SEQUENTIAL ENDS AND NONSTANDARD INFINITE BOUNDARIES OF COARSE SPACES

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Abstract. This paper is an addendum to the author's previous paper [8]. Miller et al. [10] introduced a functor $\sigma\colon \mathbf{Coarse}_* \to \mathbf{Sets}$, where $\mathbf{Coarse}_*$ is the category of pointed coarse spaces and coarse maps. DeLyser et al. [4] introduced a functor $\varepsilon\colon \mathbf{Coarse}_* \to \mathbf{Sets}$, and proved that ε coincides with σ on $\mathbf{Metr}_*$ (the full subcategory of metrisable spaces). Using techniques of nonstandard analysis, the author in [8] provided a functor $\iota\colon \mathscr{C} \subseteq \mathbf{Coarse}_* \to \mathbf{Sets}$, where $\mathscr{C}$ is an arbitrary small full subcategory, and a natural transformation $\omega\colon \sigma \restriction \mathscr{C} \Rightarrow \iota$. The surjectivity of ω has been proved for all proper geodesic metrisable spaces, while the injectivity has remained open. In this note, we first pointed out that ω is the composition of two natural transformations $\varphi \restriction \mathscr{C}\colon \sigma \restriction \mathscr{C} \Rightarrow \varepsilon \restriction \mathscr{C}$ and $\omega'\colon \varepsilon \restriction \mathscr{C} \Rightarrow \iota$, and then show that ω' is injective for all spaces in $\mathscr{C}$. As a corollary, ω is injective for all metrisable spaces in $\mathscr{C}$. This partially answers some of the problems posed in [8].

1. Introduction

Small-scale topology can be considered as the study of infinitesimal structures of spaces, maps, homotopies, and so forth from the point of view of nonstandard analysis. On the other hand, large-scale topology is of infinite (or finite) structures (cf. Protasov and Zarichnyi [11, p. 7]). The notion of a coarse space introduced by Roe [12] plays a central role in large-scale topology, as the notion of a uniform space does in small-scale topology. Recall that a *coarse structure* on a set X is an ideal $\mathcal{C}_X$ of the poset $\mathcal{P}(X \times X)$ (with respect to the inclusion) that fulfills the following axioms:

(1) $\Delta_X := \{ (x,x) \mid x \in X \} \in \mathcal{C}_X$;
(2) $E \circ F := \{ (x,y) \mid (x,z) \in E \text{ and } (z,y) \in F \text{ for some } z \in X \} \in \mathcal{C}_X$ for all $E, F \in \mathcal{C}_X$;
(3) $E^{-1} := \{ (x,y) \mid (y,x) \in E \} \in \mathcal{C}_X$ for all $E \in \mathcal{C}_X$.

The set X together with the coarse structure $\mathcal{C}_X$ is called a *coarse space*. The elements of $\mathcal{C}_X$ are called *controlled sets* or *entourages*. A subset B of X is said to be *bounded* if $B \times B$ is a controlled set.

Example 1.1. Let $d_X\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{ +\infty \}$ be a (generalised) metric on a set X. The family

$$\mathcal{C}_X := \{ E \subseteq X \times X \mid \sup d_X(E) < +\infty \}$$

forms a coarse structure on X, and is called the *bounded coarse structure* induced by d_X. A subset of X is bounded with respect to $\mathcal{C}_X$ if and only if it is bounded with respect to d_X. Throughout the paper, we assume that every metric space is endowed with the bounded coarse structure.

Let X_i $(i = 0, 1)$ be coarse spaces. A map $f\colon X_0 \to X_1$ is said to be

(1) *proper* if $f^{-1}(B)$ is bounded in X_0 for all bounded subsets B of X_1;
(2) *bornologous* if $(f \times f)(E) := \{ (f(x), f(y)) \mid (x,y) \in E \} \in \mathcal{C}_Y$ for all $E \in \mathcal{C}_X$.
(3) *coarse* if it is proper and bornologous.

Denote by $\mathbf{Coarse}_*$ the category of pointed coarse spaces and basepoint-preserving coarse maps; and by $\mathbf{Metr}_*$ its full subcategory of metrisable spaces.

Coarse maps $f, g\colon X_0 \to X_1$ are said to be *close* if $(f \times g)(\Delta_{X_0}) = \{ (f(x), g(x)) \mid x \in X \} \in \mathcal{C}_{X_1}$. A coarse map $f\colon X_0 \to X_1$ is called a *coarse equivalence* if there exists a coarse map (called a coarse inverse) $g\colon X_1 \to X_0$ such that $f \circ g$ and $g \circ f$ are close to the identity maps id_{X_1} and id_{X_0}, respectively. The closeness relation of (basepoint-preserving) coarse maps gives a congruence on the category $\mathbf{Coarse}_*$.

