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# Weighted Triangle-free 2-matching Problem with Edge-disjoint Forbidden Triangles

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**Abstract** The weighted $\mathcal{T}$-free 2-matching problem is the following problem: given an undirected graph $G$, a weight function on its edge set, and a set $\mathcal{T}$ of triangles in $G$, find a maximum weight 2-matching containing no triangle in $\mathcal{T}$. When $\mathcal{T}$ is the set of all triangles in $G$, this problem is known as the weighted triangle-free 2-matching problem, which is a long-standing open problem. A main contribution of this paper is to give the first polynomial-time algorithm for the weighted $\mathcal{T}$-free 2-matching problem under the assumption that $\mathcal{T}$ is a set of edge-disjoint triangles. In our algorithm, a key ingredient is to give an extended formulation representing the solution set, that is, we introduce new variables and represent the convex hull of the feasible solutions as a projection of another polytope in a higher dimensional space. Although our extended formulation has exponentially many inequalities, we show that the separation problem can be solved in polynomial time, which leads to a polynomial-time algorithm for the weighted $\mathcal{T}$-free 2-matching problem.

**Keywords** Restricted 2-matching · Polynomial-time algorithm · Triangle-free · Extended formulation

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## 1 Introduction

### 1.1 2-matchings without Short Cycles

In an undirected graph, an edge set $M$ is said to be a 2-*matching*[1] if each vertex is incident to at most two edges in $M$. Finding a 2-matching of maximum size is a classical combinatorial optimization problem, which can be solved efficiently by using a matching algorithm. By imposing restrictions on 2-matchings, various extensions have been introduced and studied in the literature. Among them, the problem of finding a maximum 2-matching without short cycles has attracted attention, because it has applications to approximation algorithms for the traveling salesman problem and its variants. We say that a 2-matching $M$ is $C_{\leq k}$*-free* if $M$ contains no cycle of length $k$ or less, and the $C_{\leq k}$*-free* 2-*matching problem* is to find a $C_{\leq k}$-free 2-matching of maximum size in a given graph. When $k \leq 2$, every 2-matching without self-loops and parallel edges is $C_{\leq k}$-free, and hence the $C_{\leq k}$-free 2-matching problem can be solved in polynomial time. On the other hand, when $n/2 \leq k \leq n-1$, where $n$ is the number of vertices in the input graph, the $C_{\leq k}$-free 2-matching problem is NP-hard, because it decides the existence of a Hamiltonian cycle. These facts motivate us to investigate the borderline between polynomially solvable cases and NP-hard cases of the problem. Hartvigsen [12] gave a polynomial-time algorithm for the $C_{\leq 3}$-free 2-matching problem, and Papadimitriou showed that the problem is NP-hard when $k \geq 5$ (see [6]). The polynomial solvability of the $C_{\leq 4}$-free 2-matching problem is still open, whereas some positive results are known for special cases. For the case when the input graph is restricted to be bipartite, Hartvigsen [13], Király [18], and Frank [10] gave min-max theorems, Hartvigsen [14] and Pap [27] designed polynomial-time algorithms, Babenko [1] improved the running time, and Takazawa [29] showed decomposition theorems. Recently, Takazawa [31,30] extended these results to a generalized problem. When the input graph is restricted to be subcubic, i.e., the maximum degree is at most three, Bérczi and Végh [4] gave a polynomial-time algorithm for the $C_{\leq 4}$-free 2-matching problem. Nam [25] gave a polynomial-time algorithm for the square-free case (i.e., only cycles of length four are forbidden) under the assumption that the squares in a graph are vertex-disjoint. The relationship between $C_{\leq k}$-free 2-matchings and jump systems is studied in [3,8,22].

There are a lot of studies also on the weighted version of the $C_{\leq k}$-free 2-matching problem. In the weighted problem, an input consists of a graph and a weight function on the edge set, and the objective is to find a $C_{\leq k}$-free 2-matching of maximum total weight. Király proved that the weighted $C_{\leq 4}$-free 2-matching problem is NP-hard even if the input graph is restricted to be bipartite (see [10]), and a stronger NP-hardness result was shown in [3]. Under the assumption that the weight function satisfies a certain property called *vertex-induced on every square*, Makai [24] gave a polyhedral description

[1] Although such an edge set is often called a *simple 2-matching* in the literature, we call it a 2-*matching* to simplify the description.

and Takazawa [28] designed a combinatorial polynomial-time algorithm for the weighted $C_{\leq 4}$-free 2-matching problem in bipartite graphs. The case of $k = 3$, which we call the *weighted triangle-free* 2*-matching problem*, is a long-standing open problem. For the weighted triangle-free 2-matching problem in subcubic graphs, Hartvigsen and Li [15] gave a polyhedral description and a polynomial-time algorithm, followed by a slightly generalized polyhedral description by Bérczi [2] and another polynomial-time algorithm by Kobayashi [19]. The relationship between $C_{\leq k}$-free 2-matchings and discrete convexity is studied in [19, 20, 22].

## 1.2 Our Results

The previous papers on the weighted triangle-free 2-matching problem [2, 15, 19] deal with a generalized problem in which we are given a set $\mathcal{T}$ of forbidden triangles as an input in addition to a graph and a weight function. The objective is to find a maximum weight 2-matching that contains no triangle in $\mathcal{T}$, which we call the *weighted* $\mathcal{T}$*-free* 2*-matching problem*. In this paper, we focus on the case when $\mathcal{T}$ is a set of edge-disjoint triangles, i.e., no pair of triangles in $\mathcal{T}$ shares an edge. A main contribution of this paper is to give the first polynomial-time algorithm for the weighted $\mathcal{T}$-free 2-matching problem under the assumption that $\mathcal{T}$ is a set of edge-disjoint triangles. Note that we impose an assumption only on $\mathcal{T}$, and no restriction is required for the input graph. We now describe the formal statement of our result.

Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$, which might have self-loops and parallel edges. For a vertex set $X \subseteq V$, let $\delta_G(X)$ denote the set of edges between $X$ and $V \setminus X$. For $v \in V$, $\delta_G(\{v\})$ is simply denoted by $\delta_G(v)$. For $v \in V$, let $\dot{\delta}_G(v)$ denote the multiset of edges incident to $v \in V$, that is, a self-loop incident to $v$ is counted twice. We omit the subscript $G$ if no confusion may arise. For $b \in \mathbf{Z}^V_{\geq 0}$, an edge set $M \subseteq E$ is said to be a *b-matching* (resp. *b-factor*) if $|M \cap \dot{\delta}(v)| \leq b(v)$ (resp. $|M \cap \dot{\delta}(v)| = b(v)$) for every $v \in V$. If $b(v) = 2$ for every $v \in V$, a $b$-matching and a $b$-factor are called a 2*-matching* and a 2*-factor*, respectively. Let $\mathcal{T}$ be a set of triangles in $G$, where a *triangle* is a cycle of length three. For a triangle $T$, let $V(T)$ and $E(T)$ denote the vertex set and the edge set of $T$, respectively. An edge set $M \subseteq E$ is said to be $\mathcal{T}$*-free* if $E(T) \not\subseteq M$ for every $T \in \mathcal{T}$. For a vertex set $S \subseteq V$, let $E[S]$ denote the set of all edges with both endpoints in $S$. For an edge weight vector $w \in \mathbf{R}^E$, we consider the problem of finding a $\mathcal{T}$-free $b$-matching (resp. $b$-factor) maximizing $w(M)$, which we call the *weighted* $\mathcal{T}$*-free b-matching (resp. b-factor) problem.* Note that, for a set $A$ and a vector $c \in \mathbf{R}^A$, we denote $c(A) = \sum_{a \in A} c(a)$.

Our main result is formally stated as follows.

**Theorem 1** *There exists a polynomial-time algorithm for the following problem: given a graph $G = (V, E)$, $b(v) \in \mathbf{Z}_{\geq 0}$ for each $v \in V$, a set $\mathcal{T}$ of edge-disjoint triangles, and a weight $w(e) \in \mathbf{R}$ for each $e \in E$, find a $\mathcal{T}$-free b-factor $M \subseteq E$ that maximizes the total weight $w(M)$.*

A proof of this theorem is given in Section 5. Since finding a maximum weight $\mathcal{T}$-free $b$-matching can be reduced to finding a maximum weight $\mathcal{T}$-free $b$-factor, Theorem 1 implies the following corollary.

**Corollary 1** *There exists a polynomial-time algorithm for the following problem: given a graph $G = (V, E)$, $b(v) \in \mathbf{Z}_{\geq 0}$ for each $v \in V$, a set $\mathcal{T}$ of edge-disjoint triangles, and a weight $w(e) \in \mathbf{R}$ for each $e \in E$, find a $\mathcal{T}$-free $b$-matching $M \subseteq E$ that maximizes the total weight $w(M)$.*

*Proof* Suppose we are given an instance of the maximum weight $\mathcal{T}$-free $b$-matching problem. We construct a new graph by adding a new vertex $r$ with $b(r) = b(V)$, $\lfloor \frac{b(r)}{2} \rfloor$ self-loops incident to $r$, and $b(v)$ parallel edges connecting $r$ and $v$ for each $v \in V$. The weight of each new edge is defined as 0. Then, finding a maximum weight $\mathcal{T}$-free $b$-matching in the original graph is equivalent to finding a maximum weight $\mathcal{T}$-free $b$-factor in the obtained graph. Therefore, Corollary 1 follows from Theorem 1. $\square$

In particular, we can find a $\mathcal{T}$-free 2-matching (or 2-factor) $M \subseteq E$ that maximizes the total weight $w(M)$ in polynomial time if $\mathcal{T}$ is a set of edge-disjoint triangles.

### 1.3 Key Ingredient: Extended Formulation

A natural strategy to solve the maximum weight $\mathcal{T}$-free $b$-factor problem is to give a polyhedral description of the $\mathcal{T}$-free $b$-factor polytope as Hartvigsen and Li [15] did for the subcubic case. However, as we will see in Example 1, giving a system of inequalities that represents the $\mathcal{T}$-free $b$-factor polytope seems to be quite difficult even when $\mathcal{T}$ is a set of edge-disjoint triangles. A key idea of this paper is to give an extended formulation of the $\mathcal{T}$-free $b$-factor polytope, that is, we introduce new variables and represent the $\mathcal{T}$-free $b$-factor polytope as a projection of another polytope in a higher dimensional space.

Extended formulations of polytopes arising from various combinatorial optimization problems have been intensively studied in the literature, and the main focus in this area is on the number of inequalities that are required to represent the polytope. If a polytope has an extended formulation with polynomially many inequalities, then we can optimize a linear function in the original polytope by the ellipsoid method (see e.g. [11]). On the other hand, even if a linear function on a polytope can be optimized in polynomial time, the polytope does not necessarily have an extended formulation of polynomial size. In this context, the existence of a polynomial size extended formulation has attracted attention. See survey papers [5, 17] for previous work on extended formulations.

In this paper, under the assumption that $\mathcal{T}$ is a set of edge-disjoint triangles, we give an extended formulation of the $\mathcal{T}$-free $b$-factor polytope that has exponentially many inequalities (Theorem 2). In addition, we show that the separation problem for the extended formulation is solvable in polynomial time, and hence we can optimize a linear function on the $\mathcal{T}$-free $b$-factor polytope by the ellipsoid method in polynomial time. This yields the first polynomial-time algorithm for the weighted $\mathcal{T}$-free $b$-factor (or $b$-matching) problem, where $\mathcal{T}$ is a set of edge-disjoint triangles. Note that it is rare that the first polynomial-time algorithm was designed with the aid of an extended formulation. To the best of our knowledge, the *weighted linear matroid parity problem* was the only such problem before this paper (see [16]).

### 1.4 Organization of the Paper

The rest of this paper is organized as follows. In Section 2, we introduce an extended formulation of the $\mathcal{T}$-free $b$-factor polytope, whose correctness proof is given in Section 4. In Section 3, we show a few technical lemmas that will be used in the proof. In Section 5, we give a polynomial-time algorithm for the weighted $\mathcal{T}$-factor problem and prove Theorem 1. Finally, we conclude this paper with remarks in Section 6. Some of the proofs are given in the appendix.

## 2 Extended Formulation of the $\mathcal{T}$-free $b$-factor Polytope

Let $G = (V, E)$ be a graph, $b \in \mathbf{Z}^V_{\geq 0}$ be a vector, and $\mathcal{T}$ be a set of forbidden triangles. Throughout this paper, we only consider the case when triangles in $\mathcal{T}$ are mutually edge-disjoint.

For an edge set $M \subseteq E$, define its characteristic vector $x_M \in \mathbf{R}^E$ by $x_M(e) = 1$ if $e \in M$ and $x_M(e) = 0$ otherwise. The *$\mathcal{T}$-free $b$-factor polytope* is defined as $\mathrm{conv}\{x_M \mid M \text{ is a } \mathcal{T}\text{-free } b\text{-factor in } G\}$, where conv denotes the convex hull of vectors, and the *$b$-factor polytope* is defined similarly. Edmonds [9] shows that the $b$-factor polytope is determined by the following inequalities.

$$x(\dot{\delta}(v)) = b(v) \qquad (v \in V) \tag{1}$$

$$0 \leq x(e) \leq 1 \qquad (e \in E) \tag{2}$$

$$\sum_{e \in F_0} x(e) + \sum_{e \in F_1} (1 - x(e)) \geq 1 \qquad ((S, F_0, F_1) \in \mathcal{F}) \tag{3}$$

Here, $\mathcal{F}$ is the set of all triples $(S, F_0, F_1)$ such that $S \subseteq V$, $(F_0, F_1)$ is a partition of $\delta(S)$, and $b(S) + |F_1|$ is odd. Note that $x(\dot{\delta}(v)) = \sum_{e \in \dot{\delta}(v)} x(e)$ and $x(e)$ is added twice if $e$ is a self-loop incident to $v$.

In order to deal with $\mathcal{T}$-free $b$-factors, we consider the following constraint in addition to (1)–(3).

$$x(E(T)) \leq 2 \qquad (T \in \mathcal{T}) \tag{4}$$

However, as we will see in Example 1, the system of inequalities (1)–(4) does not represent the $\mathcal{T}$-free $b$-factor polytope. In contrast, when we consider *uncapacitated* 2-factors, i.e., when we are allowed to use two copies of the same

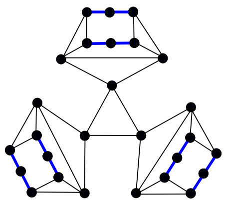

**Fig. 1** Graph $G = (V, E)$

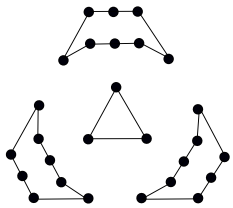

**Fig. 2** $b$-factor $M_1$

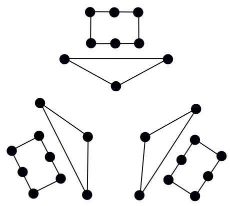

**Fig. 3** $b$-factor $M_2$

edge, we can obtain a polyhedral description by using (4). More precisely, it is shown by Cornuejols and Pulleyblank [7] that the $\mathcal{T}$-free uncapacitated 2-factor polytope is represented by $x(e) \geq 0$ for $e \in E$, $x(\dot{\delta}(v)) = 2$ for $v \in V$, and (4).

