ is "maximal".
- The corresponding exceptional curves on the minimal resolution intersect.

Today
$\rightsquigarrow$ Mainly talk about the case of the D_4-type singular surface.

Preparation : Definition of jet schemes

Setting

Sch$/\mathbb{C}$: *the category of schemes over* $\mathbb{C}$
Set : *the category of sets*
$X \in$ **Sch**$/\mathbb{C}$: *a scheme of finite type over* $\mathbb{C}$

For $m \in \mathbb{Z}_{\geq 0}$, the functor

$$F_m^X : \mathbf{Sch}/\mathbb{C} \to \mathbf{Set};\, Z \mapsto \mathrm{Hom}_Z(Z \times \mathrm{Spec}\ \mathbb{C}[t]/\langle t^{m+1}\rangle, Z \times X)$$

is a representable functor, and is represented by a scheme X_m of finite type over $\mathbb{C}$.

Definition(Jet scheme)

The scheme X_m is called the m-th jet scheme of X.

Preparation : Calculation of jet schemes

$X = (f(x, y, z) = 0) \subset \mathbb{A}^3$
The $m(\in \mathbb{Z}_{\geq 0})$-th jet schemes are calculated as follows:

Let $\mathbf{x} = x_0 + x_1 t + x_2 t^2 + \cdots + x_m t^m$, $\mathbf{y} = y_0 + y_1 t + y_2 t^2 + \cdots + y_m t^m$, $\mathbf{z} = z_0 + z_1 t + z_2 t^2 + \cdots + z_m t^m$.
Expand $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv f^{(0)} + f^{(1)} t + \cdots + f^{(m)} t^m \mod t^{m+1}$$

$(f^{(0)}, \ldots, f^{(m)} \in \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_m, z_0, \ldots, z_m])$.

Then the m-th jet scheme X_m of X is defined by

$$\langle f^{(0)}, \ldots, f^{(m)} \rangle$$

in $\mathbb{A}^{3(m+1)}$.

$\pi_m : X_m \to X$: truncation morphism(the map given by "t = 0")

Preparation

For $X = \mathbb{A}^3$, the m-th jet scheme is $X_m = \mathbb{A}^{3(m+1)}$ and the truncation morphism is as follows:

$$\pi_m : X_m \to X; (a_0, \ldots, a_m, b_0, \ldots, b_m, c_0, \ldots, c_m) \mapsto (a_0, b_0, c_0).$$

Definition(Singular fiber)

The inverse image of the singular point $\pi_m^{-1}(0)$, denoted by X_m^0, is called the singular fiber.

Singular fiber

$S = \mathbf{V}(f) \subset \mathbb{A}^3$: surface with an A_n- or D_4-type singular point at 0
The singular fiber S_m^0 is defined by the following ideal in $(\mathbb{A}^3)_m \cong \mathbb{A}^{3(m+1)}$:

$$\langle x_0, y_0, z_0, f^{(0)}, \ldots, f^{(m)} \rangle.$$

The following theorem was proven by H. Mourtada.

Theorem (H. Mourtada [1, Theorem 3.1, Theorem 3.2])

Suppose $n > 0$ and $m \gg 0$. The singular fiber S_m^0 decomposes into n irreducible components and their codimensions are 1 in S_m.

We considering the following problem about irreducible components of the singular fiber.

Problem

Fix the degree m of jets. When is the intersection of two distinct irreducible components of the singular fiber "maximal"?

Main result1

① For A_n-type singular surfaces,

Theorem(Main theorem 1) [2, Theorem 2.10, Corollary 2.14]

Suppose $m \geq 2n + 2$ and $S_m^0 = Z_m^1 \cup Z_m^2 \cup \cdots \cup Z_m^n$. For $1 \leq i < j \leq n$, the number of irreducible components $Z_m^i \cap Z_m^j$ is $n - (j - i) + 2$ and the codimension of $Z_m^i \cap Z_m^j$ in S_m is 2.
Moreover

$$Z_m^i \cap Z_m^j \text{ is maximal in } \{Z_m^{l_1} \cap Z_m^{l_2} \mid 1 \leq l_1 < l_2 \leq n\} \Leftrightarrow j - i = 1.$$

Method : Concrete calculation using the defining ideals of Z_m^i

② For a D_4-type singular surface,
⇝ The generators of defining ideal of irred. components are not known.
⇝ Find some jets explicitly and determine the inclusion relations.