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1.1. **Invariant σ.** Miller et al. [10] and DeLyser et al. [3] introduced a set-valued coarse invariant $\sigma(X,\xi)$ of a pointed *metric* space (X,ξ). Recall the simplified definition given by DeLyser et al. [4]. Let (X,ξ) be a pointed *coarse* space. (This generalisation to coarse spaces by [8] is merely a rewriting of the original definition in terms of coarse structures.) A coarse map $s\colon (\mathbb{N},0) \to (X,\xi)$ is called a *coarse sequence* in (X,ξ). Notice that a coarse map is precisely a sequence $\{ s(i) \}_{i\in\mathbb{N}}$ in X with the following properties:

- **properness:** the sequence $\{ s(i) \}_{i\in\mathbb{N}}$ diverges from ξ to infinity, i.e., for all bounded subsets $B \ni \xi$ of X, there is an $N \in \mathbb{N}$ such that $\{ s(i) \}_{i\geq N}$ is contained in $X \setminus B$;
- **bornologousness:** there is an $E \in \mathcal{C}_X$ such that $(s(i), s(i+1)) \in E$ for all $i \in \mathbb{N}$.

Denote the set of all coarse sequences in (X,ξ) by $S(X,\xi)$. Given $s,t \in S(X,\xi)$, we say that s and t are *σ-equivalent* ($s \equiv^{\sigma}_{X,\xi} t$) if there exists a finite sequence $\{ s_i \}_{i\leq n}$ in $S(X,\xi)$ such that $s_0 = s$, $s_n = t$, and either s_i or s_{i+1} is a subsequence to the other. The quotient set

$$\sigma(X,\xi) := S(X,\xi) / \equiv^{\sigma}_{X,\xi}$$

can be extended to a functor $\sigma\colon \mathbf{Coarse}_* \to \mathbf{Sets}$. More precisely, the morphism part is given by

$$\sigma f\,[s]_{\equiv^{\sigma}_{X,\xi}} := [f \circ s]_{\equiv^{\sigma}_{Y,\eta}}.$$

Moreover, σ is invariant under coarse equivalences (relative to the base points), i.e., if two coarse maps $f,g\colon (X,\xi) \to (Y,\eta)$ are close, then $\sigma f = \sigma g$. See [9] for the systematic proofs of the functoriality and the coarse invariance of σ.

The calculation of the invariant $\sigma(X,\xi)$ is quite hard because of the difficulty of the decision of $s \equiv^{\sigma}_{X,\xi} t$. On the other hand, the coarse invariants $\varepsilon(X,\xi)$ and $\iota(X,\xi)$ defined below can easily and intuitively be calculated.

1.2. **Invariant ε.** DeLyser et al. [4] provided alternative definition of σ in terms of sequential ends. Before recalling the definition of sequential ends, we first recall the definition of ends of a topological space introduced by Freudenthal [5]. Let X be a topological space. A proper continuous map $r\colon \mathbb{R}_+ \to X$ is called a *proper ray*. (A map between topological spaces is said to be *proper* if the inverse image of each compact set is compact.) Two proper rays r_0, r_1 in X are said to be *converge to the same end* if for all compact subsets K of X there exists an $N \in \mathbb{N}$ such $r_0(\mathbb{R}_{\geq N})$ and $r_1(\mathbb{R}_{\geq N})$ are contained in the same path-connected component of $X \setminus K$. This gives an equivalence relation of proper rays in X. The quotient set is denoted by $\mathrm{Ends}(X)$. See also Bridson and Häfliger [1, pp .144–148].

Theorem 1.2 (DeLyser et al. [4, Proposition 2.12])**.** *Let (X,ξ) be a proper geodesic pointed metric space. Then* $\mathrm{Ends}(X)$ *is isomorphic to $\sigma(X,\xi)$.*

Returning to the definition of sequential ends, let (X,ξ) be a pointed coarse space. Given $s,t \in S(X,\xi)$, we say that s and t are *ε-equivalent* ($s \equiv^{\varepsilon}_{X,\xi} t$) if there is an $E \in \mathcal{C}_X$ such that for all bounded subsets $B \ni \xi$ of X there exists an $N \in \mathbb{N}$ such that $\{ s(i) \}_{i\geq N}$ and $\{ t(i) \}_{i\geq N}$ are contained in the same E-connected component of $X \setminus B$. Note that a subset A of X is *E-connected* if and only if for any two points $x,y \in A$, there is a finite sequence $\{ x_i \}_{i\leq n}$ in A, called an *E-chain*, such that $x_0 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in E$ for all $i < n$. The $\equiv^{\varepsilon}_{X,\xi}$-equivalence classes are called *sequential ends* of (X,ξ). The quotient set

$$\varepsilon(X,\xi) := S(X,\xi) / \equiv^{\varepsilon}_{X,\xi}$$

can be considered as a coarsely invariant functor $\varepsilon\colon \mathbf{Coarse}_* \to \mathbf{Sets}$. The morphism part of ε is defined similarly to that of σ.

Theorem 1.3 (DeLyser et al. [4, Theorem 3.3])**.** *Let (X,ξ) be a pointed coarse space. Then $\equiv^{\sigma}_{X,\xi}$ implies $\equiv^{\varepsilon}_{X,\xi}$.*

Proof. Obvious from the fact that every coarse sequence is ε-equivalent to its subsequences. □

Corollary 1.4. *Let (X,ξ) be a pointed coarse space. The map $\varphi_{(X,\xi)}\colon \sigma(X,\xi) \to \varepsilon(X,\xi)$ defined by*

$$\varphi_{(X,\xi)}\,[s]_{\equiv^{\sigma}_{X,\xi}} := [s]_{\equiv^{\varepsilon}_{X,\xi}}$$

is well-defined, surjective and natural in (X,ξ).