*Example 1* Consider the graph $G = (V, E)$ in Figure 1. Let $b(v) = 2$ for every $v \in V$ and $\mathcal{T}$ be the set of all triangles in $G$. Then, $G$ has no $\mathcal{T}$-free $b$-factor, i.e., the $\mathcal{T}$-free $b$-factor polytope is empty. For $e \in E$, let $x(e) = 1$ if $e$ is incident to a vertex of degree 2 (i.e., if $e$ is a blue line in Figure 1) and let $x(e) = \frac{1}{2}$ otherwise. Then, we can easily check that $x$ satisfies (1), (2), and (4). Furthermore, since $x$ is represented as a linear combination of two $b$-factors $M_1$ and $M_2$ shown in Figures 2 and 3, $x$ satisfies (3).

In what follows in this section, we introduce new variables and give an extended formulation of the $\mathcal{T}$-free $b$-factor polytope. For $T \in \mathcal{T}$, we denote $\mathcal{E}_T = \{J \subseteq E(T) \mid J \neq E(T)\}$. For $T \in \mathcal{T}$ and $J \in \mathcal{E}_T$, we introduce a new variable $y(T, J)$. Roughly, $y(T, J)$ denotes the (fractional) amount of $b$-factors $M$ satisfying $M \cap E(T) = J$. In particular, when $x$ and $y$ are integral, $y(T, J) = 1$ if and only if the $b$-factor $M$ corresponding to $(x, y)$ satisfies $M \cap E(T) = J$. We consider the following inequalities.

$$\sum_{J \in \mathcal{E}_T} y(T, J) = 1 \qquad (T \in \mathcal{T}) \tag{5}$$

$$\sum_{e \in J \in \mathcal{E}_T} y(T, J) = x(e) \qquad (T \in \mathcal{T},\ e \in E(T)) \tag{6}$$

$$y(T, J) \geq 0 \qquad (T \in \mathcal{T},\ J \in \mathcal{E}_T) \tag{7}$$

If $T$ is clear, $y(T, J)$ is simply denoted by $y(J)$. Since triangles in $\mathcal{T}$ are edge-disjoint, this causes no ambiguity unless $J = \emptyset$. In addition, for $\alpha, \beta \in E(T)$, $y(\{\alpha\})$, $y(\{\alpha, \beta\})$, and $y(\emptyset)$ are simply denoted by $y_\alpha$, $y_{\alpha\beta}$, and $y_\emptyset$, respectively.

We now strengthen (3) by using $y$. For $(S, F_0, F_1) \in \mathcal{F}$, let $\mathcal{T}_S = \{T \in \mathcal{T} \mid E(T) \cap \delta(S) \neq \emptyset\}$. For $T \in \mathcal{T}_S$ with $E(T) = \{\alpha, \beta, \gamma\}$ and $E(T) \cap \delta(S) = \{\alpha, \beta\}$, we define

$$q^*(T) = \begin{cases} y_\alpha + y_{\alpha\gamma} & \text{if } \alpha \in F_0 \text{ and } \beta \in F_1, \\ y_\beta + y_{\beta\gamma} & \text{if } \beta \in F_0 \text{ and } \alpha \in F_1, \\ y_\emptyset + y_\gamma & \text{if } \alpha, \beta \in F_1, \\ y_{\alpha\beta} & \text{if } \alpha, \beta \in F_0. \end{cases}$$

Note that this value depends on $(S, F_0, F_1) \in \mathcal{F}$ and $y$, but it is simply denoted by $q^*(T)$ for a notational convenience. We consider the following inequality.

$$\sum_{e \in F_0} x(e) + \sum_{e \in F_1} (1 - x(e)) - \sum_{T \in \mathcal{T}_S} 2q^*(T) \geq 1 \qquad ((S, F_0, F_1) \in \mathcal{F}) \tag{8}$$

Let $P$ be the polytope defined by

$$P = \{(x, y) \in \mathbf{R}^E \times \mathbf{R}^Y \mid x \text{ and } y \text{ satisfy (1), (2), and (4)–(8)}\},$$

where $Y = \{(T, F) \mid T \in \mathcal{T},\ F \in \mathcal{E}_T\}$. Note that we do not need (3), because it is implied by (8). Define the projection of $P$ onto $E$ as

$$\text{proj}_E(P) = \{x \in \mathbf{R}^E \mid \text{There exists } y \in \mathbf{R}^Y \text{ such that } (x, y) \in P\}.$$

Our aim is to show that $\text{proj}_E(P)$ is equal to the $\mathcal{T}$-free $b$-factor polytope. It is not difficult to see that the $\mathcal{T}$-free $b$-factor polytope is contained in $\text{proj}_E(P)$.

**Lemma 1** *The $\mathcal{T}$-free $b$-factor polytope is contained in* $\text{proj}_E(P)$.

*Proof* Suppose that $M \subseteq E$ is a $\mathcal{T}$-free $b$-factor in $G$ and let $x_M \in \mathbf{R}^E$ be its characteristic vector. For $T \in \mathcal{T}$ and $J \in \mathcal{E}_T$, define $y_M(T, J) = 1$ if $M \cap E(T) = J$ and $y_M(T, J) = 0$ otherwise. We can easily see that $(x_M, y_M)$ satisfies (1), (2), and (4)–(7). Thus, it suffices to show that $(x_M, y_M)$ satisfies (8). Fix $(S, F_0, F_1) \in \mathcal{F}$. For $T \in \mathcal{T}_S$, let $c(T)$ be the the contribution of $E(T)$ and $T$ to the left-hand side of (8), i.e., $c(T) := \sum_{e \in F_0 \cap E(T)} x_M(e) + \sum_{e \in F_1 \cap E(T)} (1 - x_M(e)) - 2q^*(T)$. Since the triangles in $\mathcal{T}_S$ are edge-disjoint, the left-hand side of (8) is equal to

$$\sum_{e \in F_0 \setminus \bigcup_{T \in \mathcal{T}_S} E(T)} x_M(e) + \sum_{e \in F_1 \setminus \bigcup_{T \in \mathcal{T}_S} E(T)} (1 - x_M(e)) + \sum_{T \in \mathcal{T}_S} c(T). \tag{9}$$

For each $T \in \mathcal{T}_S$, we can see that $c(T) = 0$ if $|M \cap E(T) \cap \delta(S)| \equiv |F_1 \cap E(T)| \pmod 2$ and $c(T) = 1$ otherwise by the following observations, where we denote $E(T) = \{\alpha, \beta, \gamma\}$ and $E(T) \cap \delta(S) = \{\alpha, \beta\}$.

- If $\alpha \in F_0$ and $\beta \in F_1$, then (5) and (6) show that $x(\alpha) = y_\alpha + y_{\alpha\beta} + y_{\alpha\gamma}$ and $1 - x(\beta) = 1 - (y_\beta + y_{\alpha\beta} + y_{\beta\gamma}) = y_\emptyset + y_\alpha + y_\gamma + y_{\alpha\gamma}$. Therefore, $c(T) = x(\alpha) + (1 - x(\beta)) - 2q^*(T) = y_\emptyset + y_\gamma + y_{\alpha\beta}$, which is 0 if and only if $|M \cap \{\alpha, \beta\}|$ is odd.
- If $\beta \in F_0$ and $\alpha \in F_1$, then $c(T) = (1-x(\alpha))+x(\beta)-2q^*(T) = y_\emptyset+y_\gamma+y_{\alpha\beta}$, which is 0 if and only if $|M \cap \{\alpha, \beta\}|$ is odd.
- If $\alpha, \beta \in F_1$, then $c(T) = (1 - x(\alpha)) + (1 - x(\beta)) - 2q^*(T) = y_\alpha + y_\beta + y_{\alpha\gamma} + y_{\beta\gamma}$, which is 0 if and only if $|M \cap \{\alpha, \beta\}|$ is even.
- If $\alpha, \beta \in F_0$, then $c(T) = x(\alpha) + x(\beta) - 2q^*(T) = y_\alpha + y_\beta + y_{\alpha\gamma} + y_{\beta\gamma}$, which is 0 if and only if $|M \cap \{\alpha, \beta\}|$ is even.

Assume that (8) is violated for $(S, F_0, F_1)$. Then, since (9) is less than 1, we have that $x_M(e) = 0$ for every $e \in F_0 \setminus \bigcup_{T \in \mathcal{T}_S} E(T)$, $x_M(e) = 1$ for every $e \in F_1 \setminus \bigcup_{T \in \mathcal{T}_S} E(T)$, and $c(T) = 0$ for every $T \in \mathcal{T}_S$, because $x_M(e) \in \{0,1\}$ and $c(T) \in \{0,1\}$. Therefore,

$$\begin{aligned} |M \cap \delta(S)| &= |(M \cap \delta(S)) \setminus \bigcup_{T \in \mathcal{T}_S} E(T)| + \sum_{T \in \mathcal{T}_S} |M \cap E(T) \cap \delta(S)| \\ &\equiv |F_1 \setminus \bigcup_{T \in \mathcal{T}_S} E(T)| + \sum_{T \in \mathcal{T}_S} |F_1 \cap E(T)| = |F_1| \pmod 2. \end{aligned}$$

Since $M$ is a $b$-factor, it holds that $|M \cap \delta(S)| \equiv b(S) \pmod 2$, which contradicts that $b(S) + |F_1|$ is odd. □

To prove the opposite inclusion (i.e., $\mathrm{proj}_E(P)$ is contained in the $\mathcal{T}$-free $b$-factor polytope), we consider a relaxation of (8). For $T \in \mathcal{T}_S$ with $E(T) = \{\alpha, \beta, \gamma\}$ and $E(T) \cap \delta(S) = \{\alpha, \beta\}$, we define

$$q(T) = \begin{cases} y_\alpha + y_{\alpha\gamma} & \text{if } \alpha \in F_0 \text{ and } \beta \in F_1, \\ y_\beta + y_{\beta\gamma} & \text{if } \beta \in F_0 \text{ and } \alpha \in F_1, \\ y_\gamma & \text{if } \alpha, \beta \in F_1, \\ 0 & \text{if } \alpha, \beta \in F_0. \end{cases}$$

Since $q(T) \le q^*(T)$ for $T \in \mathcal{T}_S$, the following inequality is a relaxation of (8).

$$\sum_{e \in F_0} x(e) + \sum_{e \in F_1} (1 - x(e)) - \sum_{T \in \mathcal{T}_S} 2q(T) \ge 1 \qquad ((S, F_0, F_1) \in \mathcal{F}) \qquad (10)$$

Define a polytope $Q$ and its projection onto $E$ as

$$Q = \{(x, y) \in \mathbf{R}^E \times \mathbf{R}^Y \mid x \text{ and } y \text{ satisfy (1), (2), (4)–(7), and (10)}\},$$
$$\mathrm{proj}_E(Q) = \{x \in \mathbf{R}^E \mid \text{There exists } y \in \mathbf{R}^Y \text{ such that } (x, y) \in Q\}.$$

Since (10) is implied by (8), we have that $P \subseteq Q$ and $\mathrm{proj}_E(P) \subseteq \mathrm{proj}_E(Q)$. In what follows in Sections 3 and 4, we show the following proposition.

**Proposition 1** $\mathrm{proj}_E(Q)$ *is contained in the* $\mathcal{T}$*-free b-factor polytope.*

By Lemma 1, Proposition 1, and $\mathrm{proj}_E(P) \subseteq \mathrm{proj}_E(Q)$, we obtain the following theorem.

**Theorem 2** *Let* $G = (V, E)$ *be a graph,* $b(v) \in \mathbf{Z}_{\ge 0}$ *for each* $v \in V$*, and let* $\mathcal{T}$ *be a set of edge-disjoint triangles. Then, both* $\mathrm{proj}_E(P)$ *and* $\mathrm{proj}_E(Q)$ *are equal to the* $\mathcal{T}$*-free b-factor polytope.*

We remark here that we do not know how to prove directly that $\mathrm{proj}_E(P)$ is contained in the $\mathcal{T}$-free $b$-factor polytope. Introducing $\mathrm{proj}_E(Q)$ and considering Proposition 1, which is a stronger statement, is a key idea in our proof. We also note that our algorithm in Section 5 is based on the fact that the $\mathcal{T}$-free $b$-factor polytope is equal to $\mathrm{proj}_E(P)$. In this sense, both $\mathrm{proj}_E(P)$ and $\mathrm{proj}_E(Q)$ play important roles in this paper.

*Example 2* Suppose that $G = (V, E)$, $b \in \mathbf{Z}^V_{\geq 0}$, and $x \in \mathbf{R}^E$ are as in Example 1. Let $T$ be the central triangle in $G$ and let $E(T) = \{\alpha, \beta, \gamma\}$. If $y \in \mathbf{R}^Y$ satisfies (5) and (7), then $y_{\alpha\beta} + y_{\beta\gamma} + y_{\alpha\gamma} \leq 1$. Thus, without loss of generality, we may assume that $y_{\alpha\beta} \leq \frac{1}{3}$ by symmetry. Let $S$ be a vertex set with $\delta(S) = \{\alpha, \beta\}$. Then, (10) does not hold for $(S, \{\alpha\}, \{\beta\}) \in \mathcal{F}$, because $x(\alpha) + (1 - x(\beta)) - 2q(T) = 1 - x(\alpha) - x(\beta) + 2y_{\alpha\beta} \leq \frac{2}{3} < 1$. Therefore, $x$ is not in $\mathrm{proj}_E(Q)$.

## 3 Extreme Points of the Projection of $Q$

In this section, we show a property of extreme points of $\mathrm{proj}_E(Q)$, which will be used in Section 4. We begin with the following easy lemma.

**Lemma 2** *Suppose that $x \in \mathbf{R}^E$ satisfies (2) and (4). Then, there exists $y \in \mathbf{R}^Y$ that satisfies (5)–(7).*

*Proof* Let $T \in \mathcal{T}$ be a triangle with $E(T) = \{\alpha, \beta, \gamma\}$ and $x(\alpha) \geq x(\beta) \geq x(\gamma)$. For $J \in \mathcal{E}_T$, we define $y(T, J)$ as follows.

