ON THE PERSISTENCE PAIRS AND STRONG CONNECTEDNESS OF DISCRETE MORSE FUNCTIONS ON GRAPHS

CHONG ZHENG

Abstract. This article is an abstract of my presentation at RIMS, June, 2021. We show the relationship between the number of critical simplices, and Betti numbers with persistent pairs , which is an improvement to the discrete Morse inequalities. Furthermore, we prove that given two discrete Morse functions f_1, f_2 on a simple graph, the number of strongly connected number between f_1, f_2 is the Betti number. This result is also an necessary condition to the problem posed in [4] in case of graphs.

1. Introduction

Introduced by R. Forman [2] for the first time, discrete Morse theory also known as Forman theory is a combinatorial version of normal Morse theory. The idea of discrete Morse theory is to collapse pairs of simplices with adjacent dimensions. In the process of collapsing, the homotopy type of given space remains consistent. Similar to elementary collapse theory invented by J. Whitehead [8], discrete Morse theory establishs a rule, which is called *discrete Morse function*, to accomplish the goal. Since discrete Morse theory allows us to reduce the number of simplices in space by collapsing the pairs in the *gradient vector field* induced by a given discrete Morse function, it is widely used for the computation of homology of spaces. An element of a gradient vector field consists of two non-critical simplices with codimension one. Main theorems of discrete Morse theory were established on general CW complexes at first. This approach has proven to be useful to study the topology of spaces. Applications of discrete Morse theory are miscellaneous in many areas, such as calculating the persistent homology [6].

King-Knudson-Kostoc [4] introduced the birth and death theory of discrete Morse theory and showed some interesting applications in data analysis. Given a CW complex and a series of discrete Morse functions on it, they defined a "connectedness" relationship between two critical simplices in two different gradient vector fields. A series of organized discrete Morse functions on the same cellulation cut space into slices in which critical simplices appear and disappear. Connecting critical simplices in different slides by their "connectedness", one can glue those fragments to reconstruct given space. As time t varies, they traced critical points via a discrete version

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of bifurcation diagram. In this diagram one can obtain the birth and death information of each critical simplex. The "persistent time" of each critical simplex reflects the geometric information of the space under the given discrete Morse functions.

In this paper, we use another kind of pair of simplices to describe the birth and death of homology group generators. Let X be a simplicial complex and f a discrete Morse function on it. We first build a natural filtration of X by a sequence of ascending real number values of f. Then given a homology class $[h]$ of the filtration, we use a pair of simplices (σ, τ), which is called a *persistence pair*, to describe the birth and death of $[h]$. Different from the pairs in a gradient vector field, a persistence pair consists of two critical simplices or infinity in some case. We will show some fundamental facts about persistent pairs which can be used in discrete Morse theory. Furthermore, in response to the problem

Problem. *"Are there conditions that guarantee that a critical cell in one slice is strongly connected to at most one critical cell in the next slice?"*

in [4], we use persistence pairs to give a positive solution when X is a finite simple graph. Also, the number of q-dimensional strong connection pairs is the q-th Betti number of the graph.

2. Definitions and results

We first introduce basic definitions of discrete Morse theory. One can refer [7] and [5] for more details and theorems.

Definition 2.1. A *discrete Morse function* on M is a real-valued function

$$f : K \longrightarrow \mathbb{R}$$

satisfying for all $\sigma \in K_p$

(1) $\#\{\tau^{(p+1)} \prec \sigma | f(\tau) \leq f(\sigma)\} \leq 1$,
(2) $\#\{v^{(p-1)} \succ \sigma | f(\sigma) \leq f(v)\} \leq 1$.

Definition 2.2. Given a discrete Morse function f on M, we say that $\sigma \in K_p$ is a *critical* p-dimensional simplex (of f) if

(1) $\#\{\tau^{(p+1)} \prec \sigma \mid f(\tau) \leq f(\sigma)\} = 0$;
(2) $\#\{v^{(p-1)} \succ \sigma \mid f(\sigma) \leq f(v)\} = 0$.