On the other hand, the converse of Theorem 1.3 holds for metrisable spaces.

Theorem 1.5 (DeLyser et al. [4, Theorem 3.3])**.** *Let (X,ξ) be a pointed metric space. Then $\equiv^{\varepsilon}_{X,\xi}$ implies $\equiv^{\sigma}_{X,\xi}$.*

Proof. Let $s, t \in S(X,\xi)$ and suppose $s \equiv^{\varepsilon}_{X,\xi} t$. Let $K > 0$ be a witness of the ε-equivalence. Then for each $r \in \mathbb{N}$, there is an $N_r \in \mathbb{N}$ such that $\{ s(i) \}_{i \geq N_r}$ and $\{ t(i) \}_{i \geq N_r}$ are contained in the same K-connected component outside the r-ball $B_X(\xi; r)$. In particular, there exists a K-chain $\{ u^r_i \}_{i \leq M_r}$ that connects $s(N_r)$ and $t(N_r)$ outside $B_X(\xi; r)$. We may assume that $N_r < N_{r+1}$ for all $r \in \mathbb{N}$. Then the concatenation of the sequences

$$\begin{gathered} \{ s(i) \}_{i \leq N_0}, \{ u^0_i \}_{i \leq M_0}, \{ t(N_0 - i) \}_{i \leq N_0} \\ \{ t(i) \}_{i \leq N_1}, \{ u^1_{M_1 - i} \}_{i \leq M_1}, \{ s(M_1 - i) \}_{i \leq M_1 - M_0}, \\ \{ s(M_0 + i) \}_{i \leq M_2 - M_0}, \{ u^2_i \}_{i \leq M_2}, \{ t(M_2 - i) \}_{i \leq M_2 - M_1}, \\ \vdots \end{gathered}$$

is proper bornologous, and has s and t as subsequences, whence $s \equiv^{\sigma}_{X,\xi} t$. □

Corollary 1.6. *The functors σ and ε are equal on* $\mathbf{Metr}_*$. *The natural map $\varphi_{(X,\xi)} \colon \sigma(X,\xi) \to \varepsilon(X,\xi)$ is therefore the identity map for all metrisable spaces (X,ξ).*

Remark 1.7. In the proof of Theorem 1.5, the properness of the resulting sequence depends on the fact that each bounded subset $B \ni \xi$ of X is contained in some r-ball $B(\xi; r)$. In other words, the bornology of X is generated by the countable family $\{ B(\xi; r) \mid r \in \mathbb{N} \}$. Such a countability property fails in general pointed coarse spaces.

1.3. **Invariant ι.** In [8], we introduced a set-valued invariant $\iota(X,\xi)$ of a (standard) pointed coarse space (X,ξ) via nonstandard analysis. First, let us recall the nonstandard treatment of coarse spaces. We assume the reader to be familiar with the terminology of nonstandard analysis. Fix a small full subcategory $\mathscr{C}$ of $\mathbf{Coarse}_*$. By the reflection principle, there exists a transitive set $\mathbb{U} \ni \mathcal{C}$, the *standard universe*, such that all classical objects we need belong to $\mathbb{U}$ and all (but finitely many) set-theoretic formulae we need are absolute with respect to $\mathbb{U}$. We also fix a sufficiently saturated elementary extension $* \colon \mathbb{U} \hookrightarrow {}^*\mathbb{U}$; $x \mapsto {}^*x$, the *nonstandard extension*. See Chang and Keisler [2, Section 4.4] for more detailed account of nonstandard analysis.

Let X be a standard coarse space. The *finite proximity* of ${}^*X$ is the binary relation on ${}^*X$ defined by

$$x \sim_X y :\iff \exists E \in \mathcal{C}_X . (x,y) \in {}^*E.$$

For each $\xi \in {}^*X$, the set of all points infinitely far away from ξ is denoted by

$$\mathrm{INF}(X,\xi) := \{ x \in {}^*X \mid x \nsim_X \xi \}.$$

The finite proximity relation $\sim_X$ completely characterises the coarse structure of X. We have indeed the following characterisations (see [7]).

1. (1) A subset E of $X \times X$ is controlled if and only if ${}^*E \subseteq \sim_X$.
2. (2) A subset B of X is bounded if and only if ${}^*B \subseteq {}^*X \setminus \mathrm{INF}(X,\xi)$ for all $\xi \in B$.
3. (3) A subset A of X is E-connected for some $E \in \mathcal{C}_X$ if and only if for any two points $x, y \in {}^*A$, there is an internal hyperfinite sequence $\{ x_i \}_{i \leq n}$ in ${}^*A$, where $n \in {}^*\mathbb{N}$, such that $x_0 = x$, $x_n = y$, and $x_i \sim_X x_{i+1}$ for all $i < n$.
4. (4) A map $f \colon X \to Y$ between standard coarse spaces is proper if and only if ${}^*f(\mathrm{INF}(X,\xi)) \subseteq \mathrm{INF}(Y, f(\xi))$ for all $\xi \in X$.
5. (5) A map $f \colon X \to Y$ between standard coarse spaces is bornologous if and only if $x \sim_X y$ implies ${}^*f(x) \sim_Y {}^*f(y)$ for all $x, y \in {}^*X$.