- If $x(\alpha) \geq x(\beta) + x(\gamma)$, then $y_{\alpha\beta} = x(\beta)$, $y_{\alpha\gamma} = x(\gamma)$, $y_\emptyset = 1 - x(\alpha)$, $y_\alpha = x(\alpha) - x(\beta) - x(\gamma)$, and $y_\beta = y_\gamma = y_{\beta\gamma} = 0$.
- If $x(\alpha) < x(\beta) + x(\gamma)$, then $y_{\alpha\beta} = \frac{1}{2}(x(\alpha) + x(\beta) - x(\gamma))$, $y_{\alpha\gamma} = \frac{1}{2}(x(\alpha) + x(\gamma) - x(\beta))$, $y_{\beta\gamma} = \frac{1}{2}(x(\beta) + x(\gamma) - x(\alpha))$, $y_\emptyset = 1 - \frac{1}{2}(x(\alpha) + x(\beta) + x(\gamma))$, and $y_\alpha = y_\beta = y_\gamma = 0$.

Then, $y$ satisfies (5)–(7). □

By using this lemma, we show the following.

**Lemma 3** *For an extreme point $x$ of $\mathrm{proj}_E(Q)$, one of the following holds.*

*(i) $x = x_M$ for some $\mathcal{T}$-free $b$-factor $M \subseteq E$.*
*(ii) (4) is tight for some $T \in \mathcal{T}$.*
*(iii) There exists a vector $y \in \mathbf{R}^Y$ with $(x, y) \in Q$ such that (10) is tight for some $(S, F_0, F_1) \in \mathcal{F}$ with $\mathcal{T}_S^+ \neq \emptyset$, where we define $\mathcal{T}_S^+ = \{T \in \mathcal{T} \mid E(T) \cap F_1 \neq \emptyset\}$.*

*Proof* We prove (i) by assuming that (ii) and (iii) do not hold. Since for any $y \in \mathbf{R}^Y$ with $(x, y) \in Q$, (10) is not tight for any $(S, F_0, F_1) \in \mathcal{F}$ with $\mathcal{T}_S^+ \neq \emptyset$, $x$ is an extreme point of

$$\{x \in \mathbf{R}^E \mid \text{There exists } y \in \mathbf{R}^Y \text{ such that } (x, y) \text{ satisfies (1)–(7)}\},$$

because (3) is a special case of (10) in which $\mathcal{T}_S^+ = \emptyset$. By Lemma 2, this polytope is equal to $\{x \in \mathbf{R}^E \mid x \text{ satisfies (1)–(4)}\}$. Since (4) is not tight for any $T \in \mathcal{T}$, $x$ is an extreme point of $\{x \in \mathbf{R}^E \mid x \text{ satisfies (1)–(3)}\}$, which is the $b$-factor polytope. Thus, $x$ is a characteristic vector of a $b$-factor. Since $x$ satisfies (4), it holds that $x = x_M$ for some $\mathcal{T}$-free $b$-factor $M \subseteq E$. □

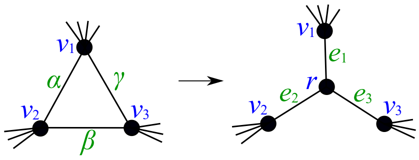


**Fig. 4** Construction of $G'$

## 4 Proof of Proposition 1

In this section, we prove Proposition 1 by induction on $|\mathcal{T}|$. If $|\mathcal{T}| = 0$, then $Y = \emptyset$ and (10) is equivalent to (3). Thus, $\mathrm{proj}_E(Q)$ is the $b$-factor polytope, which shows the base case of the induction.

Fix an instance $(G, b, \mathcal{T})$ with $|\mathcal{T}| \geq 1$ and assume that Proposition 1 holds for instances with smaller $|\mathcal{T}|$. Suppose that $Q \neq \emptyset$, which implies that $b(V)$ is even as $(V, \emptyset, \emptyset) \notin \mathcal{F}$ by (10). Let $x$ be an extreme point of $\mathrm{proj}_E(Q)$ and let $y \in \mathbf{R}^Y$ be a vector with $(x, y) \in Q$. Our aim is to show that $x$ is contained in the $\mathcal{T}$-free $b$-factor polytope.

In what follows in this section, we prove this as follows. We apply Lemma 3 to obtain one of (i), (ii), and (iii). If (i) holds, that is, $x = x_M$ for some $\mathcal{T}$-free $b$-factor $M \subseteq E$, then $x$ is obviously in the $\mathcal{T}$-free $b$-factor polytope. If (ii) holds, that is, (4) is tight for some $T \in \mathcal{T}$, then we replace $T$ with a certain graph and apply the induction hypothesis, which will be discussed in Section 4.1. If (iii) holds, that is, (10) is tight for some $(S, F_0, F_1) \in \mathcal{F}$ with $\mathcal{T}_S^+ \neq \emptyset$, then we divide $G$ into two graphs and apply the induction hypothesis for each graph, which will be discussed in Section 4.2.

### 4.1 When (4) is Tight

In this subsection, we consider the case when (4) is tight for some $T \in \mathcal{T}$. Fix a triangle $T \in \mathcal{T}$ with $x(E(T)) = 2$, where we denote $V(T) = \{v_1, v_2, v_3\}$, $E(T) = \{\alpha, \beta, \gamma\}$, $\alpha = v_1v_2$, $\beta = v_2v_3$, and $\gamma = v_3v_1$ (Figure 4). Since (4) is tight, we obtain $2 = x(\alpha) + x(\beta) + x(\gamma) = 2 - (y_\alpha + y_\beta + y_\gamma) - 2y_\emptyset$, and hence $y_\alpha = y_\beta = y_\gamma = y_\emptyset = 0$. Therefore, $x(\alpha) = y_{\alpha\beta} + y_{\alpha\gamma}$, $x(\beta) = y_{\alpha\beta} + y_{\beta\gamma}$, $x(\gamma) = y_{\alpha\gamma} + y_{\beta\gamma}$, and $y_{\alpha\beta} + y_{\alpha\gamma} + y_{\beta\gamma} = 1$.

We construct a new instance of the $\mathcal{T}$-free $b$-factor problem as follows. Let $G' = (V', E')$ be the graph obtained from $G = (V, E)$ by removing $E(T)$ and adding a new vertex $r$ together with three new edges $e_1 = rv_1$, $e_2 = rv_2$, and $e_3 = rv_3$ as in Figure 4. Define $b' \in \mathbf{Z}_{\geq 0}^{V'}$ as $b'(r) = 1$, $b'(v) = b(v) - 1$ for $v \in \{v_1, v_2, v_3\}$, and $b'(v) = b(v)$ for $v \in V \setminus \{v_1, v_2, v_3\}$. Define $x' \in R^{E'}$ as $x'(e_1) = y_{\alpha\gamma}$, $x'(e_2) = y_{\alpha\beta}$, $x'(e_3) = y_{\beta\gamma}$, and $x'(e) = x(e)$ for $e \in E' \cap E$. Let $\mathcal{T}' = \mathcal{T} \setminus \{T\}$, and let $Y'$ and $\mathcal{F}'$ be the objects for the obtained instance $(G', b', \mathcal{T}')$ that are defined in the same way as $Y$ and $\mathcal{F}$. Define $y'$ as the restriction of $y$ to $Y'$. We now show the following lemma.

**Lemma 4** *$(x', y')$ satisfies (1), (2), (4)–(7), and (10) with respect to the new instance $(G', b', \mathcal{T}')$.*

*Proof* We can easily see that $(x', y')$ satisfies (1), (2), (4)–(7). Consider (10) for $(S', F'_0, F'_1) \in \mathcal{F}'$. By changing the roles of $S'$ and $V' \setminus S'$ if necessary, we may assume that $r \in S'$. For $(S', F'_0, F'_1) \in \mathcal{F}'$ (resp. $(S, F_0, F_1) \in \mathcal{F}$), we denote the left-hand side of (10) by $h'(S', F'_0, F'_1)$ (resp. $h(S, F_0, F_1)$).

Then, we obtain $h'(S', F'_0, F'_1) \geq 1$ for each $(S', F'_0, F'_1) \in \mathcal{F}'$ by the following case analysis and by the symmetry of $v_1$, $v_2$, and $v_3$.

1. Suppose that $v_1, v_2, v_3 \in S'$. Since $(S' \setminus \{r\}, F'_0, F'_1) \in \mathcal{F}$, we obtain $h'(S', F'_0, F'_1) = h(S' \setminus \{r\}, F'_0, F'_1) \geq 1$.
2. Suppose that $v_2, v_3 \in S'$ and $v_1 \notin S'$.
   - If $e_1 \in F'_0$, then define $(S, F_0, F_1) \in \mathcal{F}$ as $S = S' \setminus \{r\}$, $F_0 = (F'_0 \setminus \{e_1\}) \cup \{\alpha\}$, and $F_1 = F'_1 \cup \{\gamma\}$. Since $x(\alpha) + (1 - x(\gamma)) - 2q(T) = y_{\alpha\gamma} = x'(e_1)$, we obtain $h'(S', F'_0, F'_1) = h(S, F_0, F_1) \geq 1$.
   - If $e_1 \in F'_1$, then define $(S, F_0, F_1) \in \mathcal{F}$ as $S = S' \setminus \{r\}$, $F_0 = F'_0$, and $F_1 = (F'_1 \setminus \{e_1\}) \cup \{\alpha, \gamma\}$. Since $(1 - x(\alpha)) + (1 - x(\gamma)) - 2q(T) = y_{\beta\gamma} + y_{\alpha\beta} = 1 - x'(e_1)$, we obtain $h'(S', F'_0, F'_1) = h(S, F_0, F_1) \geq 1$.
3. Suppose that $v_1 \in S'$ and $v_2, v_3 \notin S'$.
   - If $e_2, e_3 \in F'_0$, then define $(S, F_0, F_1) \in \mathcal{F}$ as $S = S' \setminus \{r\}$, $F_0 = F'_0 \setminus \{e_2, e_3\}$, and $F_1 = F'_1 \cup \{\alpha, \gamma\}$. Since $(1 - x(\alpha)) + (1 - x(\gamma)) - 2q(T) = y_{\beta\gamma} + y_{\alpha\beta} = x'(e_2) + x'(e_3)$, we obtain $h'(S', F'_0, F'_1) = h(S, F_0, F_1) \geq 1$.
   - If $e_2 \in F'_0$ and $e_3 \in F'_1$, then define $(S, F_0, F_1) \in \mathcal{F}$ as $S = S' \setminus \{r\}$, $F_0 = (F'_0 \setminus \{e_2\}) \cup \{\alpha\}$, and $F_1 = (F'_1 \setminus \{e_3\}) \cup \{\gamma\}$. Since $x(\alpha) + (1 - x(\gamma)) - 2q(T) = y_{\alpha\gamma} \leq 1 - x'(e_3)$, we obtain $h'(S', F'_0, F'_1) \geq h(S, F_0, F_1) \geq 1$.
   - If $e_2, e_3 \in F'_1$, then $h'(S', F'_0, F'_1) \geq 2 - x'(e_2) - x'(e_3) \geq 1$.
4. Suppose that $v_1, v_2, v_3 \notin S'$.
   - If $F'_1 \cap \delta_{G'}(r) = \emptyset$, then $h'(S', F'_0, F'_1) \geq x'(e_1) + x'(e_2) + x'(e_3) = 1$.
   - If $|F'_1 \cap \delta_{G'}(r)| \geq 2$, then $h'(S', F'_0, F'_1) \geq 2 - (x'(e_1) + x'(e_2) + x'(e_3)) = 1$.
   - If $|F'_1 \cap \delta_{G'}(r)| = 1$, then define $(S, F_0, F_1) \in \mathcal{F}$ as $S = S' \setminus \{r\}$, $F_0 = F'_0 \setminus \delta_{G'}(r)$, and $F_1 = F'_1 \setminus \delta_{G'}(r)$. Then, we obtain $h'(S', F'_0, F'_1) \geq h(S, F_0, F_1) \geq 1$. □

By this lemma and by the induction hypothesis, $x'$ is in the $\mathcal{T}'$-free $b'$-factor polytope. That is, there exist $\mathcal{T}'$-free $b'$-factors $M'_1, \ldots, M'_t$ in $G'$ and non-negative coefficients $\lambda_1, \ldots, \lambda_t$ such that $\sum_{i=1}^t \lambda_i = 1$ and $x' = \sum_{i=1}^t \lambda_i x_{M'_i}$, where $x_{M'_i} \in \mathbf{R}^{E'}$ is the characteristic vector of $M'_i$.

For a $\mathcal{T}'$-free $b'$-factor $M' \subseteq E'$ in $G'$, we define a corresponding $\mathcal{T}$-free $b$-factor $\varphi(M') \subseteq E$ in $G$ as $\varphi(M') = (M' \cap E) \cup \{\alpha, \gamma\}$ if $e_1 \in M'$, $\varphi(M') = (M' \cap E) \cup \{\alpha, \beta\}$ if $e_2 \in M'$, and $\varphi(M') = (M' \cap E) \cup \{\beta, \gamma\}$ if $e_3 \in M'$. By the choice of $\lambda_i$ and $M_i$, we obtain $x(e) = \sum_{i=1}^t \lambda_i x_{\varphi(M'_i)}(e)$ for each $e \in E \cap E'$. We can also see that $\sum_{i=1}^t \lambda_i x_{\varphi(M'_i)}(\alpha) = \sum\{\lambda_i \mid e_1 \in M'_i\} + \sum\{\lambda_i \mid e_2 \in M'_i\} = x'(e_1) + x'(e_2) = y_{\alpha\gamma} + y_{\alpha\beta} = x(\alpha)$, and similar equalities hold for $\beta$ and $\gamma$. Therefore, we obtain $x = \sum_{i=1}^t \lambda_i x_{\varphi(M'_i)}$, which shows that $x$ is in the $\mathcal{T}$-free $b$-factor polytope.

## 4.2 When (10) is Tight

In this subsection, we consider the case when (10) is tight for $(S^*, F_0^*, F_1^*) \in \mathcal{F}$ with $\mathcal{T}_{S^*}^+ \neq \emptyset$, where $\mathcal{T}_{S^*}^+ = \{T \in \mathcal{T} \mid E(T) \cap F_1^* \neq \emptyset\}$. In this case, we divide the original instance into two instances $(G_1, b_1, \mathcal{T}_1)$ and $(G_2, b_2, \mathcal{T}_2)$, apply the induction hypothesis for each instance, and combine the two parts. We denote $\tilde{F}_0^* = F_0^* \setminus \bigcup_{T \in \mathcal{T}_{S^*}^+} E(T)$ and $\tilde{F}_1^* = F_1^* \setminus \bigcup_{T \in \mathcal{T}_{S^*}^+} E(T)$.