Definition 2.3. For $c \in \mathbb{R}$, we define the *sub-level complex*

$$K(c) = \bigcup_{\sigma \in K, f(\sigma) \leq c} \bigcup_{\tau \leq \sigma} \tau.$$

$K(c)$ is the sub-complex of M consisting of all simplices τ with $f(\tau) \leq c$ and their faces.

Given a discrete Morse function f and its corresponding gradient vector field V, we have a natural filtration consisting of subcomplexes of K.

Definition 2.4. Let $c_1 < c_2 < \cdots < c_n$ be a sequence of real numbers and $K(c_i)$ be the sub-level complex at value c_i. We say that

$$K(c_1) \subset K(c_2) \subset \cdots \subset K(c_n)$$

is a *filtration* of K. For convenience, if $f(\sigma) = c$, we both use $K(c)$ and $K(\sigma)$ to represent the sub-level complex at value c or $f(\sigma)$. It is easy to see that when $c_1 < \min\{f(\sigma) \mid \sigma \in K\}$, $K(c_1) = \emptyset$ and when $c_n \geq \max\{f(\sigma) \mid \sigma \in K\}$, $K(c_n) = K$.

We always take the filtration ending at the whole space as following

$$K(c_1) \subset K(c_2) \subset \cdots \subset K(c_n) = K.$$

Definition 2.5. A pair of simplices (σ, τ) is said to be a *persistence pair* if there is a homology class $[h]$ that is born at σ and dies at τ. If $[h]$ is born at σ and does not die eventually, we say that σ is paired with infinity, denoted (σ, ∞).

Let $[h]$ be a homology class, we say the *dimension* of $[h]$ is the dimension of simplices of its representative chain $\sum_i n_i \sigma_i$. It is not hard to find the birth simplex of a cycle if we arrange the real number in ascending order. The death simplex of a cycle, however, may be tricky sometimes because a cycle may split into more than one cycles. Note that the birth simplex has not to be a face of the death simplex, unless the dimension of the birth simplex is 0. Hence, we use *the elder rule* to describe the persistence of a cycle precisely.

The elder rule is that the death simplex is always paired with the youngest birth simplex.

The following statements about persistent pairs hold under the conditions of discrete Morse function and the elder rule.

Lemma 2.6. *Let f be a discrete Morse function on K. Suppose that $[h]$ is a q-dimensional homology class with persistent pair (σ, τ). Then*

(1) $\dim \sigma = q$;
(2) $\dim \tau = q + 1$;
(3) $[h]$ *is represented by a cycle* $z = \sum_i \sigma_i n_i$ *such that* $\sigma = \sigma_j$ *for some* j.

Lemma 2.7. *Let f be a discrete Morse function on K. If $(\sigma^{(q)}, \tau)$ is a persistence pair of K then both σ and τ are f-critical.*

Hence we can classify critical simplices by the structure of their persistence pairs.

Let $\sigma^{(q)}$ be a critical simplex, then $\sigma^{(q)}$ is in exactly one of the three following persistence pairs

(1) a q-dimensional persistence pair $(\sigma^{(q)}, \tau^{(q+1)})$;
(2) a $(q-1)$-dimensional persistence pair $(\alpha^{(q-1)}, \sigma^{(q)})$;
(3) a q-dimensional persistence pair consisting of infinity $(\sigma^{(q)}, \infty)$.

Let P_q be the set of persistence pairs whose birth simplices have dimension q. Let $\hat{P}_q$ be the set of persistence pairs whose birth simplices have dimension q and death simplices are not infinity. Then $\hat{P}_q \subset P_q$. Note that the q-th Betti number $\beta_q = \#P_q - \#\hat{P}_q$.

Theorem 2.8. *The Morse inequality can be re-written as an equation*

$$C_q = \beta_q + \#\hat{P}_{q-1} + \#\hat{P}_q.$$

Moreover, the strong and weak Morse inequalities become more explicit as well.