Now, recall the definition of the invariant ι. Let (X,ξ) be a pointed coarse space in $\mathscr{C}$. Given $x, y \in \mathrm{INF}(X,\xi)$, we say that x and y are *ι-equivalent* ($x \equiv^{\iota}_{X,\xi} y$) if there exists an internal hyperfinite sequence $\{ x_i \}_{i \leq n}$ in $\mathrm{INF}(X,\xi)$, called a *macrochain*, such that $x_0 = x$, $x_n = y$, and $x_i \sim_X x_{i+1}$ for all $i < n$. The quotient set

$$\iota(X,\xi) := \mathrm{INF}(X,\xi) / \equiv^{\iota}_{X,\xi}$$

can be considered as a coarsely invariant functor $\iota \colon \mathscr{C} \subseteq \mathbf{Coarse}_* \to \mathbf{Sets}$. More precisely, the morphism part of ι is given by

$$\iota f\, [x]_{\equiv_{X,\xi}} := [{}^*f(x)]_{\equiv_{Y,\eta}}$$

for each coarse map $f \colon (X,\xi) \to (Y,\eta)$ in $\mathscr{C}$. The well-definedness follows from the nonstandard characterisation of coarseness mentioned above.

Remark 1.8. Goldbring [6] introduced a similar invariant $\mathrm{IPC}(X)$ for a metric space X, where the metric function is assumed to be finite-valued. Rougly speaking, $\mathrm{IPC}(X)$ is the set of internal path components of $\mathrm{INF}(X,\xi)$; more precisely, two points x and y of $\mathrm{INF}(X,\xi)$ are said to *belong to the same internal path component* if there exists an internally continuous map $\gamma\colon {}^*[0,1]\to{}^*X$ such that $\gamma\left({}^*[0,1]\right)\subseteq\mathrm{INF}(X,\xi)$, $\gamma(0)=x$ and $\gamma(1)=y$. If X is proper geodesic, then $\mathrm{IPC}(X)\cong\mathrm{Ends}(X)\cong\iota(X,\xi)$ by [6, Lemmas 3.6 and 3.9]. On the other hand, if X is (topologically) discrete, then $\mathrm{IPC}(X)\cong\mathrm{INF}(X,\xi)$; hence, the coarse invariance of IPC holds only for proper geodesic spaces.

The relationship between σ and ι is given by a natural transformation $\omega\colon\sigma\restriction\mathscr{C}\Rightarrow\iota$.

Theorem 1.9 ([8, Lemma 4.1])**.** *Let (X,ξ) be a pointed coarse space in $\mathscr{C}$ and $s,t\in S(X,\xi)$. If $s\equiv^{\sigma}_{X,\xi}t$, then ${}^*s(i)\equiv^{\iota}_{X,\xi}{}^*t(j)$ for all $i,j\in{}^*\mathbb{N}\setminus\mathbb{N}$.*

Corollary 1.10. *Let (X,ξ) be a coarse space in $\mathscr{C}$. The map $\omega_{(X,\xi)}\colon\sigma(X,\xi)\to\iota(X,\xi)$ defined by*

$$\omega_{(X,\xi)}[s]_{\equiv^{\sigma}_{X,\xi}}:=[{}^*s(i)]_{\equiv^{\iota}_{X,\xi}},\ i\in{}^*\mathbb{N}\setminus\mathbb{N}$$

is well-defined and natural in (X,ξ).

It has been known from [8, Theorem 4.2] that $\omega_{(X,\xi)}$ is surjective for all proper geodesic metrisable spaces (X,ξ) in $\mathscr{C}$. However, no general results on the injectivity have been given in [8]. The aim of this paper is to solve the injectivity problem for metrisable spaces. To do this, we first observe that ω is decomposed into two natural transformations $\varphi\restriction\mathscr{C}\colon\sigma\restriction\mathscr{C}\Rightarrow\varepsilon\restriction\mathscr{C}$ and $\omega'\colon\varepsilon\restriction\mathscr{C}\Rightarrow\iota$, where φ is the same as in Corollary 1.4 and ω' is injective for all spaces. As a consequence, ω is injective for all metrisable spaces and bijective for all proper geodesic metrisable spaces. This partially answers the problems posed in [8]. Using the bijectivity result, we give a calculation of the above-mentioned invariants of finitely branching trees in Section 3. Finally, we conclude the paper with some remarks concerning non-metrisable spaces in Section 4.

2. Main Results

We require a special case of the underspill principle.

Lemma 2.1. *Let X be a standard coarse space and $\mathcal{A}$ an internal subset of ${}^*\mathcal{C}_X$. If $\mathcal{A}$ contains all $E\in{}^*\mathcal{C}_X$ with $\sim_X\subseteq E$, then it contains ${}^*E$ for some $E\in\mathcal{C}_X$.*

Proof. Suppose, on the contrary, that ${}^*E\notin\mathcal{A}$ for any $E\in\mathcal{C}_X$. For each $E\in\mathcal{C}_X$, consider the internal set

$$\mathcal{F}_E:=\{\,F\in{}^*\mathcal{C}_X\mid{}^*E\subseteq F\notin\mathcal{A}\,\}.$$

Since the family $\{\,\mathcal{F}_E\mid E\in\mathcal{C}_X\,\}$ has the finite intersection property, its intersection $\bigcap_{E\in\mathcal{C}_X}\mathcal{F}_E$ is non-empty by the saturation principle. Let F be an element of the intersection. Then $F\in{}^*\mathcal{C}_X\setminus\mathcal{A}$ and $\sim_X\subseteq F$. □

The ε-equivalence admits the following nonstandard characterisation, a generalisation of [6, Lemma 3.8] to coarse spaces.

