The degree of the irrationality of Fano complete intersections.

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Introduction

Purpose

Obtain a lower bound of **the degree of the irrationality** of Fano complete intersection$/\mathbb{C}$.

k: a field, X: a variety$/k$

Definition 1 (Bastianelli. et. al.[1])

An invariant $\mathrm{irr}_k(X)$ is defined as the minimal degree of dominant generically finite rational maps from X to the projective space.

Definition 2

An invariant $\mathrm{m}_k(X)$ is defined as the minimal degree of dominant generically finite rational maps from X to a ruled variety.

Remark 0.1

$\mathrm{irr}_k(X) \geq \mathrm{m}_k(X)$

Previous work

Previous work

	Hypersurfaces	Complete Intersections
Rationality	Kollár('95) [4]	Braune('17) [2]
Lower bound of $\mathrm{irr}_k(-)$	Chen-Stapleton('20) [3]	???

Theorem 3 (Y.)

p: a prime number. $e, n \in \mathbb{Z}_{>0}$ s.t. $e \leq n-1, e \leq \frac{1}{2}(n+3)$, $(e,n) \notin \mathscr{E}_p$.
$d_1, d_2, \ldots, d_e \in \mathbb{Z}_{>0}$,

$$I := \left\lfloor \frac{\sum_{i=1}^{e}(p+1)\left\lfloor \frac{d_i}{p} \right\rfloor - (n+e-2)}{2} \right\rfloor$$

Then $\mathrm{irr}_{\mathbb{C}}(X_{d_1,d_2,\ldots,d_e}) \geq \min\{I, p\}$, for X: a very general complete intersection.

Sketch of proof of main theorem

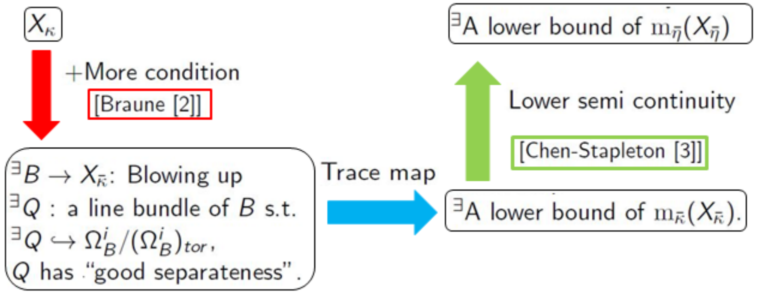
Setting 1.1

R: a DVR of mixed characteristic,
η: gen. pt. κ: cl. pt. of $\mathrm{Spec}(R)$, $\mathrm{char}(\kappa) = p$.
$d_1, \ldots, d_e \in \mathbb{Z}_{>0}$.
$X := \mathrm{V}(\{Y_j^p - f_j(\mathbb{X})\}, \{pY_j - g_j(\mathbb{X})\}) \subset \mathbb{P}_R^{n+2e}(d_1, d_2, \ldots, d_e, 1, 1, \ldots, 1)$
where $f_j, g_j \in R[\mathbb{X}]^h, \deg(f_j) = pd_j, \deg(g_j) = d_j$.

Remark 1.2

X_η : a $(pd_1, \ldots, pd_e)$-complete intersection in $\mathbb{P}_\eta^{n+e}$.
X_κ: a p-cyclic cover of a $(d_1, \ldots, d_e)$-complete intersection in $\mathbb{P}_\kappa^{n+e}$.

Sketch of proof of main theorem



Separateness

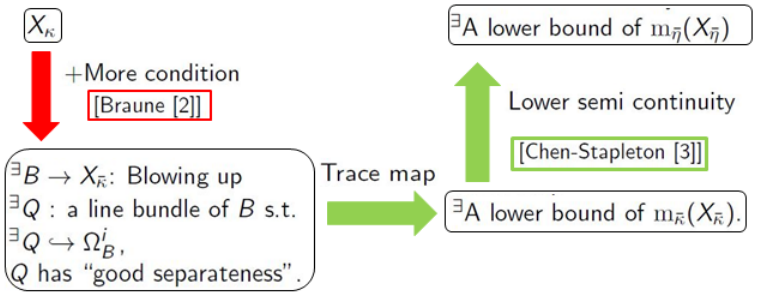
Definition 4 (Chen-Stapleton [3]2.1, "separeteness")

$k = \bar{k}$, X: a variety $/k$, L: a line bundle of X.
We say that L separates almost all l points of X
$:\Leftrightarrow$
$\emptyset \neq \exists U$:open subset of X
s.t. $\forall x_1, \ldots, x_l \in U(k)$,
$\exists s \in \Gamma(X, L)$ s.t. s vanishes at $x_1, \ldots, x_{l-1}$ but not at x_l.

Example 5

$X = \mathbb{P}^n_k, L = \mathcal{O}(d)$,
L separates almost all $d+1$ points.

Sketch of proof of main theorem



How to obtain a lower bound of $\mathrm{m}_k(X)$

Proposition 1.3 (Chen-Stapleton[3] 2.3)

$f: X \to Y \times \mathbb{P}^1$: *finite étale*, Y: *smooth*$/k$.
$^\exists L \hookrightarrow \Omega^i_X$, *$L$ separates $2l$ points. Then* $\deg(f) > l$.