Theorem 2.9. *Let* $\dim(X) = K$. *Then*

$$C_0 - C_1 + C_2 - \cdots \pm C_k = \beta_0 - \beta_1 + \cdots + \beta_k.$$

Also, for $i = 0, 1, \cdots, \dim(X)$,

$$C_i - C_{i-1} + \cdots \pm C_0 = \beta_i - \beta_{i-1} + \cdots \pm \beta_0 + \#\hat{P}_i.$$

The definition of connectedness between the critical simplices in different gradient vector field is introduced by King-Knudson-Kostoc [4]. We fix it slightly and introduce it as followings.

Definition 2.10. Suppose that α and β are k-simplices of M and α is critical for V_i and β is critical for V_j, $j \neq i$. We say that α is *connected* to β if there is a k-simplex γ and a V_i path $\alpha, \cdots, \gamma$ of k- and $(k-1)$-cells and a V_j path $\gamma, \cdots, \tau^{(k-1)}$ of k- and $(k+1)$-cells to a face $\tau^{(k-1)} \prec \beta$, denoted $\alpha \longrightarrow \beta$.

We say that α is *strongly connected* to β if α is connected to β and β is connected to α, denoted $\alpha \longleftrightarrow \beta$.

Recall the problem posed in [4], we give the following theorem as a solution in the case of graph.

Theorem 2.11. *Let* $G = (\mathcal{V}, \mathcal{E})$ *be a graph, and*

$$f_1, f_2 : \mathcal{V} \cup \mathcal{E} \to \mathbb{R}$$

be discrete Morse functions, $q = 0, 1$. *Then for any* f_1*- critical* $\sigma_1^{(q)}$, *there is at most one* f_2*-critical simplex* $\sigma_2^{(q)}$ *that is strongly connected to* $\sigma_1^{(q)}$.

Also, let $A_q^{f_1, f_2}(G)$ *be the number of* q*-dimensional strongly connection pairs of* f_1 *and* f_2 *critical simplices in* G. *Then*

$$A_q^{f_1, f_2}(G) \geq \beta_q(G).$$

For a graph, we mean a finite graph $G = (\mathcal{V}, \mathcal{E})$ with no multi-edges and loops.

M. Chari and M. Joswig [1] introduced a method to represent the set of discrete Morse functions as a simplicial complex.

Definition 2.12. Let G be a graph. Construct a simplicial complex $\Delta(G)$ as follows: the vertices of $\Delta(G)$ are given by the edges of G and faces are all directed forests which are subgraphs of G.

Lemma 2.13. [1] *The set of discrete Morse functions on a graph* G *is in one-to-one correspondence with the set of rooted forests of* G.

We show the relationship between the path-connectedness on the simplicial complex above and connectedness of discrete Morse functions defined above. This is also an approach to represent the relationship among all discrete Morse functions of a given space.

Definition 2.14. Let X be a simplicial complex. We say that one-dimensional simplices σ_1 and σ_2 of X are **one-dimensional simplicial path-connected in** X if there is a path consisting of one-dimensional simplicial simplices of X that joins σ_1 and σ_2.

Theorem 2.15. *Let $G = (\mathcal{V}, \mathcal{E})$ be a graph, and*

$$f_1, f_2 : \mathcal{V} \cup \mathcal{E} \to \mathbb{R}$$

be discrete Morse functions. Suppose that σ_1 is f_1-critical and σ_2 is f_2-critical, then the followings hold.

(1) $\sigma_1 \longleftrightarrow \sigma_1$ *if and only if* $\sigma_1 \notin \Delta_{f_2}(G)$.

(2) *Assume that* $\sigma_1, \sigma_2 \in \Delta_{f_1}(G) \cup \Delta_{f_2}(G)$, *then* $\sigma_1 \longleftrightarrow \sigma_2$ *if and only if* σ_1 *and* σ_2 *are one-dimensional simplicial path-connected in* $\Delta_{f_1}(G) \cup \Delta_{f_2}(G)$.

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Faculty of Science and Engineering, Waseda University, Ohkubo, Shinkuju-ku, Tokyo, 169-8555 Japan

Email address: engzhengchong@ruri.waseda.jp