Lemma 2.2. *Let (X,ξ) be a pointed coarse space in $\mathscr{C}$ and $s,t\in S(X,\xi)$. The following are equivalent:*

(1) $s\equiv^{\varepsilon}_{X,\xi}t$;
(2) ${}^*s(i)\equiv^{\iota}_{X,\xi}{}^*t(j)$ *for all* $i,j\in{}^*\mathbb{N}\setminus\mathbb{N}$;
(3) ${}^*s(i)\equiv^{\iota}_{X,\xi}{}^*t(j)$ *for some* $i,j\in{}^*\mathbb{N}\setminus\mathbb{N}$.

Proof. (1) ⇒ (3): Let E be a witness of $s\equiv^{\varepsilon}_{X,\xi}t$. Let B be an internal bounded subset of ${}^*X$ such that ${}^*X\setminus B\subseteq\mathrm{INF}(X,\xi)$ by the saturation principle. (In fact, it suffices to apply the enlargement principle, a weaker version of saturation. See also [7, Lemma 2.5].) By transfer, there exists an $N\in{}^*\mathbb{N}$ such that $\{\,{}^*s(i)\,\}_{i\geq N}$ and $\{\,{}^*t(i)\,\}_{i\geq N}$ are contained in the same internally ${}^*E$-connected component of ${}^*X\setminus B$. Choose an $i\in{}^*\mathbb{N}\setminus\mathbb{N}$ so that $i\geq N$. Then there exists an internal ${}^*E$-chain in ${}^*X\setminus B$ that connects ${}^*s(i)$ and ${}^*t(i)$. Such a chain witnesses that ${}^*s(i)\equiv^{\iota}_{X,\xi}{}^*t(i)$.

(3) ⇒ (2): Let $i,j\in{}^*\mathbb{N}\setminus\mathbb{N}$ with $i\leq j$. Since s is coarse, the sequence $\{\,{}^*s(k)\,\}_{k=i}^{j}$ is a macrochain in $\mathrm{INF}(X,\xi)$, so ${}^*s(i)\equiv^{\iota}_{X,\xi}{}^*s(j)$. Similarly, we have that ${}^*t(i)\equiv^{\iota}_{X,\xi}{}^*t(j)$. The desired implication is now obvious.

(2) ⇒ (1): Suppose that ${}^*s(i) \equiv^{\iota}_{X,\xi} {}^*t(j)$ for some $i, j \in {}^*\mathbb{N}\setminus\mathbb{N}$, i.e., there exists an internal hyperfinite sequence $\{u_k\}_{k\le n}$ in $\mathrm{INF}(X,\xi)$ such that $u_0 = {}^*s(i)$, $u_n = {}^*t(j)$, and $u_k \sim_X u_{k+1}$ holds for all $k < n$. Consider the following internal subset of ${}^*\mathcal{C}_X$:

$$\mathcal{A} := \left\{ E \in {}^*\mathcal{C}_X \;\middle|\; \begin{array}{c} (u_k, u_{k+1}) \in E \cap E^{-1} \text{ for all } k < n, \\ ({}^*s(k), {}^*s(k+1)) \in E \cap E^{-1} \text{ for all } k \in {}^*\mathbb{N}, \\ ({}^*t(k), {}^*t(k+1)) \in E \cap E^{-1} \text{ for all } k \in {}^*\mathbb{N} \end{array} \right\}.$$

Since s and t are bornologous, ${}^*s(k) \sim_X {}^*s(k+1)$ and ${}^*t(k) \sim_X {}^*t(k+1)$ hold for all $k \in {}^*\mathbb{N}$. Hence $\mathcal{A}$ contains all $E \in {}^*\mathcal{C}_X$ with $\sim_X \subseteq E$. By Lemma 2.1, there exists a standard $E \in \mathcal{C}_X$ such that ${}^*E \in \mathcal{A}$. Fix such an E.

Now, let B be a bounded subset of X containing ξ. Let $N = \min\{i, j\}$. Since s and t are proper, $S := \{{}^*s(k)\}_{k\ge N}$ and $T := \{{}^*t(k)\}_{k\ge N}$ are contained in $\mathrm{INF}(X,\xi) \subseteq {}^*X \setminus {}^*B$. By the choice of E, any two points of S can be connected by an internal ${}^*E$-chain in $S \subseteq {}^*X \setminus {}^*B$; any two points of T can be connected by an internal ${}^*E$-chain in $T \subseteq {}^*X \setminus {}^*B$; and the points ${}^*s(i) \in S$ and ${}^*t(j) \in T$ can be connected by the internal ${}^*E$-chain $\{u_i\}_{i\le n}$ in $\mathrm{INF}(X,\xi) \subseteq {}^*X \setminus {}^*B$. Combining them, any two points of $S \cup T$ can be connected by an internal ${}^*E$-chain in ${}^*X \setminus {}^*B$. In other words, there exists an $N \in {}^*\mathbb{N}$ such that $\{{}^*s(k)\}_{k\ge N}$ and $\{{}^*t(k)\}_{k\ge N}$ are contained in the same internal ${}^*E$-connected component of ${}^*X \setminus {}^*B$. By transfer, there exists an $N \in \mathbb{N}$ such that $\{s(k)\}_{k\ge N}$ and $\{t(k)\}_{k\ge N}$ are contained in the same E-connected component of $X \setminus B$. Because B is arbitrary and E does not depend on B, we have that $s \equiv^{\varepsilon}_{X,\xi} t$. □