### *4.2.1 Construction of* $(G_j, b_j, \mathcal{T}_j)$

We first construct $(G_1, b_1, \mathcal{T}_1)$ and its feasible LP solution $x_1$. Starting from the subgraph $G[S^*] = (S^*, E[S^*])$ induced by $S^*$, we add a new vertex $r$ corresponding to $V^* \setminus S^*$, set $b_1(r) = 1$, and apply the following procedure.

- For each $f = uv \in \tilde{F}_0^*$ with $u \in S^*$, we add a new edge $e^f = ur$ (Figure 5). Let $x_1(e^f) = x(f)$.
- For each $f = uv \in \tilde{F}_1^*$ with $u \in S^*$, we add a new vertex $p_u^f$ and new edges $e_u^f = up_u^f$ and $e_r^f = p_u^f r$ (Figure 6). Let $b_1(p_u^f) = 1$, $x_1(e_u^f) = x(f)$, and $x_1(e_r^f) = 1 - x(f)$.
- For each $T \in \mathcal{T}_{S^*}^+$ with $|E(T) \cap F_1^*| = 2$ and $|V(T) \cap S^*| = 2$, which we call *a triangle of type (A)*, add new vertices $p_1, p_2$ and new edges $e_1, \ldots, e_6$ as in Figure 7. Define $b_1(p_1) = b_1(p_2) = 1$ and

$$x_1(e_1) = y_\emptyset + y_\gamma, \quad x_1(e_2) = y_\emptyset + y_\alpha, \quad x_1(e_3) = y_{\alpha\beta},$$
$$x_1(e_4) = y_{\beta\gamma}, \quad x_1(e_5) = 1 - y_\emptyset - y_\gamma, \quad x_1(e_6) = 1 - y_\emptyset - y_\alpha,$$

  where $\alpha, \beta$, and $\gamma$ are as in Figure 7.
- For each $T \in \mathcal{T}_{S^*}^+$ with $|E(T) \cap F_1^*| = 2$ and $|V(T) \cap S^*| = 1$, which we call *a triangle of type (A')*, add a new vertex $p_3$ and new edges $e_1, e_2, e_3, e_4, e_7, e_8$, and $e_9$ as in Figure 8. Define $b_1(p_3) = 2$ and

$$x_1(e_1) = y_\emptyset + y_\gamma, \quad x_1(e_2) = y_\emptyset + y_\alpha, \quad x_1(e_3) = y_{\alpha\beta}, \quad x_1(e_4) = y_{\beta\gamma},$$
$$x_1(e_7) = y_\beta, \quad x_1(e_8) = y_{\alpha\gamma}, \quad x_1(e_9) = 1 - y_\emptyset - y_\beta,$$

  where $\alpha, \beta$, and $\gamma$ are as in Figure 8.
- For each $T \in \mathcal{T}_{S^*}^+$ with $|E(T) \cap F_1^*| = 1$ and $|V(T) \cap S^*| = 2$, which we call *a triangle of type (B)*, add new vertices $p_1, p_2, p_3$ and new edges $e_1, \ldots, e_9$ as in Figure 9. Define $b_1(p_i) = 1$ for $i \in \{1, 2, 3\}$, and

$$x_1(e_1) = y_\emptyset + y_\beta, \quad x_1(e_2) = y_{\alpha\gamma}, \quad x_1(e_3) = y_\gamma,$$
$$x_1(e_4) = y_\alpha, \quad x_1(e_5) = y_{\alpha\beta}, \quad x_1(e_6) = y_{\beta\gamma},$$
$$x_1(e_7) = y_\emptyset, \quad x_1(e_8) = 1 - y_\emptyset - y_\gamma, \quad x_1(e_9) = 1 - y_\emptyset - y_\alpha,$$

  where $\alpha, \beta$, and $\gamma$ are as in Figure 9.

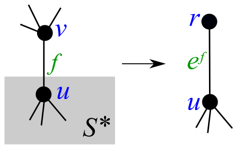


**Fig. 5** An edge in $\tilde{F}_0^*$

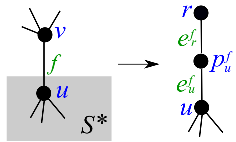


**Fig. 6** An edge in $\tilde{F}_1^*$

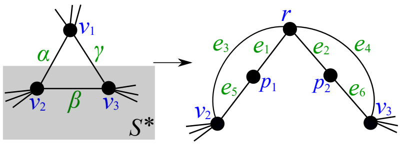


**Fig. 7** A triangle of type (A)

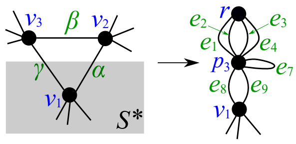


**Fig. 8** A triangle of type (A')

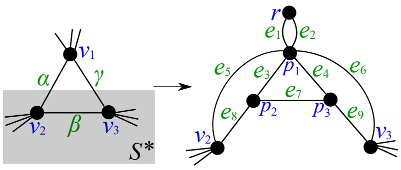


**Fig. 9** A triangle of type (B)

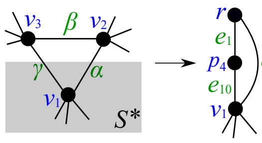


**Fig. 10** A triangle of type (B')

– For each $T \in \mathcal{T}_{S^*}^{+}$ with $|E(T) \cap F_1^*| = 1$ and $|V(T) \cap S^*| = 1$, which we call *a triangle of type (B')*, add a new vertex $p_4$ and new edges $e_1, e_2$, and $e_{10}$ as in Figure 10. Define $b_1(p_4) = 1$, and

$$x_1(e_1) = y_\emptyset + y_\beta, \qquad x_1(e_2) = y_{\alpha\gamma}, \qquad x_1(e_{10}) = 1 - y_\emptyset - y_\beta,$$

where $\alpha, \beta$, and $\gamma$ are as in Figure 10.

In order to make it clear that $p_i$ and $e_i$ are associated with $T \in \mathcal{T}_{S^*}^{+}$, we sometimes denote $p_i^T$ and $e_i^T$. Let $G_1 = (V_1, E_1)$ be the obtained graph. Define $b_1 \in \mathbf{Z}_{\geq 0}^{V_1}$ by $b_1(v) = b(v)$ for $v \in S^*$ and $b_1(v)$ is as above for $v \in V_1 \setminus S^*$. Define $x_1 \in \mathbf{R}^{E_1}$ by $x_1(e) = x(e)$ for $e \in E[S^*]$ and $x_1(e)$ is as above for $e \in E_1 \setminus E[S^*]$.

For each $T \in \mathcal{T}_{S^*} \setminus \mathcal{T}_{S^*}^{+}$ with $|V(T) \cap S^*| = 2$, say $V(T) \cap S^* = \{u, v\}$, let $\psi(T)$ be the corresponding triangle in $G_1$ whose vertex set is $\{u, v, r\}$. Let

$$\mathcal{T}_1 = \{T \in \mathcal{T} \mid V(T) \subseteq S^*\} \cup \{\psi(T) \mid T \in \mathcal{T}_{S^*} \setminus \mathcal{T}_{S^*}^{+} \text{ with } |V(T) \cap S^*| = 2\},$$

and let $Y_1$ and $\mathcal{F}_1$ be the objects for the obtained instance $(G_1, b_1, \mathcal{T}_1)$ that are defined in the same way as $Y$ and $\mathcal{F}$. Define $y_1$ as the restriction of $y$ to $Y_1$, where we identify $f \in F_0^*$ with $e^f$ and identify $T \in \mathcal{T}_{S^*} \setminus \mathcal{T}_{S^*}^{+}$ with $\psi(T)$.

Similarly, by changing the roles of $S^*$ and $V \setminus S^*$, we construct a graph $G_2 = (V_2, E_2)$ and an instance $(G_2, b_2, \mathcal{T}_2)$, where the new vertex corresponding to $S^*$ is denoted by $r'$. Define $x_2, y_2, Y_2$, and $\mathcal{F}_2$ in the same way as above.

Note that a triangle $T \in \mathcal{T}^+_{S^*}$ is of type (A) (resp. type (B)) for $(G_1, b_1, \mathcal{T}_1)$ if and only if it is of type (A') (resp. type (B')) for $(G_2, b_2, \mathcal{T}_2)$.

We use the following lemma, whose proof is given in Appendix A.

**Lemma 5** *For $j \in \{1,2\}$, $(x_j, y_j)$ satisfies (1), (2), (4)–(7), and (10) with respect to the new instance $(G_j, b_j, \mathcal{T}_j)$.*

*4.2.2 Pairing up $\mathcal{T}_j$-free $b_j$-factors*

Since $|\mathcal{T}_j| \le |\mathcal{T}| - |\mathcal{T}^+_{S^*}| < |\mathcal{T}|$ for $j \in \{1,2\}$, by Lemma 5 and by the induction hypothesis, $x_j$ is in the $\mathcal{T}_j$-free $b_j$-factor polytope. That is, there exists a set $\mathcal{M}_j$ of $\mathcal{T}_j$-free $b_j$-factors in $G_j$ and a non-negative coefficient $\lambda_M$ for each $M \in \mathcal{M}_j$ such that $\sum_{M \in \mathcal{M}_j} \lambda_M = 1$ and $x_j = \sum_{M \in \mathcal{M}_j} \lambda_M x_M$, where $x_M \in \mathbf{R}^{E_j}$ is the characteristic vector of $M$.

Let $j \in \{1,2\}$ and consider $(G_j, b_j, \mathcal{T}_j)$. Since $x_j(e^T_1) \ge x_j(e^T_7)$ for each triangle $T \in \mathcal{T}^+_{S^*}$ of type (B), by swapping parallel edges $e^T_1$ and $e^T_2$ if necessary, we may assume that $\{e^T_2, e^T_7\} \not\subseteq M$ for each $M \in \mathcal{M}_j$ and for each $T \in \mathcal{T}^+_{S^*}$ of type (B). In what follows, we construct a collection of $\mathcal{T}$-free $b$-factors in $G$ by combining $\mathcal{M}_1$ and $\mathcal{M}_2$.

Since there is a one-to-one correspondence between $\delta_{G_1}(r)$ and $\delta_{G_2}(r')$, we identify them and denote $E_0$, that is, $E_0 = E_1 \cap E_2 = \delta_{G_1}(r) = \delta_{G_2}(r')$. Note that $e^f_r \in E_1$ and $e^f_{r'} \in E_2$ are identified for each $f \in \tilde{F}^*_1$. Since $b_1(r) = b_2(r') = 1$, it holds that $|M_1 \cap E_0| = |M_2 \cap E_0| = 1$ for every $M_1 \in \mathcal{M}_1$ and for every $M_2 \in \mathcal{M}_2$. Define $\mathcal{M} = \{(M_1, M_2) \mid M_1 \in \mathcal{M}_1,\ M_2 \in \mathcal{M}_2,\ M_1 \cap E_0 = M_2 \cap E_0\}$. Since $x_1(e) = x_2(e)$ for $e \in E_0$ by the definitions of $x_1$ and $x_2$, we can pair up a $b_1$-factor $M_1$ in $\mathcal{M}_1$ and a $b_2$-factor $M_2$ in $\mathcal{M}_2$ so that $(M_1, M_2) \in \mathcal{M}$. More precisely, we can assign a non-negative coefficient $\lambda_{(M_1,M_2)}$ for each pair $(M_1, M_2) \in \mathcal{M}$ such that $\sum\{\lambda_{(M_1,M_2)} \mid (M_1, M_2) \in \mathcal{M}\} = 1$ and $\sum\{\lambda_{(M_1,M_2)} \mid (M_1, M_2) \in \mathcal{M},\ e' \in M_i\} = x_i(e')$ for $i \in \{1,2\}$ and for $e' \in E_i$.

Let $(M_1, M_2) \in \mathcal{M}$. For a triangle $T \in \mathcal{T}^+_{S^*}$ of type (A) or (A'), denote $M_T = (M_1 \cup M_2) \cap \{e^T_1, \dots, e^T_9\}$ and define $\varphi(M_1, M_2, T) \subseteq E(T)$ as

$$\varphi(M_1, M_2, T) = \begin{cases} \{\alpha\} & \text{if } M_T = \{e_2, e_5, e_8\} \text{ or } M_T = \{e_2, e_5, e_9\}, \\ \{\gamma\} & \text{if } M_T = \{e_1, e_6, e_8\} \text{ or } M_T = \{e_1, e_6, e_9\}, \\ \{\alpha, \beta\} & \text{if } M_T = \{e_3, e_5, e_6, e_8\} \text{ or } M_T = \{e_3, e_5, e_6, e_9\}, \\ \{\beta, \gamma\} & \text{if } M_T = \{e_4, e_5, e_6, e_8\} \text{ or } M_T = \{e_4, e_5, e_6, e_9\}, \\ \{\alpha, \gamma\} & \text{if } M_T = \{e_5, e_6, e_8, e_9\}, \\ \{\beta\} & \text{if } M_T = \{e_5, e_6, e_7\}, \end{cases}$$

where the superscript $T$ is omitted here. Note that $M_T$ satisfies one of the above conditions, because $M_j$ is a $b_j$-factor for $j \in \{1,2\}$.

For a triangle $T \in \mathcal{T}^+_{S^*}$ of type (B) or (B'), denote $M_T = (M_1 \cup M_2) \cap \{e^T_1, \dots, e^T_{10}\}$ and define $\varphi(M_1, M_2, T) \subseteq E(T)$ as

$$\varphi(M_1, M_2, T) = \begin{cases} \emptyset & \text{if } M_T = \{e_1, e_7\}, \\ \{\alpha\} & \text{if } M_T = \{e_4, e_8, e_{10}\} \text{ or } M_T = \{e_5, e_7, e_{10}\}, \\ \{\gamma\} & \text{if } M_T = \{e_3, e_9, e_{10}\} \text{ or } M_T = \{e_6, e_7, e_{10}\}, \\ \{\alpha, \beta\} & \text{if } M_T = \{e_5, e_8, e_9, e_{10}\}, \\ \{\beta, \gamma\} & \text{if } M_T = \{e_6, e_8, e_9, e_{10}\}, \\ \{\alpha, \gamma\} & \text{if } M_T = \{e_2, e_8, e_9, e_{10}\}, \\ \{\beta\} & \text{if } M_T = \{e_1, e_8, e_9\}, \end{cases}$$

where the superscript $T$ is omitted here again. Note that $M_T$ satisfies one of the above conditions, because we are assuming that $M_j$ is a $b_j$-factor with $\{e_2, e_7\} \not\subseteq M_j$ for $j \in \{1, 2\}$.