Proof.

Assume $d := \deg(f) \leq l$. $y_1, y_2 \in Y \times \mathbb{P}^1$: in a same fiber of $Y \times \mathbb{P}^1 \to Y$
s.t. $f^{-1}(\{y_1, y_2\})$ are distinct $2d$ points.
By the separateness of L,
$^\exists \alpha \in \Gamma(X, \Omega^i_X)$ s.t. $\mathrm{Tr}^i_f(\alpha)$ vanishes at y_1 but not at y_2.
However, $\Gamma(Y \times \mathbb{P}^1, \Omega^i_{Y \times \mathbb{P}^1}) \cong \Gamma(Y, \Omega^i_Y)$, Contradiction. □

$f: X \to Y$, $f_*(\Omega^i_X) \to \Omega^i_Y$

Proposition 1.4 (de Jong-Starr[6] 3.3)

k: a field, X: normal integral variety/k. Y: smooth variety/k.
$f: X \to Y$: a dominant, gen. fin., separated, proper morphism over k.
Then $^{\exists}$ a morphism ψ: $f_*(\Omega^i_{X/k}) \to \Omega^i_{Y/k}$ s.t.

$$\begin{array}{ccc} f_*(\Omega^i_{K(X)/k}) & \xrightarrow{\varphi} & \Omega^i_{K(Y)/k} \\ \uparrow & & \uparrow \\ f_*(\Omega^i_{X/k}) & \xrightarrow{\psi} & \Omega^i_{Y/k} \end{array}$$

the following diagram where φ is the trace map of the function fields is commutative.

Corollary 6 (Y.)

k: a field, X: normal integral variety/k. Y: smooth variety/k.
$f: X \to Y$: a dominant, gen. fin., separated, proper morphism over k.
Then $^{\exists}$ a morphism ψ: $f_*(\Omega^i_{X/k}/(\Omega^i_{X/k})_{tor}) \to \Omega^i_{Y/k}$ s.t.

$$\begin{array}{ccc} f_*(\Omega^i_{K(X)/k}) & \xrightarrow{\varphi} & \Omega^i_{K(Y)/k} \\ \uparrow & & \uparrow \\ f_*(\Omega^i_{X/k}/(\Omega^i_{X/k})_{tor}) & \xrightarrow{\psi} & \Omega^i_{Y/k} \end{array}$$

the following diagram where φ is the trace map of the function fields is commutative.

Sketch of proof of the corollary

Proof.

There exists a $\mathscr{O}_Y$-hom $f_*(\Omega^i_{X/k}/(\Omega^i_{X/k})_{tor}) \to \Omega^i_{K(Y)/k}$.
ETS: For any $y \in Y$ s.t. $\overline{\{y\}}$ is a divisor, the image of this map at y is in $\Omega^i_{Y/k,y}$.
Fix such $y \in Y$. There exists an affine nbd. V of y s.t. $f|_{f^{-1}(V)}$ is finite.
By [Garel, Theorem, [5]], there exists following the diagram of $\mathscr{O}_V$-mod.

$$\begin{array}{ccc} f_*(\Omega^i_{K(X)/k}) & \longrightarrow & \Omega^i_{K(Y)/k} \\ \uparrow & & \uparrow \\ f_*(\Omega^i_{f^{-1}(V)/k}) & \longrightarrow & \Omega^i_{V/k} \end{array}$$

□

Sketch of proof of the corollary

Proof.

There exists a $\mathscr{O}_Y$-hom $f_*(\Omega^i_{X/k}/(\Omega^i_{X/k})_{tor}) \to \Omega^i_{K(Y)/k}$.
ETS: For any $y \in Y$ s.t. $\overline{\{y\}}$ is a divisor, the image of this map at y is in $\Omega^i_{Y/k,y}$.
Fix such $y \in Y$. There exists an affine nbd. V of y s.t. $f|_{f^{-1}(V)}$ is finite.
By [[5] Theorem], there exists following the diagram of $\mathscr{O}_V$-mod.

$$\begin{array}{ccc} f_*(\Omega^i_{K(X)/k}) & \longrightarrow & \Omega^i_{K(Y)/k} \\ \uparrow & & \uparrow \\ f_*(\Omega^i_{f^{-1}(V)/k}/(\Omega^i_{f^{-1}(V)/k})_{tor}) & \searrow & \\ \uparrow & & \\ f_*(\Omega^i_{f^{-1}(V)/k}) & \longrightarrow & \Omega^i_{V/k} \end{array}$$

- F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld & B. Ullery, *Measures of irrationality for hypersurfaces of large degree*, Compos. Math. 153, 2017, 2368 2393.
- L. Braune, *Irrational Complete Intersection*, arXiv:1909.05723v1, 2019.
- N. Chen & D.Stapleton, *Fano hypersurfaces with arbitrarily large degrees of irrationality*, Forum of Mathematics, Sigma, 2020.
- J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete 32, Springer Verlag, Berlin, 1996.
- E. Garel, *An extension of the trace map*, Journal of Pure and Applied Algebra, Volume 32, Issue 3, (1984), 301-313,
- A. J. de Jong & J. M. Starr, *Cubic fourfolds and spaces of rational curves*, Illinois J. Math. 52 (2008), no. 1, 345 346

Thank you for listening.