Remark 2.3. The use of the saturation principle (Lemma 2.1) in the implication (2) ⇒ (1) is avoidable if (X,ξ) is metrisable. Suppose that ${}^*s(i) \equiv^{\iota}_{X,\xi} {}^*t(j)$ for some $i, j \in {}^*\mathbb{N} \setminus \mathbb{N}$, i.e., there exists an internal hyperfinite sequence $\{u_k\}_{i\le n}$ in $\mathrm{INF}(X,\xi)$ such that $u_0 = {}^*s(i)$, $u_k = {}^*t(j)$, and ${}^*d_X(u_k, u_{k+1})$ is finite for all $k < n$. The maximum $L := \max_{k<n} {}^*d_X(u_k, u_{k+1})$ exists by transfer, and is finite. Since s and t are bornologous, the suprema $M := \sup_{k\in\mathbb{N}} d_X(s(k), s(k+1))$ and $N := \sup_{k\in\mathbb{N}} d_X(k(i), k(i+1))$ exist. Fix a standard $K > 0$ such that $L, M, N \le K$ and let $E := \{(x,y) \in X \times X \mid d_X(x,y) \le K\} \in \mathcal{C}_X$. The remaining proof is the same as above.

Theorem 2.4. *Let (X,ξ) be a pointed coarse space in $\mathscr{C}$. The map $\omega'_{(X,\xi)} \colon \varepsilon(X,\xi) \to \iota(X,\xi)$ defined by*

$$\omega'_{(X,\xi)} [s]_{\equiv^{\varepsilon}_{X,\xi}} := [{}^*s(i)]_{\equiv^{\iota}_{X,\xi}}, \; i \in {}^*\mathbb{N} \setminus \mathbb{N}$$

is well-defined, injective and natural in (X,ξ).

Proof. The well-definedness follows from the implication (1) ⇒ (2) and the injectivity does from the implication (3) ⇒ (1) of Lemma 2.2. The naturality is trivial by definition. □

Theorem 2.5. *$\omega = \omega' \circ (\varphi \upharpoonright \mathscr{C})$.*

Proof. Trivial. □

Theorem 2.6. *The map $\omega_{(X,\xi)} \colon \sigma(X,\xi) \to \iota(X,\xi)$ is injective for all metrisable spaces (X,ξ) in $\mathscr{C}$.*

Proof. Combine Corollary 1.6, Theorem 2.4 and Theorem 2.5. □

This answers [8, Problems 5.3 and 5.5] for metrisable spaces, and gives a partial answer to [8, Problem 5.4].

Corollary 2.7. *The map $\omega_{(X,\xi)} \colon \sigma(X,\xi) \to \iota(X,\xi)$ is bijective for all proper geodesic metrisable spaces (X,ξ) in $\mathscr{C}$.*

Corollary 2.8. *Let (X,ξ) be a proper geodesic metrisable space in $\mathscr{C}$. Then $\sigma(X,\xi) \cong \varepsilon(X,\xi) \cong \iota(X,\xi) \cong \mathrm{Ends}(X)$.*

3. Case study

Throughout this section, we identify a graph with its geometric realisation. We also assume that $\mathscr{C}$ contains (asymorphic copies of) all pointed coarse spaces we consider. Let $G := (V, E)$ be a locally finite connected graph. The metric $d_G \colon V \times V \to \mathbb{R}_{\ge 0}$ is defined as usual:

$$d_G(v, w) := \text{the length of the shortest path between } v \text{ and } w.$$

This makes the graph G a proper geodesic metric space where each edge is isometric to the unit interval $[0, 1]$. The vertex set V is coarsely equivalent to the whole graph G, so we obtain the isomorphisms:

$$\begin{array}{ccccccc} & & \sigma(V, v) & \xrightarrow[\cong]{\varphi} & \varepsilon(V, v) & \xrightarrow[\cong]{\omega'} & \iota(V, v) \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathrm{Ends}(G) & \xrightarrow[\cong]{} & \sigma(G, v) & \xrightarrow[\cong]{\varphi} & \varepsilon(G, v) & \xrightarrow[\cong]{\omega'} & \iota(G, v) \end{array}$$

where the vertical maps are induced by the inclusion $V \hookrightarrow G$.

Theorem 3.1. *Let T be a finitely branching tree with root r. Then $\iota(T, r)$ is equipotent to the set of all infinite branches of T, and so are $\sigma(T, r)$, $\varepsilon(T, r)$ and* $\mathrm{Ends}(T)$.