The irreducible components for D_4-singularity (Mourtada)

Let $f = x^2 - y^2 z + z^3$, $S = \mathbb{V}(f)$, $m \geq 5$ and
$R_m = \mathbb{C}[x_0, ..., x_m, y_0, ..., y_m, z_0, ..., z_m]$.
The defining ideals of the irreducible components of S_m^0 are as follows:

$$\begin{aligned}
I_m^0 &:= \langle x_0, x_1, x_2, \quad y_0, y_1 \quad z_0, z_1, \quad f^{(0)}, ..., f^{(m)} \rangle \\
J_m^1 &:= \langle x_0, x_1, \quad y_0, \quad z_0, \quad z_1, \quad f^{(0)}, ..., f^{(m)} \rangle \\
J_m^2 &:= \langle x_0, x_1, \quad y_0, \quad z_0, \quad y_1 - z_1, \quad f^{(0)}, ..., f^{(m)} \rangle \\
J_m^3 &:= \langle x_0, x_1, \quad y_0, \quad z_0, \quad y_1 + z_1, \quad f^{(0)}, ..., f^{(m)} \rangle
\end{aligned}$$

$I_m^0 \rightsquigarrow$ the defining ideal of an irreducible component
$J_m^i \rightsquigarrow$ the defining ideals of irreducible components in $y_1 \neq 0$

$$I_m^i := J_m^i \cdot (R_m)_{y_1} \cap R_m \ (i = 1, 2, 3),$$

then

$$Z_m^i = \mathbb{V}(I_m^i) \ (i = 0, 1, 2, 3).$$

Main result2

Theorem(Main theorem 2) [2, Theorem 3.17]

Suppose $m \geq 5$ and Z_m^i's are as above. Then $S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3$ and the maximal elements of $\{Z_m^{l_1} \cap Z_m^{l_2} \mid 0 \leq l_1 < l_2 \leq 3\}$ are

$$Z_m^0 \cap Z_m^1, Z_m^0 \cap Z_m^2, Z_m^0 \cap Z_m^3.$$

We outline the proof of $Z_m^0 \cap Z_m^1 \not\subseteq Z_m^0 \cap Z_m^2$ i.e.
$Z_m^0 \cap Z_m^1 \supsetneq Z_m^0 \cap Z_m^1 \cap Z_m^2$, in the case $m \geq 6$.
Consider the following two elements.

$$\begin{aligned}
h_1 =& -4y_2^2 z_2^2 + y_1^2 z_3^2 + 4x_3^2 z_2 - 4x_2 x_3 z_3 \\
h_2 =& -y_2^4 - 4y_2^3 z_2 + 2y_2^2 z_2^2 + 12 y_2 z_2^3 - 9z_2^4 + 4y_3^2 z_1^2 - 8y_3 z_1^2 z_3 \\
& + 4z_1^2 z_3^2 + 8x_3^2 y_2 - 8x_3^2 z_2 - 8x_2 x_3 y_3 + 8x_2 x_3 z_3.
\end{aligned}$$

We can prove $h_1 \in I_m^1$ and $h_2 \in I_m^2$, and using these we can prove
$y_2 \in \sqrt{I_m^0 + I_m^1 + I_m^2}$.
On the other hand, we have $(0, t^2, 0) \in Z_m^0 \cap Z_m^1$ and so $y_2 \notin \sqrt{I_m^0 + I_m^1}$.

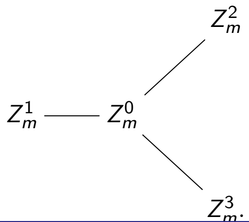
Main result2

For $m \gg 0$, we construct the graph as follows:
vertices : Irreducible components of S_m^0
edges : When two irreducible components are maximal in $\{Z_m^i \cap Z_m^j \mid 1 \leq i < j \leq n\}$.

Corollary [2, Corollary 2.16, Corollary 3.18]

For an A_n- or D_4-type singularity, the graph constructed as above is isomorphic to the resolution graph of the minimal resolution of singularity.

In the D_4 case, the graph is as follows;



References

H. Mourtada, Jet schemes of rational double point singularities, Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2014), 373–388.

Y. Koreeda, On the configuration of the singular fibers of jet schemes of rational double points, arXiv:math/2012.08144, to appear in Communications in Algebra.