For $(M_1, M_2) \in \mathcal{M}$, define $M_1 \oplus M_2 \subseteq E$ as

$$\begin{aligned} M_1 \oplus M_2 = (M_1 \cap E[S^*]) \cup (M_2 \cap E[V \setminus S^*]) \cup \{f \in \tilde{F}^*_0 \mid e^f \in M_1 \cap M_2\} \\ \cup \{f \in \tilde{F}^*_1 \mid e^f_r \notin M_1 \cap M_2\} \cup \bigcup \{\varphi(M_1, M_2, T) \mid T \in \mathcal{T}^+_{S^*}\}. \end{aligned}$$

We now use the following lemma, whose proof is postponed to Appendix B.

**Lemma 6** *For any $(M_1, M_2) \in \mathcal{M}$, $M_1 \oplus M_2$ forms a $\mathcal{T}$-free $b$-factor. Furthermore, it holds that $x = \sum_{(M_1,M_2)\in\mathcal{M}} \lambda_{(M_1,M_2)} x_{M_1\oplus M_2}$, where $x_{M_1\oplus M_2} \in \mathbf{R}^E$ is the characteristic vector of $M_1 \oplus M_2$.*

By this lemma, it holds that $x$ is in the $\mathcal{T}$-free $b$-factor polytope. This completes the proof of Proposition 1.

## 5 Algorithm

In this section, we give a polynomial-time algorithm for the weighted $\mathcal{T}$-free $b$-factor problem and prove Theorem 1. Our algorithm is based on the ellipsoid method using the fact that the $\mathcal{T}$-free $b$-factor polytope is equal to $\text{proj}_E(P)$ (Theorem 2). In order to apply the ellipsoid method, we need a polynomial-time algorithm for the separation problem. That is, for $(x, y) \in \mathbf{R}^E \times \mathbf{R}^Y$, we need a polynomial-time algorithm that concludes $(x, y) \in P$ or returns a violated inequality.

Let $(x, y) \in \mathbf{R}^E \times \mathbf{R}^Y$. We can easily check whether $(x, y)$ satisfies (1), (2), and (4)–(7) or not in polynomial time. In order to solve the separation problem for (8), we use the following theorem, which implies that the separation problem for (3) can be solved in polynomial time.

**Theorem 3 (Padberg-Rao [26] (see also [23]))** *Suppose we are given a graph $G' = (V', E')$, $b' \in \mathbf{Z}^{V'}_{\geq 0}$, and $x' \in [0, 1]^{E'}$. Then, in polynomial time, we can compute $S' \subseteq V'$ and a partition $(F'_0, F'_1)$ of $\delta_{G'}(S')$ that minimize $\sum_{e\in F'_0} x'(e) + \sum_{e\in F'_1}(1 - x'(e))$ subject to $b'(S') + |F'_1|$ is odd.*

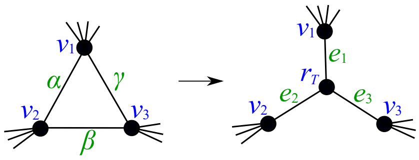


**Fig. 11** Replacement of a triangle $T \in \mathcal{T}$

In what follows, we reduce the separation problem for (8) to that for (3) and utilize Theorem 3. Suppose that $(x, y) \in \mathbf{R}^E \times \mathbf{R}^Y$ satisfies (1), (2), and (4)–(7). For each triangle $T \in \mathcal{T}$, we remove $E(T)$ and add a vertex $r_T$ together with three new edges $e_1 = r_T v_1$, $e_2 = r_T v_2$, and $e_3 = r_T v_3$ (Figure 11). Let $E'_T = \{e_1, e_2, e_3\}$ and define $x'(e_1) = x(\alpha) + x(\gamma) - 2y_{\alpha\gamma}$, $x'(e_2) = x(\alpha) + x(\beta) - 2y_{\alpha\beta}$, and $x'(e_3) = x(\beta) + x(\gamma) - 2y_{\beta\gamma}$. Let $G' = (V', E')$ be the graph obtained from $G$ by applying this procedure for every $T \in \mathcal{T}$. Define $b' \in \mathbf{Z}^{V'}_{\geq 0}$ as $b'(v) = b(v)$ for $v \in V$ and $b'(v) = 0$ for $v \in V' \setminus V$. By setting $x'(e) = x(e)$ for $e \in E' \cap E$ and by defining $x'(e)$ as above for $e \in E' \setminus E$, we obtain $x' \in [0, 1]^{E'}$. We now show the following lemma.

**Lemma 7** *Suppose that $(x, y) \in \mathbf{R}^E \times \mathbf{R}^Y$ satisfies (1), (2), and (4)–(7). Define $G' = (V', E')$, $b'$, and $x'$ as above. Then, $(x, y)$ violates (8) for some $(S, F_0, F_1) \in \mathcal{F}$ if and only if there exist $S' \subseteq V'$ and a partition $(F'_0, F'_1)$ of $\delta_{G'}(S')$ such that $b'(S') + |F'_1|$ is odd and $\sum_{e \in F'_0} x'(e) + \sum_{e \in F'_1} (1 - x'(e)) < 1$.*

*Proof* First, to show the "only if" part, assume that $(x, y)$ violates (8) for some $(S, F_0, F_1) \in \mathcal{F}$. Recall that $\mathcal{T}_S = \{T \in \mathcal{T} \mid E(T) \cap \delta_G(S) \neq \emptyset\}$. Define $S' \subseteq V'$ by $S' = S \cup \{r_T \mid T \in \mathcal{T},\ |V(T) \cap S| \geq 2\}$. Then, for each $T \in \mathcal{T}_S$, $E'_T \cap \delta_{G'}(S')$ consists of a single edge, which we denote by $e_T$. Define $F'_0$ and $F'_1$ as follows:

$$F'_0 = (F_0 \cap E') \cup \{e_T \mid T \in \mathcal{T}_S,\ |E(T) \cap F_1| = 0 \text{ or } 2\},$$
$$F'_1 = (F_1 \cap E') \cup \{e_T \mid T \in \mathcal{T}_S,\ |E(T) \cap F_1| = 1\}.$$

It is obvious that $(F'_0, F'_1)$ is a partition of $\delta_{G'}(S')$ and $b'(S') + |F'_1| \equiv b(S) + |F_1| \equiv 1 \pmod 2$.

To show that $\sum_{e \in F'_0} x'(e) + \sum_{e \in F'_1} (1 - x'(e)) < 1$, we evaluate $x'(e_T)$ or $1 - x'(e_T)$ for each $T \in \mathcal{T}_S$. Let $T \in \mathcal{T}_S$ be a triangle such that $E(T) = \{\alpha, \beta, \gamma\}$ and $E(T) \cap \delta_G(S) = \{\alpha, \beta\}$. Then, we obtain the following by the definition of $q^*(T)$.

- If $T \in \mathcal{T}_S$ and $\alpha, \beta \in F_0$, then $x(\alpha) + x(\beta) - 2q^*(T) = x'(e_T)$.
- If $T \in \mathcal{T}_S$ and $\alpha, \beta \in F_1$, then $(1 - x(\alpha)) + (1 - x(\beta)) - 2q^*(T) = x'(e_T)$.
- If $T \in \mathcal{T}_S$, $\alpha \in F_0$, and $\beta \in F_1$, then $x(\alpha) + (1 - x(\beta)) - 2q^*(T) = y_\emptyset + y_\gamma + y_{\alpha\beta} = 1 - x'(e_T)$.
- If $T \in \mathcal{T}_S$, $\beta \in F_0$, and $\alpha \in F_1$, then $(1 - x(\alpha)) + x(\beta) - 2q^*(T) = y_\emptyset + y_\gamma + y_{\alpha\beta} = 1 - x'(e_T)$.

With these observations, we obtain

$$\sum_{e \in F'_0} x'(e) + \sum_{e \in F'_1} (1 - x'(e)) = \sum_{e \in F_0} x(e) + \sum_{e \in F_1} (1 - x(e)) - \sum_{T \in \mathcal{T}_S} 2q^*(T) < 1,$$

which shows the "only if" part.

We next show the "if" part. For edge sets $F'_0, F'_1 \subseteq E'$, we denote $g(F'_0, F'_1) = \sum_{e \in F'_0} x'(e) + \sum_{e \in F'_1} (1 - x'(e))$ to simplify the notation. Let $(S', F'_0, F'_1)$ be a minimizer of $g(F'_0, F'_1)$ subject to $(F'_0, F'_1)$ is a partition of $\delta_{G'}(S')$ and $b'(S') + |F'_1|$ is odd. Among minimizers, we choose $(S', F'_0, F'_1)$ so that $F'_0 \cup F'_1$ is inclusion-wise minimal. To derive a contradiction, assume that $g(F'_0, F'_1) < 1$. We show the following claim.

*Claim* Let $T \in \mathcal{T}$ be a triangle as shown in Figure 11 and denote $\hat{F}_0 = F'_0 \cap E'_T$ and $\hat{F}_1 = F'_1 \cap E'_T$. Then, we obtain the following.

(i) If $v_1, v_2, v_3 \notin S'$, then $r_T \notin S'$.
(ii) If $v_1, v_2, v_3 \in S'$, then $r_T \in S'$.
(iii) If $v_1 \in S'$, $v_2, v_3 \notin S'$, and $|\hat{F}_1|$ is even, then $g(\hat{F}_0, \hat{F}_1) = x'(e_1) = x(\alpha) + x(\gamma) - 2y_{\alpha\gamma}$.
(iv) If $v_1 \in S'$, $v_2, v_3 \notin S'$, and $|\hat{F}_1|$ is odd, then $g(\hat{F}_0, \hat{F}_1) = 1 - x'(e_1) = y_\emptyset + y_\beta + y_{\alpha\gamma}$.

*Proof of the claim* (i) Assume that $v_1, v_2, v_3 \notin S'$ and $r_T \in S'$, which implies that $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_3\}$. Then, we derive a contradiction by the following case analysis and by the symmetry of $e_1, e_2$, and $e_3$.

- If $\hat{F}_0 = \{e_1, e_2\}$ and $\hat{F}_1 = \{e_3\}$, then $g(F'_0, F'_1) \geq g(\hat{F}_0, \hat{F}_1) = (x(\alpha)+x(\gamma)-2y_{\alpha\gamma})+(x(\alpha)+x(\beta)-2y_{\alpha\beta})+(1-x(\beta)-x(\gamma)+2y_{\beta\gamma}) = 1+2y_\alpha+2y_{\beta\gamma} \geq 1$, which is a contradiction.
- If $\hat{F}_0 = \emptyset$ and $\hat{F}_1 = \{e_1, e_2, e_3\}$, then $(F'_0, F'_1) \geq g(\hat{F}_0, \hat{F}_1) = (1 - x(\alpha) - x(\gamma) + 2y_{\alpha\gamma}) + (1 - x(\alpha) - x(\beta) + 2y_{\alpha\beta}) + (1 - x(\beta) - x(\gamma) + 2y_{\beta\gamma}) = 1 + 2(1 - x(\alpha) - x(\beta) - x(\gamma) + y_{\alpha\beta} + y_{\beta\gamma} + y_{\alpha\gamma}) = 1 + 2y_\emptyset \geq 1$, which is a contradiction.
- Suppose that $|\hat{F}_1|$ is even. Since $b'(S' \setminus \{r_T\}) + |F'_1 \setminus \delta_{G'}(r_T)|$ is odd and $g(F'_0 \setminus \delta_{G'}(r_T), F'_1 \setminus \delta_{G'}(r_T)) \leq g(F'_0, F'_1)$, $(S' \setminus \{r_T\}, F'_0 \setminus \delta_{G'}(r_T), F'_1 \setminus \delta_{G'}(r_T))$ is also a minimizer of $g$. This contradicts that a minimizer $(S', F'_0, F'_1)$ is chosen so that $F'_0 \cup F'_1$ is inclusion-wise minimal.

(ii) Assume that $v_1, v_2, v_3 \in S'$ and $r_T \notin S'$, which implies that $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_3\}$. Then, we derive a contradiction by the same argument as (i).

(iii) Suppose that $v_1 \in S'$, $v_2, v_3 \notin S'$, and $|\hat{F}_1|$ is even. Then, we have one of the following cases.

- If $\hat{F}_0 = \{e_1\}$ and $\hat{F}_1 = \emptyset$, then $g(\hat{F}_0, \hat{F}_1) = x'(e_1) = x(\alpha) + x(\gamma) - 2y_{\alpha\gamma}$.
- If $\hat{F}_0 = \{e_2, e_3\}$ and $\hat{F}_1 = \emptyset$, then $g(\hat{F}_0, \hat{F}_1) = (x(\alpha) + x(\beta) - 2y_{\alpha\beta}) + (x(\beta) + x(\gamma) - 2y_{\beta\gamma}) = x(\alpha) + x(\gamma) + 2(x(\beta) - y_{\alpha\beta} - y_{\beta\gamma}) \geq x(\alpha) + x(\gamma) \geq x(\alpha) + x(\gamma) - 2y_{\alpha\gamma}$.

– If $\hat{F}_0 = \emptyset$ and $\hat{F}_1 = \{e_2, e_3\}$, then $g(\hat{F}_0, \hat{F}_1) = (1 - x(\alpha) - x(\beta) + 2y_{\alpha\beta}) + (1 - x(\beta) - x(\gamma) + 2y_{\beta\gamma}) = x(\alpha) + x(\gamma) - 2y_{\alpha\gamma} + 2(1 - x(\alpha) - x(\beta) - x(\gamma) + y_{\alpha\beta} + y_{\beta\gamma} + y_{\alpha\gamma}) \geq x(\alpha) + x(\gamma) - 2y_{\alpha\gamma}$.

Since $(S', F'_0, F'_1)$ is a minimizer of $g(F'_0, F'_1)$, $g(\hat{F}_0, \hat{F}_1) = x'(e_1) = x(\alpha) + x(\gamma) - 2y_{\alpha\gamma}$.

(iv) Suppose that $v_1 \in S'$, $v_2, v_3 \notin S'$, and $|\hat{F}_1|$ is odd. Then, we have one of the following cases by changing the labels of $e_2$ and $e_3$ if necessary.