Proof. We denote the set of all infinite branches by $[T]$. It suffices to prove that $[T] \cong \iota(T, r)$. Define a map $\psi\colon [T] \to \iota(T, r)$ as follows. Let $f \in [T]$. Choose an arbitrary $i \in {}^*\mathbb{N} \setminus \mathbb{N}$. By transfer, ${}^*f(i) \in {}^*T$ and $d_T(r, {}^*f(i)) = i$, so ${}^*f(i) \in \mathrm{INF}(T, r)$. Now, define $\psi(f) := [{}^*f(i)]_{\equiv^{\iota}_{T,r}}$, where the right hand side is independent of the choice of i by the following fact.

Claim. ${}^*f(i) \equiv^{\iota}_{T,r} {}^*f(j)$ for all $i, j \in {}^*\mathbb{N} \setminus \mathbb{N}$.

Proof. We may assume without loss of generality that $i \le j$. The sequence $\{\, {}^*f(k) \,\}_{k=i}^{j}$ witnesses that ${}^*f(i) \equiv^{\iota}_{T,r} {}^*f(j)$.

Claim. ψ is surjective.

Proof. Let $x \in \mathrm{INF}(T, r)$. By transfer, there exists a (unique) hyperfinite sequence $\{\, t_i \,\}_{i \le n}$ in ${}^*T$ such that $t_0 = r$, $t_n = x$ and t_{i+1} is a child of t_i for each $i < n$. Notice that ${}^*d_T(r, t_i) = i$ and ${}^*d_T(r, x) = n \in {}^*\mathbb{N} \setminus \mathbb{N}$. For each (standard) $i \in \mathbb{N}$, since T is finitely branching, the set $T_i := \{\, t \in T \mid d_T(r, t) = i \,\}$ is finite, so $t_i \in {}^*T_i = T_i$. Hence the map $f\colon \mathbb{N} \to T$ defined by $f(i) := t_i$ is an infinite branch of T.

Let us verify that $\psi(f) = [x]_{\equiv^{\iota}_{X,\xi}}$. First note that ${}^*f(i) = t_i$ holds for all $i \in \mathbb{N}$ by definition. By the countable saturation, ${}^*f(i) = t_i$ holds for some (infinite) $i \in {}^*\mathbb{N} \setminus \mathbb{N}$ with $i \le n$. The sequence $\{\, t_k \,\}_{k=i}^{n}$ witnesses that ${}^*f(i) \equiv^{\iota}_{T,r} x$. Therefore $\psi(f) = [x]_{\equiv^{\iota}_{T,r}}$. See also the proof of [8, Theorem 4.2].

Claim. ψ is injective.

Proof. Let $\{\, t_i \,\}_{i \le n}$ be a macrochain in $\mathrm{INF}(T, r)$ and $n \in \mathbb{N}$. Any two adjacent nodes of $\{\, t_i \,\}_{i \le n}$ have the same ancestor of level n. (Otherwise, the distance ${}^*d_T(t_i, t_{i+1})$ would be infinite, which is a contradiction.) By the transferred induction principle, all nodes of $\{\, t_i \,\}_{i \le n}$ have the same ancestor of level n.

Now, let $f, g \in [T]$ and suppose $\psi(f) = \psi(g)$, i.e., there exists a macrochain in $\mathrm{INF}(T, r)$ connecting ${}^*f(i)$ and ${}^*g(i)$ for $i \in {}^*\mathbb{N} \setminus \mathbb{N}$. All nodes of the macrochain have the same ancestors of standard level. In particular, ${}^*f(i)$ and ${}^*g(i)$ have the same ancestors of standard level. In other words, $f(j) = g(j)$ for all standard $j \in \mathbb{N}$. Hence $f = g$. □

Example 3.2. Let G be a finitely generated group endowed with a finite generating set S. The Cayley graph $\Gamma_S(G)$ is locally finite and connected; hence $\varepsilon(G, e_G) \cong \varepsilon(G, e_G) \cong \iota(G, e_G) \cong \mathrm{Ends}(\Gamma_S(G)) =: \mathrm{Ends}(G)$.

(1) Consider the Abelian group $\mathbb{Z}$ with the generating set $\{\, \pm 1 \,\}$. The infinite part $\mathrm{INF}(\mathbb{Z}, 0)$ consists of positive and negative infinite hypernatural numbers. Two infinite points are ι-equivalent if and only if they have the same sign. To see the "only if" part, let $\{\, x_i \,\}_{i \le n}$ be a macrochain in $\mathrm{INF}(\mathbb{Z}, 0)$. The internal set $A := \{\, i \in {}^*\mathbb{N} \mid \mathrm{sgn}(x_i) = \mathrm{sgn}(x_0) \text{ or } i > n \,\}$ contains 0, and is closed under successor. By the transferred induction principle, $A = {}^*\mathbb{N}$, i.e., the components of $\{\, x_i \,\}_{i \le n}$ have the same sign. On the other hand, the "if" part is evident. Thus we have that $|\sigma(\mathbb{Z}, 0)| = |\varepsilon(\mathbb{Z}, 0)| = |\iota(\mathbb{Z}, 0)| = |\mathrm{Ends}(\mathbb{Z})| = 2$. In fact, the Cayley graph of $\mathbb{Z}$ with respect to $\{\, \pm 1 \,\}$ can be considered as a tree, where the root has exactly two children, and each node other than the root has exactly one child. Such a tree has exactly two infinite branches.

