Birational geometry of moduli spaces of parabolic connections

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Introduction (1)

- An regular singular parabolic connection (E, ∇, l_*) over a smooth complex projective curve C with marked points $\boldsymbol{t} = (t_1, \ldots, t_n)$ is the pair of a parabolic bundle (E, l_*) and a logarithmic connection ∇ compatible with the parabolic structure l_*.
- **moduli space of parabolic connections**

$$M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, r, d) := \left\{ (E, \nabla, l_*) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha}\text{-stable } \boldsymbol{\nu}\text{-parabolic connec-} \\ \text{tions of rank } r \text{ and degree } d \\ \text{over } (C, \boldsymbol{t}). \end{array} \right\} / \sim$$

- ◦ constructed by Arinkin-Lysenko and Inaba-Iwasaki-Saito.
- ◦ motivated by the search for the differential equation determined by the isomonodromic deformation (e.g. Painlevé equation) and the geometric Langlands correspondence.

Introduction (2)

Theorem (Loray-Saito, 2015)

In the case $C = \mathbb{P}^1$, assume $n, d, \boldsymbol{\nu}, \boldsymbol{\alpha}$ satisfy appropriate conditions. Then the rational map

$$\mathrm{App} \times \mathrm{Bun} : M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, 2, d) \cdots \to |\mathcal{O}_C(n-3)| \times \mathcal{P}^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(2, d)$$

is birational. In particular, $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, 2, d)$ is a rational variety.

Problem Is $\mathrm{App} \times \mathrm{Bun}$ birational for any rank and any genus?
In the case of fixing the determinant and the trace connection;

- ◦ $r = 2, g = 0$: Loray-Saito
- ◦ $r = 2, g = 1$: Fassarella-Loray-Muniz (arXiv: 2008.11767)
- ◦ $r = 2, g \geq 2$: M $\longleftarrow$ Today's talk
- ◦ $r \geq 3$: in progress

Moduli of parabolic connections with fixed determinant

- determinant map

$$\det : M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, r, d) \longrightarrow M_{(C,\boldsymbol{t})}(\mathrm{tr}\,(\boldsymbol{\nu}), 1, d)$$

$$(E, \nabla, l_*) \longmapsto (\det E, \mathrm{tr}\, \nabla)$$

- **moduli space of parabolic connections with fixed determinant**

$$M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, r, (L, \nabla_L))$$
$$:= \{(E, \nabla, l_*) \in M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, r, d) \mid (\det E, \mathrm{tr}\, \nabla) \simeq (L, \nabla_L)\}$$

- ◦ (Inaba, 2013)
When
$$g = 0, r \geq 2, rn - 2(r+1) > 0 \text{ or } g = 1, n \geq 2 \text{ or } g \geq 2, \geq 1,$$
$M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, r, (L, \nabla_L))$ is an irreducible smooth quasi-projective variety of dimension
$$2r^2(g-1) + rn(n-1) + 2 - 2g.$$

Apparent map

- An apparent singularity is a singular point of a differential equation with rational coefficients such that there is a basis of regular solutions in a neighborhood of the point.

a differential equation	$\longrightarrow$	the set of apparent singularities
a connection	$\longrightarrow$	the apparent singularity divisor

- (Saito-Szabó)
  Suppose $d = r(g-1)+1$. Then we define the apparent map

$$\mathrm{App} : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d) \cdots \to S^N(C).$$

  - fixed determinant case

$$\mathrm{App} : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, 2, (L, \nabla_L)) \cdots \to |L \otimes \Omega^1_C(D)|.$$

Birational structure of moduli spaces (1)

- $\mathcal{P}^{\boldsymbol{\alpha}}(L)$: moduli space of rank 2 $\boldsymbol{\alpha}$-semistable parabolic bundles with determinant L over $(C, \boldsymbol{t})$
- $\mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) = \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, 2, (L, \nabla_L))$
- Bundle map

$$\begin{aligned} \mathrm{Bun} : \mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) &\cdots \to \mathcal{P}^{\boldsymbol{\alpha}}(L) \\ (E, \nabla, l_*) &\longmapsto (E, l_*) \end{aligned}$$

Theorem (M)

Assume $g \geq 1$, $d = 2g-1$, $\sum_{i=1}^n \nu_0^{(i)} \neq 0$ and $\sum_{i=1}^n (\alpha_2^{(i)} - \alpha_1^{(i)}) < 1$. Then the rational map

$$\mathrm{App} \times \mathrm{Bun} : \mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \cdots \to |L \otimes \Omega^1_C(D)| \times \mathcal{P}^{\boldsymbol{\alpha}}(L)$$

is birational. In particular, $\mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ is a rational variety.

Birational structure of moduli spaces (2)

- $V_0 \subset \mathcal{P}^{\boldsymbol{\alpha}}(L)$: the distinguished open subset
  - There is an open immersion

$$V_0 \hookrightarrow \mathbb{P}\,\mathrm{Ext}^1((L, D), (\mathcal{O}_C, \emptyset)) \simeq |L \otimes \Omega^1_C(D)|^*.$$

- Σ: incidence variety

$$\Sigma \subset |L \otimes \Omega^1_C(D)| \times |L \otimes \Omega^1_C(D)|^*$$

- $\mathcal{M}^0 \subset \mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$: a nonempty open subset

$$\mathcal{M}^0 := \{(E, \nabla, l_*) \in \mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \mid (E, l_*) \in V_0\}$$

Tha map

$$\mathrm{App} \times \mathrm{Bun} : \mathcal{M}^0 \longrightarrow (|L \otimes \Omega^1_C(D)| \times V_0) \setminus \Sigma$$

is an isomorphism.

Birational structure of moduli spaces (3)

Proposition

Assume $g \geq 1$, $d = 2g-1$, $\sum_{i=1}^n \nu_0^{(i)} = 0$ and $\sum_{i=1}^n (\alpha_2^{(i)} - \alpha_1^{(i)}) < 1$. Then $\mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ is birational to $T^*\mathcal{P}^{\boldsymbol{\alpha}}(L)$. In particular, $\mathcal{M}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ is a rational variety.

$$T^*_{(E,l_*)}\mathcal{P}^{\boldsymbol{\alpha}}(L) \simeq \left\{ \begin{array}{l} \text{parabolic Higgs fields over} \\ (E, l_*) \text{ such that tr } = 0 \end{array} \right\}$$

$$\mathrm{Bun}^{-1}((E, l_*)) = \nabla_0 + \left\{ \begin{array}{l} \text{parabolic Higgs fields over} \\ (E, l_*) \text{ such that tr } = 0 \end{array} \right\}$$

$$\begin{array}{ccc} \mathcal{M}^0 & \xrightarrow{\sim} & T^*V_0 \\ & {\scriptstyle \mathrm{Bun}} \searrow \quad \swarrow {\scriptstyle \text{projection}} & \\ & V_0 & \end{array} \qquad \begin{array}{ccc} \mathcal{M}^0 & \xrightarrow{\mathrm{App} \times \mathrm{Bun}} & \Sigma \\ \downarrow \wr & & \downarrow \wr \\ T^*V_0 & \xrightarrow{\text{projectivization}} & \mathbb{P}T^*V_0 \end{array}$$