– If $\hat{F}_0 = \emptyset$ and $\hat{F}_1 = \{e_1\}$, then $g(\hat{F}_0, \hat{F}_1) = 1 - x'(e_1) = 1 - x(\alpha) - x(\gamma) + 2y_{\alpha\gamma}$.
– If $\hat{F}_0 = \{e_2\}$ and $\hat{F}_1 = \{e_3\}$, then $g(\hat{F}_0, \hat{F}_1) = (x(\alpha) + x(\beta) - 2y_{\alpha\beta}) + (1 - x(\beta) - x(\gamma) + 2y_{\beta\gamma}) \geq 1 - x(\alpha) - x(\gamma) + 2(x(\alpha) - y_{\alpha\beta}) \geq 1 - x(\alpha) - x(\gamma) + 2y_{\alpha\gamma}$.

Since $(S', F'_0, F'_1)$ is a minimizer of $g(F'_0, F'_1)$, $g(\hat{F}_0, \hat{F}_1) = 1 - x'(e_1) = 1 - x(\alpha) - x(\gamma) + 2y_{\alpha\gamma} = y_\emptyset + y_\beta + y_{\alpha\gamma}$.

(End of the proof of the claim)

Note that each $T \in \mathcal{T}$ satisfies exactly one of (i)–(iv) of this claim by changing the labels of $v_1$, $v_2$, and $v_3$ if necessary.

In what follows, we construct $(S, F_0, F_1) \in \mathcal{F}$ for which $(x, y)$ violates (8). We initialize $(S, F_0, F_1)$ as $S = S' \cap V$, $F_0 = F'_0 \cap E$, and $F_1 = F'_1 \cap E$, and apply the following procedures for each triangle $T \in \mathcal{T}$.

– If $T$ satisfies the condition (i) or (ii) of the claim, then we do nothing.
– If $T$ satisfies the condition (iii) of the claim, then we add $\alpha$ and $\gamma$ to $F_0$.
– If $T$ satisfies the condition (iv) of the claim, then we add $\alpha$ to $F_0$ and add $\gamma$ to $F_1$.

Then, we obtain that $(F_0, F_1)$ is a partition of $\delta_G(S)$, $b(S) + |F_1| \equiv b'(S') + |F'_1| \equiv 1 \pmod 2$, and

$$\sum_{e \in F_0} x(e) + \sum_{e \in F_1} (1 - x(e)) - \sum_{T \in \mathcal{T}_S} 2q^*(T) = \sum_{e \in F'_0} x'(e) + \sum_{e \in F'_1} (1 - x'(e)) < 1$$

by the claim. This shows that $(x, y)$ violates (8) for $(S, F_0, F_1) \in \mathcal{F}$, which completes the proof of "if" part. □

Since the proof of Lemma 7 is constructive, given $S' \subseteq V'$ and $F'_0, F'_1 \subseteq E'$ such that $(F'_0, F'_1)$ is a partition of $\delta_{G'}(S')$, $b'(S') + |F'_1|$ is odd, and $\sum_{e \in F'_0} x'(e) + \sum_{e \in F'_1} (1 - x'(e)) < 1$, we can construct $(S, F_0, F_1) \in \mathcal{F}$ for which $(x, y)$ violates (8) in polynomial time. By combining this with Theorem 3, it holds that the separation problem for $P$ can be solved in polynomial time. Therefore, the ellipsoid method can maximize a linear function on $P$ in polynomial time (see e.g. [11]), and hence we can maximize $\sum_{e \in E} w(e)x(e)$ subject to $x \in \mathrm{proj}_E(P)$. By perturbing the objective function if necessary, we can obtain a maximizer $x^*$ that is an extreme point of $\mathrm{proj}_E(P)$. Since each extreme point of $\mathrm{proj}_E(P)$ corresponds to a $\mathcal{T}$-free $b$-factor by Theorem 2, $x^*$ is a characteristic vector of a maximum weight $\mathcal{T}$-free $b$-factor. This completes the proof of Theorem 1. Note that the running time of our algorithm is very large, because it is based on the ellipsoid method.

## 6 Concluding Remarks

This paper gives the first polynomial-time algorithm for the weighted $\mathcal{T}$-free $b$-matching problem where $\mathcal{T}$ is a set of edge-disjoint triangles. A key ingredient is an extended formulation of the $\mathcal{T}$-free $b$-factor polytope with exponentially many inequalities. As we mentioned in Section 1.3, it is rare that the first polynomial-time algorithm was designed with the aid of an extended formulation. This approach has a potential to be used for other combinatorial optimization problems for which no polynomial-time algorithm is known.

Some interesting problems remain open. By projecting our extended formulation to the original space, we can obtain a polyhedral description of the $\mathcal{T}$-free $b$-factor polytope without using $y$, and so it is interesting to analyze the obtained polyhedral description. Another open problem is to design a combinatorial and faster polynomial-time algorithm, because the algorithm proposed in this paper relies on the ellipsoid method. It is also open whether our approach can be applied to the weighted $C_{\leq 4}$-free $b$-matching problem in general graphs under the assumption that the forbidden cycles are edge-disjoint and the weight is vertex-induced on every square. In addition, the weighted $C_{\leq 3}$-free 2-matching problem and the $C_{\leq 4}$-free 2-matching problem are big open problems in this area.

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## A Proof of Lemma 5

By symmetry, it suffices to consider $(G_1, b_1, \mathcal{T}_1)$. Since the tightness of (10) for $(S^*, F_0^*, F_1^*)$ implies that $x_1(\delta_{G_1}(r)) = 1$, we can easily see that $(x_1, y_1)$ satisfies (1), (2), (4)–(7). In what follows, we consider (10) for $(x_1, y_1)$ in $(G_1, b_1, \mathcal{T}_1)$. For edge sets $F_0', F_1' \subseteq E_1$, we denote $g(F_0', F_1') = \sum_{e \in F_0'} x_1(e) + \sum_{e \in F_1'} (1 - x_1(e))$ to simplify the notation. For $(S', F_0', F_1') \in \mathcal{F}_1$, let $h(S', F_0', F_1')$ denote the left-hand side of (10). To derive a contradiction, let $(S', F_0', F_1') \in \mathcal{F}_1$ be a minimizer of $h(S', F_0', F_1')$ and assume that $h(S', F_0', F_1') < 1$. By changing the roles of $S'$ and $V' \setminus S'$ if necessary, we may assume that $r \notin S'$.

For $T \in \mathcal{T}_{S^*}^+$, let $v_1, v_2, v_3, \alpha, \beta$, and $\gamma$ be as in Figures 7–10. Let $G_T' = (V_T', E_T')$ be the subgraph of $G_1$ corresponding to $T$, that is, the subgraph induced by $\{r, p_1, p_2, v_2, v_3\}$ (Figure 7), $\{r, p_3, v_1\}$ (Figure 8), $\{r, p_1, p_2, p_3, v_2, v_3\}$ (Figure 9), or $\{r, p_4, v_1\}$ (Figure 10). Let $\hat{S} = S' \cap (V_T' \setminus \{v_1, v_2, v_3\})$, $\hat{F}_0 = F_0' \cap E_T'$, and $\hat{F}_1 = F_1' \cap E_T'$.

We show the following properties (P1)–(P9) in Section A.1, and show that $(x_1, y_1)$ satisfies (10) by using these properties in Section A.2.

(P1) If $T$ is of type (A) or (B) and $v_2, v_3 \notin S'$, then $b_1(\hat{S}) + |\hat{F}_1|$ is even.
(P2) If $T$ is of type (A), $v_2, v_3 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even, then $g(\hat{F}_0, \hat{F}_1) \geq \min\{x(\alpha) + x(\gamma), 2 - x(\alpha) - x(\gamma) - 2y_\beta\}$.
(P3) If $T$ is of type (B), $v_2, v_3 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even, then $g(\hat{F}_0, \hat{F}_1) \geq y_\alpha + y_\gamma + y_{\alpha\beta} + y_{\beta\gamma}$.
(P4) If $T$ is of type (A) or (B), $v_2, v_3 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd, then $g(\hat{F}_0, \hat{F}_1) \geq y_\emptyset + y_\beta + y_{\alpha\gamma}$.
(P5) If $T$ is of type (A) or (B), $v_2 \in S'$, $v_3 \notin S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even, then $g(\hat{F}_0, \hat{F}_1) \geq \min\{x(\alpha) + x(\beta), 2 - x(\alpha) - x(\beta) - 2y_\gamma\}$.
(P6) If $T$ is of type (A) or (B), $v_2 \in S'$, $v_3 \notin S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd, then $g(\hat{F}_0, \hat{F}_1) \geq y_\emptyset + y_\gamma + y_{\alpha\beta}$.
(P7) If $T$ is of type (A') or type (B') and $v_1 \notin S'$, then $b_1(\hat{S}) + |\hat{F}_1|$ is even.
(P8) If $T$ is of type (A') or type (B'), $v_1 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even, then $g(\hat{F}_0, \hat{F}_1) = \min\{x(\alpha) + x(\gamma), 2 - x(\alpha) - x(\gamma) - 2y_\beta\}$.
(P9) If $T$ is of type (A') or type (B'), $v_1 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd, then $g(\hat{F}_0, \hat{F}_1) = y_\emptyset + y_\beta + y_{\alpha\gamma}$.

Note that each $T \in \mathcal{T}_{S^*}^+$ satisfies exactly one of (P1)–(P9) by changing the labels of $v_2$ and $v_3$ if necessary.

## A.1 Proofs of (P1)–(P9)

### *A.1.1 When T is of type (A)*

We first consider the case when $T$ is of type (A).

**Proof of (P1).** Suppose that $T$ is of type (A) and $v_2, v_3 \notin S'$. If $b_1(\hat{S}) + |\hat{F}_1|$ is odd, then either $p_1 \in \hat{S}$ and $|\hat{F}_1 \cap \delta_{G_1}(p_1)|$ is even or $p_2 \in \hat{S}$ and $|\hat{F}_1 \cap \delta_{G_1}(p_2)|$ is even. In the former case, $h(S', F_0', F_1') \geq \min\{x_1(e_1) + x_1(e_5), 2 - x_1(e_1) - x_1(e_5)\} = 1$, which is a contradiction. The same argument can be applied to the latter case. Therefore, $b_1(\hat{S}) + |\hat{F}_1|$ is even.

**Proof of (P2).** Suppose that $T$ is of type (A), $v_2, v_3 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even. If $p_1 \notin S'$, then we define $(S'', F_0'', F_1'') \in \mathcal{F}_1$ as $(S'', F_0'', F_1'') = (S' \cup \{p_1\}, F_0' \setminus \{e_5\}, F_1' \cup \{e_1\})$ if $e_5 \in F_0'$ and $(S'', F_0'', F_1'') = (S' \cup \{p_1\}, F_0' \cup \{e_1\}, F_1' \setminus \{e_5\})$ if $e_5 \in F_1'$. Since $h(S'', F_0'', F_1'') = h(S', F_0', F_1')$ holds, by replacing $(S', F_0', F_1')$ with $(S'', F_0'', F_1'')$, we may assume that $p_1 \in S'$. Similarly, we may assume that $p_2 \in S'$, which implies that $\hat{S} = \{p_1, p_2\}$, $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_3, e_4\}$, and $|\hat{F}_1|$ is even. Then, $g(\hat{F}_0, \hat{F}_1) \geq \min\{x(\alpha) + x(\gamma), 2 - x(\alpha) - x(\gamma) - 2y_\beta\}$ by the following case analysis.

- If $\hat{F}_1 = \emptyset$, then $g(\hat{F}_0, \hat{F}_1) = x_1(e_1) + x_1(e_2) + x_1(e_3) + x_1(e_4) = 2 - x(\alpha) - x(\gamma) - 2y_\beta$.
- If $|\hat{F}_1| \geq 2$, then $g(\hat{F}_0, \hat{F}_1) \geq 2 - (x_1(e_1) + x_1(e_2) + x_1(e_3) + x_1(e_4)) = x(\alpha) + x(\gamma) + 2y_\beta \geq x(\alpha) + x(\gamma)$.

**Proof of (P4).** Suppose that $T$ is of type (A), $v_2, v_3 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd. In the same way as (P2), we may assume that $\hat{S} = \{p_1, p_2\}$, $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_3, e_4\}$, and $|\hat{F}_1|$ is odd. Then, $g(\hat{F}_0, \hat{F}_1) \geq y_\emptyset + y_\beta + y_{\alpha\gamma}$ by the following case analysis and by the symmetry of $v_2$ and $v_3$.

- If $|\hat{F}_1| = 3$, then $g(\hat{F}_0, \hat{F}_1) \geq 3-(x_1(e_1)+x_1(e_2)+x_1(e_3)+x_1(e_4)) \geq 1 \geq y_\emptyset+y_\beta+y_{\alpha\gamma}$.
- If $\hat{F}_1 = \{e_1\}$, then $g(\hat{F}_0, \hat{F}_1) \geq (1 - x_1(e_1)) + x_1(e_2) \geq y_\emptyset + y_\beta + y_{\alpha\gamma}$.
- If $\hat{F}_1 = \{e_3\}$, then $g(\hat{F}_0, \hat{F}_1) \geq 1 - x_1(e_3) \geq y_\emptyset + y_\beta + y_{\alpha\gamma}$.

**Proof of (P5).** Suppose that $T$ is of type (A), $v_2 \in S'$, $v_3 \notin S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even. In the same way as (P2), we may assume that $p_1 \in S'$. If $p_2 \in S'$, then $b_1(p_2)+|F'_1 \cap \delta_{G_1}(p_2)|$ is even by the same calculation as (P1). Therefore, we may assume that $p_2 \notin S'$, since otherwise we can replace $(S', F'_0, F'_1)$ with $(S' \setminus \{p_2\}, F'_0 \setminus \delta_{G_1}(p_2), F'_1 \setminus \delta_{G_1}(p_2))$ without increasing the value of $h(S', F'_0, F'_1)$. That is, we may assume that $\hat{S} = \{p_1\}$, $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_3\}$, and $|\hat{F}_1|$ is odd. Then, $g(\hat{F}_0, \hat{F}_1) \geq \min\{(1 - x_1(e_1)) + x_1(e_3), x_1(e_1) + (1 - x_1(e_3))\} = \min\{x(\alpha) + x(\beta), 2 - x(\alpha) - x(\beta)\} \geq \min\{x(\alpha) + x(\beta), 2 - x(\alpha) - x(\beta) - 2y_\gamma\}$.

**Proof of (P6).** Suppose that $T$ is of type (A), $v_2 \in S'$, $v_3 \notin S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd. In the same way as (P5), we may assume that $\hat{S} = \{p_1\}$, $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_3\}$, and $|\hat{F}_1|$ is even. Then, $g(\hat{F}_0, \hat{F}_1) \geq \min\{x_1(e_1)+x_1(e_3), 2-x_1(e_1)-x_1(e_3)\} = \min\{y_\emptyset+y_\gamma+y_{\alpha\beta}, 2-(y_\emptyset + y_\gamma + y_{\alpha\beta})\} = y_\emptyset + y_\gamma + y_{\alpha\beta}$.