(2) Let S be a finite set of cardinality ≥ 2 and F_S the free group generated by S. The Cayley graph $\Gamma_{S \cup S^{-1}}(F_S)$ is a finitely branching tree with the root e_{F_S}. The root has $2|S|$ children and each node other than the root has $2|S| - 1$ children. So $\Gamma_{S \cup S^{-1}}(F_S)$ has exactly $2^{\aleph_0}$ infinite branches. Hence $|\sigma(F_S, e_{F_S})| = |\varepsilon(F_S, e_{F_S})| = |\iota(F_S, e_{F_S})| = |\mathrm{Ends}(F_S)| = 2^{\aleph_0}$.

The following examples cannot be obtained as a Cayley graph of a finitely generated group.

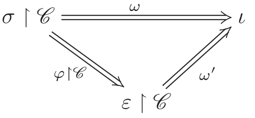
Example 3.3. Let $\Sigma := \{ a, b \}$ be an alphabet. The set $\Sigma^{<\omega}$ of all finite words over Σ can be considered as a binary tree with the root $\emptyset$. Similarly to the above example F_S, we have that $|\sigma(\Sigma^{<\omega}, \emptyset)| = |\varepsilon(\Sigma^{<\omega}, \emptyset)| = |\iota(\Sigma^{<\omega}, \emptyset)| = |\mathrm{Ends}(\Sigma^{<\omega})| = 2^{\aleph_0}$. On the other hand, the subtree

$$T := \{ a^n b^m \in \Sigma^{<\omega} \mid n, m \in \mathbb{N} \}$$

has exactly $\aleph_0$ infinite branches. Hence $|\sigma(T, \emptyset)| = |\varepsilon(T, \emptyset)| = |\iota(T, \emptyset)| = |\mathrm{Ends}(T)| = \aleph_0$. This affirmatively answers [8, Problem 5.2], which asks if $\iota(X, \xi)$ is countably infinite for some pointed coarse space (X, ξ) in $\mathscr{C}$ (provided that $\mathscr{C}$ is sufficiently rich).

4. Concluding remarks

In summary, the coarsely invariant functors $\sigma \upharpoonright \mathscr{C}, \varepsilon \upharpoonright \mathscr{C}, \iota \colon \mathscr{C} \subseteq \mathbf{Coarse}_* \to \mathbf{Sets}$ are related with the following commutative diagram of natural transformations:



where $\varphi \colon \sigma \Rightarrow \varepsilon$ is bijective on $\mathbf{Metr}_*$, ω' injective on $\mathscr{C}$, and ω injective on $\mathbf{Metr}_* \cap \mathscr{C}$. Moreover, ω' and ω are bijective for proper geodesic metrisable spaces in $\mathscr{C}$. This enables us to calculate, for a large class of spaces, the invariants σ, ε and Ends in an intuitive way, as we demonstrated in Section 3.

Our proof of the injectivity of ω does not apply to general pointed coarse spaces, because the proof of the implication from $\equiv^{\varepsilon}_{X,\xi}$ to $\equiv^{\sigma}_{X,\xi}$ depends on the metrisability as we pointed out in Remark 1.7. Hence, the following problem remains open.

Problem 4.1. Is the map $\omega_{(X,\xi)} \colon \sigma(X, \xi) \to \iota(X, \xi)$ injective for all non-metrisable spaces (X, ξ) in $\mathscr{C}$?

This is equivalent to the following purely standard problem.

Problem 4.2. Is the map $\varphi_{(X,\xi)} \colon \sigma(X, \xi) \to \varepsilon(X, \xi)$ injective for all non-metrisable spaces (X, ξ) in $\mathscr{C}$?

References

[1] M. R. Bridson and A. Häfliger, *Metric Spaces of Non-Positive Curvature*. Springer, 1999.
[2] C. C. Chang and H. J. Keisler, *Model Theory*, 3rd ed. North-Holland, 1990.
[3] M. DeLyser, B. LaBuz, and B. Wetsell, “A coarse invariant for all metric spaces,” *Mathematics Exchange*, vol. 8, no. 1, pp. 7–13, 2011.
[4] M. DeLyser, B. LaBuz, and M. Tobash, “Sequential ends of metric spaces,” 2013, arXiv:1303.0711.
[5] H. Freudenthal, “Über die Enden topologischer Räume und Gruppen,” *Mathematische Zeitschrift*, vol. 33, no. 1, pp. 692–713, 1931.
[6] I. Goldbring, “Ends of groups: a nonstandard perspective,” *Journal of Logic & Analysis*, vol. 3, no. 7, pp. 1–28, 2011.
[7] T. Imamura, “Nonstandard methods in large-scale topology,” *Topology and its Applications*, vol. 257, pp. 67–84, 2019.
[8] ——, “A nonstandard invariant of coarse spaces,” *The Graduate Journal of Mathematics*, vol. 5, no. 1, pp. 1–8, 2020.
[9] ——, “Another view of the coarse invariant σ,” *Mathematics Exchange*, vol. 14, no. 1, pp. 14–24, 2020.
[10] B. Miller, L. Stibich, and J. Moore, “An invariant of metric spaces under bornologous equivalences,” *Mathematics Exchange*, vol. 7, no. 1, pp. 12–19, 2010.
[11] I. Protasov and M. Zarichnyi, *General Asymptology*. VNTL Publishers, 2007, vol. 12.
[12] J. Roe, *Lectures on Coarse Geometry*. American Mathematical Society, 2003.

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