### *A.1.2 When T is of type (A')*

Second, we consider the case when $T$ is of type (A').

**Proof of (P7).** Suppose that $T$ is of type (A') and $v_1 \notin S'$. If $b_1(\hat{S}) + |\hat{F}_1|$ is odd, then $\hat{S} = \{p_3\}$ and $|\hat{F}_1|$ is odd. This shows that $h(S', F'_0, F'_1) \geq g(\hat{F}_0, \hat{F}_1) \geq 1$ by the following case analysis, which is a contradiction.

- If $\hat{F}_1 = \{e_1\}$, then $g(\hat{F}_0, \hat{F}_1) \geq (1 - x_1(e_1)) + x_1(e_2) + x_1(e_9) \geq 1$. The same argument can be applied to the case of $\hat{F}_1 = \{e_2\}$ by the symmetry of $\alpha$ and $\gamma$.
- If $\hat{F}_1 = \{e_i\}$ for some $i \in \{3, 4, 8\}$, then $g(\hat{F}_0, \hat{F}_1) \geq (1 - x_1(e_i)) + x_1(e_9) \geq 1$.
- If $\hat{F}_1 = \{e_9\}$, then $g(\hat{F}_0, \hat{F}_1) = 1 + 2y_\emptyset \geq 1$.
- If $|\hat{F}_1| \geq 3$, then $g(\hat{F}_0, \hat{F}_1) \geq 3-(x_1(e_1)+x_1(e_2)+x_1(e_3)+x_1(e_4)+x_1(e_8)+x_1(e_9)) \geq 1$.

Therefore, $b_1(\hat{S}) + |\hat{F}_1|$ is even.

**Proof of (P8).** Suppose that $T$ is of type (A'), $v_1 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even. Then, $g(\hat{F}_0, \hat{F}_1) \geq \min\{x(\alpha) + x(\gamma), 2 - x(\alpha) - x(\gamma) - 2y_\beta\}$ by the following case analysis.

- If $\hat{F}_0 = \{e_8, e_9\}$ and $\hat{F}_1 = \emptyset$, then $g(\hat{F}_0, \hat{F}_1) = x_1(e_8) + x_1(e_9) = x(\alpha) + x(\gamma)$.
- If $\hat{F}_0 = \emptyset$ and $\hat{F}_1 = \{e_8, e_9\}$, then $g(\hat{F}_0, \hat{F}_1) = (1 - x_1(e_8)) + (1 - x_1(e_9)) = 2 - x(\alpha) - x(\gamma) \geq 2 - x(\alpha) - x(\gamma) - 2y_\beta$.
- If $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_3, e_4\}$, then $g(\hat{F}_0, \hat{F}_1) \geq \min\{x(\alpha) + x(\gamma), 2 - x(\alpha) - x(\gamma) - 2y_\beta\}$ by the same calculation as (P2) in Section A.1.1.

**Proof of (P9).** Suppose that $T$ is of type (A'), $v_1 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd. Then, $g(\hat{F}_0, \hat{F}_1) \geq y_\emptyset + y_\beta + y_{\alpha\gamma}$ by the following case analysis.

- If $\hat{F}_0 = \{e_8\}$ and $\hat{F}_1 = \{e_9\}$, then $g(\hat{F}_0, \hat{F}_1) = x_1(e_8) + (1 - x_1(e_9)) = y_\emptyset + y_\beta + y_{\alpha\gamma}$.
- If $\hat{F}_0 = \{e_9\}$ and $\hat{F}_1 = \{e_8\}$, then $g(\hat{F}_0, \hat{F}_1) = (1-x_1(e_8))+x_1(e_9) \geq 1 \geq y_\emptyset+y_\beta+y_{\alpha\gamma}$.
- If $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_3, e_4\}$, then $g(\hat{F}_0, \hat{F}_1) \geq y_\emptyset + y_\beta + y_{\alpha\gamma}$ by the same calculation as (P4) in Section A.1.1.

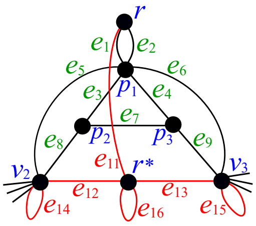


**Fig. 12** Construction of $G^+$

### *A.1.3 When T is of type (B)*

Third, we consider the case when $T$ is of type (B). Let $G^+ = (V^+, E^+)$ be the graph obtained from $G'_T = (V'_T, E'_T)$ in Figure 9 by adding a new vertex $r^*$, edges $e_{11} = rr^*$, $e_{12} = v_2r^*$, $e_{13} = v_3r^*$, and self-loops $e_{14}, e_{15}, e_{16}$ that are incident to $v_2$, $v_3$, and $r^*$, respectively (Figure 12). We define $b_T : V^+ \to \mathbf{Z}_{\geq 0}$ as $b_T(v) = 1$ for $v \in \{r, p_1, p_2, p_3\}$ and $b_T(v) = 2$ for $v \in \{r^*, v_2, v_3\}$. We also define $x_T : E^+ \to \mathbf{Z}_{\geq 0}$ as $x_T(e) = x_1(e)$ for $e \in E'_T$ and $x_T(e_{11}) = y_\alpha + y_\gamma + y_{\alpha\beta} + y_{\beta\gamma}$, $x_T(e_{12}) = y_\alpha + y_\beta + y_{\alpha\gamma} + y_{\beta\gamma}$, $x_T(e_{13}) = y_\beta + y_\gamma + y_{\alpha\beta} + y_{\alpha\gamma}$, $x_T(e_{14}) = y_\emptyset + y_\gamma$, $x_T(e_{15}) = y_\emptyset + y_\alpha$, and $x_T(e_{16}) = y_\emptyset$. For $J \in \mathcal{E}_T$, define $b_T$-factors $M_J$ in $G^+$ as follows:

$$M_\emptyset = \{e_1, e_7, e_{14}, e_{15}, e_{16}\}, \quad M_\alpha = \{e_4, e_8, e_{11}, e_{12}, e_{15}\}, \quad M_\beta = \{e_1, e_8, e_9, e_{12}, e_{13}\},$$
$$M_\gamma = \{e_3, e_9, e_{11}, e_{13}, e_{14}\}, \quad M_{\alpha\beta} = \{e_5, e_8, e_9, e_{11}, e_{13}\}, \quad M_{\alpha\gamma} = \{e_2, e_8, e_9, e_{12}, e_{13}\},$$
$$M_{\beta\gamma} = \{e_6, e_8, e_9, e_{11}, e_{12}\}.$$

Then, we obtain $\sum_{J \in \mathcal{E}_T} y_1(J) = 1$ and $\sum_{J \in \mathcal{E}_T} y_1(J) x_{M_J} = x_T$, where $x_{M_J} \in \mathbf{R}^{E^+}$ is the characteristic vector of $M_J$. This shows that $x_T$ is in the $b_T$-factor polytope in $G^+$. Therefore, $x_T$ satisfies (3) with respect to $G^+$ and $b_T$. We now show (P1), (P3), (P4), (P5), and (P6).

**Proof of (P1).** Suppose that $T$ is of type (B) and $v_2, v_3 \notin S'$. If $b_1(\hat{S}) + |\hat{F}_1|$ is odd, then $b_T(\hat{S}) + |\hat{F}_1|$ is also odd. Since $x_T$ satisfies (3) with respect to $G^+$ and $b_T$, we obtain $g(\hat{F}_0, \hat{F}_1) \geq 1$. This shows that $h(S', F'_0, F'_1) \geq 1$, which is a contradiction. Therefore, $b_1(\hat{S}) + |\hat{F}_1|$ is even.

**Proof of (P3).** Suppose that $T$ is of type (B), $v_2, v_3 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even. Since $b_T(\hat{S} \cup \{r^*, v_2, v_3\}) + |\hat{F}_1 \cup \{e_{11}\}|$ is odd and $x_T$ satisfies (3), we obtain $g(\hat{F}_0, \hat{F}_1) + (1 - x_T(e_{11})) \geq 1$. Therefore, $g(\hat{F}_0, \hat{F}_1) \geq x_T(e_{11}) = y_\alpha + y_\gamma + y_{\alpha\beta} + y_{\beta\gamma}$.

**Proof of (P4).** Suppose that $T$ is of type (B), $v_2, v_3 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd. Since $b_T(\hat{S} \cup \{r^*, v_2, v_3\}) + |\hat{F}_1|$ is odd and $x_T$ satisfies (3), we obtain $g(\hat{F}_0, \hat{F}_1) + x_T(e_{11}) \geq 1$. Therefore, $g(\hat{F}_0, \hat{F}_1) \geq 1 - x_T(e_{11}) = y_\emptyset + y_\beta + y_{\alpha\gamma}$.

**Proof of (P5).** Suppose that $T$ is of type (B), $v_2 \in S'$, $v_3 \notin S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even. If $\hat{S} \cap \{p_1, p_3\} \neq \emptyset$ and $p_2 \notin \hat{S}$, then we can add $p_2$ to $S'$ without decreasing the value of $h(S', F'_0, F'_1)$. Therefore, we can show $g(\hat{F}_0, \hat{F}_1) \geq \{x(\alpha) + x(\beta), 2 - x(\alpha) - x(\beta) - 2y_\gamma\}$ by the following case analysis.

- Suppose that $\hat{S} = \{p_1, p_2, p_3\}$, which implies that $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_6, e_9\}$ and $|\hat{F}_1|$ is odd.
  - If $\hat{F}_1 = \{e_i\}$ for $i \in \{1, 2, 6\}$, then $g(\hat{F}_0, \hat{F}_1) \geq (1 - x_1(e_i)) + x_1(e_9) \geq x(\alpha) + x(\beta)$.
  - If $\hat{F}_1 = \{e_9\}$, then $g(\hat{F}_0, \hat{F}_1) = y_\alpha + y_\beta + y_{\alpha\gamma} + y_{\beta\gamma} + 2y_\emptyset = 2 - x(\alpha) - x(\beta) - 2y_\gamma$.
  - If $|\hat{F}_1| = 3$, then $g(\hat{F}_0, \hat{F}_1) \geq 3 - (x_1(e_1) + x_1(e_2) + x_1(e_6) + x_1(e_9)) \geq x(\alpha) + x(\beta)$.

- Suppose that $\hat{S} = \{p_1, p_2\}$, which implies that $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2, e_4, e_6, e_7\}$ and $|\hat{F}_1|$ is even.
  - If $\hat{F}_1 = \emptyset$, then $g(\hat{F}_0, \hat{F}_1) = x_1(e_1) + x_1(e_2) + x_1(e_4) + x_1(e_6) + x_1(e_7) = 2 - x(\alpha) - x(\beta) - 2y_\gamma$.
  - If $|\hat{F}_1| \geq 2$, then $g(\hat{F}_0, \hat{F}_1) \geq 2 - (x_1(e_1) + x_1(e_2) + x_1(e_4) + x_1(e_6) + x_1(e_7)) \geq x(\alpha) + x(\beta)$.
- Suppose that $\hat{S} = \{p_2\}$, which implies that $\hat{F}_0 \cup \hat{F}_1 = \{e_3, e_5, e_7\}$ and $|\hat{F}_1|$ is odd.
  - If $\hat{F}_1 = \{e_i\}$ for $i \in \{3, 7\}$, then $g(\hat{F}_0, \hat{F}_1) \geq (1 - x_1(e_i)) + x_1(e_5) \geq x(\alpha) + x(\beta)$.
  - If $\hat{F}_1 = \{e_5\}$, then $g(\hat{F}_0, \hat{F}_1) \geq (1 - x_1(e_5)) + x_1(e_7) \geq 2 - x(\alpha) - x(\beta) - 2y_\gamma$.
  - If $\hat{F}_1 = \{e_3, e_5, e_7\}$, then $g(\hat{F}_0, \hat{F}_1) = 3 - (x_1(e_3) + x_1(e_5) + x_1(e_7)) \geq x(\alpha) + x(\beta)$.
- Suppose that $\hat{S} = \{p_2, p_3\}$, which implies that $\hat{F}_0 \cup \hat{F}_1 = \{e_3, e_4, e_5, e_9\}$ and $|\hat{F}_1|$ is even.
  - If $\hat{F}_1 = \emptyset$, then $g(\hat{F}_0, \hat{F}_1) = x_1(e_3) + x_1(e_4) + x_1(e_5) + x_1(e_9) = x(\alpha) + x(\beta) + 2y_\gamma \geq x(\alpha) + x(\beta)$.
  - If $|\hat{F}_1| \geq 2$, then $g(\hat{F}_0, \hat{F}_1) \geq 2 - (x_1(e_3) + x_1(e_4) + x_1(e_5) + x_1(e_9)) = 2 - x(\alpha) - x(\beta) - 2y_\gamma$.
- If $\hat{S} = \emptyset$, then $\hat{F}_0 \cup \hat{F}_1 = \{e_5, e_8\}$ and $|\hat{F}_1|$ is even. Therefore, $g(\hat{F}_0, \hat{F}_1) \geq \min\{x_1(e_5) + x_1(e_8), 2 - x_1(e_5) - x_1(e_8)\} \geq \min\{x(\alpha) + x(\beta), 2 - x(\alpha) - x(\beta) - 2y_\gamma\}$.

**Proof of (P6).** Suppose that $T$ is of type (B), $v_2 \in S'$, $v_3 \notin S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd. Since $b_T(\hat{S} \cup \{v_2\}) + |\hat{F}_1|$ is odd and $x_T$ satisfies (3), we obtain $g(\hat{F}_0, \hat{F}_1) + x_T(e_{12}) \geq 1$. Therefore, $g(\hat{F}_0, \hat{F}_1) \geq 1 - x_T(e_{12}) = y_\emptyset + y_\gamma + y_{\alpha\beta}$.

### *A.1.4 When $T$ is of type (B')*

Finally, we consider the case when $T$ is of type (B').

**Proof of (P7).** Suppose that $T$ is of type (B') and $v_1 \notin S'$. If $b_1(\hat{S}) + |\hat{F}_1|$ is odd, then $\hat{S} = \{p_4\}$ and $h(S', F'_0, F'_1) \geq \min\{x_1(e_1) + x_1(e_{10}), 2 - x_1(e_1) - x_1(e_{10})\} = 1$, which is a contradiction. Therefore, $b_1(\hat{S}) + |\hat{F}_1|$ is even.

**Proof of (P8).** Suppose that $T$ is of type (B'), $v_1 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is even. If $p_4 \notin S'$, then we define $(S'', F''_0, F''_1) \in \mathcal{F}_1$ as $(S'', F''_0, F''_1) = (S' \cup \{p_4\}, F'_0 \setminus \{e_{10}\}, F'_1 \cup \{e_1\})$ if $e_{10} \in F'_0$ and $(S'', F''_0, F''_1) = (S' \cup \{p_4\}, F'_0 \cup \{e_1\}, F'_1 \setminus \{e_{10}\})$ if $e_{10} \in F'_1$. Since $h(S'', F''_0, F''_1) = h(S', F'_0, F'_1)$, by replacing $(S', F'_0, F'_1)$ with $(S'', F''_0, F''_1)$, we may assume that $p_4 \in S'$. Then, since $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2\}$ and $|\hat{F}_1|$ is odd, we obtain $g(\hat{F}_0, \hat{F}_1) \geq \min\{(1 - x_1(e_1)) + x_1(e_2), x_1(e_1) + (1 - x_1(e_2))\} \geq \min\{x(\alpha) + x(\gamma), 2 - x(\alpha) - x(\gamma) - 2y_\beta\}$.

**Proof of (P9).** Suppose that $T$ is of type (B'), $v_1 \in S'$, and $b_1(\hat{S}) + |\hat{F}_1|$ is odd. In the same way as (P8), we may assume that $\hat{S} = \{p_4\}$, $\hat{F}_0 \cup \hat{F}_1 = \{e_1, e_2\}$, and $|\hat{F}_1|$ is even. Then, $g(\hat{F}_0, \hat{F}_1) \geq \min\{x_1(e_1) + x_1(e_2), (1 - x_1(e_1)) + (1 - x_1(e_2))\} = \min\{y_\emptyset + y_\beta + y_{\alpha\gamma}, 2 - (y_\emptyset + y_\beta + y_{\alpha\gamma})\} = y_\emptyset + y_\beta + y_{\alpha\gamma}$.

## A.2 Condition (10)

Recall that $r \notin S'$ is assumed and note that $x_1(\delta_{G_1}(r)) = 1$. Let $\mathcal{T}_{(P3)} \subseteq \mathcal{T}^+_{S^*}$ be the set of triangles satisfying the conditions in (P3), i.e., the set of triangles of type (B) such that $v_2, v_3 \in S'$ and $b_1(\hat{S}) + |\hat{F}_1|$ is even. Since $y_\alpha + y_\gamma + y_{\alpha\beta} + y_{\beta\gamma} = 1 - x_1(e_1^T) - x_1(e_2^T)$ holds for each triangle $T \in \mathcal{T}^+_{S^*}$ of type (B), if there exist two triangles $T, T' \in \mathcal{T}_{(P3)}$, then $h(S', F'_0, F'_1) \geq (1 - x_1(e_1^T) - x_1(e_2^T)) + (1 - x_1(e_1^{T'}) - x_1(e_2^{T'})) \geq 2 - x_1(\delta_{G_1}(r)) = 1$, which is a contradiction. Similarly, if there exists a triangle $T \in \mathcal{T}_{(P3)}$ and an edge $e \in (\delta_{G_1}(r) \setminus E'_T) \cap F'_1$, then $h(S', F'_0, F'_1) \geq (1 - x_1(e_1^T) - x_1(e_2^T)) + (1 - x_1(e)) \geq 2 - x_1(\delta_{G_1}(r)) = 1$, which is a contradiction. Therefore, either $\mathcal{T}_{(P3)} = \emptyset$ holds or $\mathcal{T}_{(P3)}$ consists of exactly one triangle, say $T$, and $(\delta_{G_1}(r) \setminus E'_T) \cap F'_1 = \emptyset$.

Assume that $\mathcal{T}_{(P3)} = \{T\}$ and $(\delta_{G_1}(r) \setminus E'_T) \cap F'_1 = \emptyset$. Define $(S'', F''_0, F''_1) \in \mathcal{F}_1$ as $S'' = S' \cup V'_T$, $F''_0 = (F'_0 \triangle \delta_{G_1}(r)) \setminus E'_T$, and $F''_1 = F'_1 \setminus E'_T$, where $\triangle$ denotes the symmetric difference. Note that $(F''_0, F''_1)$ is a partition of $\delta_{G_1}(S'')$, $b_1(S'') + |F''_1| = (b_1(S') + b_1(\hat{S})) + (|F'_1| - |\hat{F}_1|) \equiv 1 \pmod 2$, and $h(S', F'_0, F'_1) - h(S'', F''_0, F''_1) \geq (1 - x_1(e^T_1) - x_1(e^T_2)) - x_1(\delta_{G_1}(r) \setminus \{x_1(e^T_1), x_1(e^T_2)\}) = 0$. By these observations, $(S'', F''_0, F''_1) \in \mathcal{F}_1$ is also a minimizer of $h$. This shows that $(V'' \setminus S'', F''_0, F''_1) \in \mathcal{F}_1$ is a minimizer of $h$ such that $r \in V'' \setminus S''$. Furthermore, if a triangle $T' \in \mathcal{T}^+_{S^*}$ satisfies the conditions in (P3) with respect to $(V'' \setminus S'', F''_0, F''_1)$, then $T'$ is a triangle of type (B) such that $v_2, v_3 \notin S'$ and $b_1(\hat{S}) + |\hat{F}_1|$ is odd with respect to $(S', F'_0, F'_1)$, which contradicts (P1). Therefore, by replacing $(S', F'_0, F'_1)$ with $(V'' \setminus S'', F''_0, F''_1)$, we may assume that $\mathcal{T}_{(P3)} = \emptyset$.

In what follows, we construct $(S, F_0, F_1) \in \mathcal{F}$ for which $(x, y)$ violates (10) to derive a contradiction. We initialize $(S, F_0, F_1)$ as $S = S' \cap V$, $F_0 = F'_0 \cap E$, and $F_1 = F'_1 \cap E$, and apply the following procedures for each triangle $T \in \mathcal{T}^+_{S^*}$.

- Suppose that $T$ satisfies the condition in (P1) or (P7). In this case, we do nothing.
- Suppose that $T$ satisfies the condition in (P2) or (P8). If $g(\hat{F}_0, \hat{F}_1) \geq x(\alpha) + x(\gamma)$, then add $\alpha$ and $\gamma$ to $F_0$. Otherwise, since $g(\hat{F}_0, \hat{F}_1) \geq 2 - x(\alpha) - x(\gamma) - 2y_\beta$, add $\alpha$ and $\gamma$ to $F_1$.
- Suppose that $T$ satisfies the condition in (P4) or (P9). In this case, add $\alpha$ to $F_0$ and add $\gamma$ to $F_1$.
- Suppose that $T$ satisfies the condition in (P5). If $g(\hat{F}_0, \hat{F}_1) \geq x(\alpha) + x(\beta)$, then add $\alpha$ and $\beta$ to $F_0$. Otherwise, since $g(\hat{F}_0, \hat{F}_1) \geq 2 - x(\alpha) - x(\beta) - 2y_\gamma$, add $\alpha$ and $\beta$ to $F_1$.
- Suppose that $T$ satisfies the condition in (P6). In this case, add $\alpha$ to $F_0$ and add $\beta$ to $F_1$.

Note that exactly one of the above procedures is applied for each $T \in \mathcal{T}^+_{S^*}$, because $\mathcal{T}_{(P3)} = \emptyset$.

Then, we see that $(S, F_0, F_1) \in \mathcal{F}$ holds and the left-hand side of (10) with respect to $(S, F_0, F_1)$ is at most $h(S', F'_0, F'_1)$ by (P1)–(P9). Since $h(S', F'_0, F'_1) < 1$ is assumed, $(x, y)$ violates (10) for $(S, F_0, F_1) \in \mathcal{F}$, which is a contradiction. ◻

## B Proof of Lemma 6

We first show that $M_1 \oplus M_2$ forms a $\mathcal{T}$-free $b$-factor. We can easily see that replacing $(M_1 \cup M_2) \cap \{e^f \mid f \in \tilde{F}^*_0\}$ with $\{f \in \tilde{F}^*_0 \mid e^f \in M_1 \cap M_2\}$ does not affect the degrees of vertices in $V$. Since $M_1 \cup M_2$ contains exactly one of $\{e^f_u, e^f_v\}$ or $e^f_r (= e^f_{r'})$ for $f = uv \in \tilde{F}^*_1$, replacing $(M_1 \cup M_2) \cap \{e^f_u, e^f_r, e^f_v \mid f = uv \in \tilde{F}^*_1\}$ with $\{f \in \tilde{F}^*_1 \mid e^f_r \notin M_1 \cap M_2\}$ does not affect the degrees of vertices in $V$.

For every $T \in \mathcal{T}^+_{S^*}$ of type (A) or (A'), since $|\varphi(M_1, M_2, T) \cap \{\alpha, \gamma\}| = |M_T \cap \{e_8, e_9\}|$, $|\varphi(M_1, M_2, T) \cap \{\alpha, \beta\}| = |M_T \cap \{e_3, e_5\}|$, and $|\varphi(M_1, M_2, T) \cap \{\beta, \gamma\}| = |M_T \cap \{e_4, e_6\}|$ hold by the definition of $\varphi(M_1, M_2, T)$, replacing $M_T$ with $\varphi(M_1, M_2, T)$ does not affect the degrees of vertices in $V$.

Furthermore, for every $T \in \mathcal{T}^+_{S^*}$ of type (B) or (B'), since $|\varphi(M_1, M_2, T) \cap \{\alpha, \gamma\}| = |M_T \cap \{e_2, e_{10}\}|$, $|\varphi(M_1, M_2, T) \cap \{\alpha, \beta\}| = |M_T \cap \{e_5, e_8\}|$, and $|\varphi(M_1, M_2, T) \cap \{\beta, \gamma\}| = |M_T \cap \{e_6, e_9\}|$ hold by the definition of $\varphi(M_1, M_2, T)$, replacing $M_T$ with $\varphi(M_1, M_2, T)$ does not affect the degrees of vertices in $V$.

Since $b(v) = b_1(v)$ for $v \in S^*$ and $b(v) = b_2(v)$ for $v \in V^* \setminus S^*$, this shows that $M_1 \oplus M_2$ forms a $b$-factor. Since $M_j$ is $\mathcal{T}_j$-free for $j \in \{1, 2\}$, $M_1 \oplus M_2$ is a $\mathcal{T}$-free $b$-factor.

We next show that $x = \sum_{(M_1, M_2) \in \mathcal{M}} \lambda_{(M_1, M_2)} x_{M_1 \oplus M_2}$. By the definitions of $x_1, x_2, M_1 \oplus M_2$, and $\lambda_{(M_1, M_2)}$, it holds that

$$x(e) = \sum_{(M_1, M_2) \in \mathcal{M}} \lambda_{(M_1, M_2)} x_{M_1 \oplus M_2}(e) \tag{11}$$

for $e \in E \setminus \bigcup_{T \in \mathcal{T}^+_{S^*}} E(T)$.

Let $T \in \mathcal{T}^+_{S^*}$ be a triangle of type (A) for $(G_1, b_1, \mathcal{T}_1)$ and let $\alpha, \beta$, and $\gamma$ be as in Figures 7 and 8. By the definition of $\varphi(M_1, M_2, T)$, we obtain

$$\sum_{(M_1,M_2)\in\mathcal{M}} \lambda_{(M_1,M_2)} x_{M_1\oplus M_2}(\beta) = \sum\{\lambda_{(M_1,M_2)} \mid \varphi(M_1, M_2, T) = \{\alpha, \beta\}, \{\beta, \gamma\}, \text{ or } \{\beta\}\}$$
$$= x_1(e_3) + x_1(e_4) + x_2(e_7) = y_{\alpha\beta} + y_{\beta\gamma} + y_\beta = x(\beta).$$

We also obtain

$$\sum_{(M_1,M_2)\in\mathcal{M}} \lambda_{(M_1,M_2)} x_{M_1\oplus M_2}(\alpha) = \sum\{\lambda_{(M_1,M_2)} \mid \varphi(M_1, M_2, T) \neq \{\gamma\}, \{\beta, \gamma\}, \{\beta\}\}$$
$$= 1 - x_1(e_1) - x_1(e_4) - x_2(e_7) = 1 - y_\emptyset - y_\gamma - y_{\beta\gamma} - y_\beta = x(\alpha).$$

Since a similar equality holds for $\gamma$ by symmetry, (11) holds for $e \in \{\alpha, \beta, \gamma\}$. Since $T$ is a triangle of type (A') for $(G_1, b_1, \mathcal{T}_1)$ if and only if it is of type (A) for $(G_2, b_2, \mathcal{T}_2)$, the same argument can be applied when $T$ is a triangle of type (A') for $(G_1, b_1, \mathcal{T}_1)$.

Let $T \in \mathcal{T}^+_{S^*}$ be a triangle of type (B) for $(G_1, b_1, \mathcal{T}_1)$ and let $\alpha, \beta$, and $\gamma$ be as in Figures 9 and 10. By the definition of $\varphi(M_1, M_2, T)$, we obtain

$$\sum_{(M_1,M_2)\in\mathcal{M}} \lambda_{(M_1,M_2)} x_{M_1\oplus M_2}(\beta) = \sum\{\lambda_{(M_1,M_2)} \mid \varphi(M_1, M_2, T) \neq \emptyset, \{\alpha\}, \{\gamma\}, \{\alpha, \gamma\}\}$$
$$= 1 - x_1(e_2) - x_1(e_3) - x_1(e_4) - x_1(e_7) = 1 - y_{\alpha\gamma} - y_\gamma - y_\alpha - y_\emptyset = x(\beta).$$

We also obtain

$$\sum_{(M_1,M_2)\in\mathcal{M}} \lambda_{(M_1,M_2)} x_{M_1\oplus M_2}(\alpha) = \sum\{\lambda_{(M_1,M_2)} \mid \varphi(M_1, M_2, T) = \{\alpha\}, \{\alpha, \beta\}, \text{ or } \{\alpha, \gamma\}\}$$
$$= x_1(e_2) + x_1(e_4) + x_1(e_5) = y_{\alpha\gamma} + y_\alpha + y_{\alpha\beta} = x(\alpha).$$

Since a similar equality holds for $\gamma$ by symmetry, (11) holds for $e \in \{\alpha, \beta, \gamma\}$. The same argument can be applied when $T$ is a triangle of type (B') for $(G_1, b_1, \mathcal{T}_1)$.

Therefore, (11) holds for every $e \in E$, which complete the proof. □