COMBINATORIAL ROBOT MOTION PLANNING ON FINITE SPACES

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1. Introduction

This article summarizes a combinatorial approach for the computation of topological complexity based on the existing study [Tan18].

The robot motion planning problem considers the assignment of a path from an initial position to a final position. In the context of topology, this problem can be represented as follows: For a space X, the (free) path space

$$X^I = \{\gamma\colon\ I = [0,1] \to X\}$$

consists of paths in X equipped with the compact-open topology. The path fibration $\pi\colon\ X^I \to X \times X$ is defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$. The *motion planning algorithm* in X is a continuous section of π, that is, a continuous map

$$s\colon\ X \times X \to X^I$$

such that $\pi \circ s$ agrees with the identity map $\mathrm{id}_{X\times X}$. For each pair (x, y) of points (initial position x and final position y) in X, a motion planning algorithm presents a path connecting x and y. The topological motion planning problem questions whether we can construct a motion planning algorithm.

Theorem 1.1 ([Far03])**.** *A space X admits a motion planning algorithm if and only if X is contractible.*

The above theorem indicates that motion planning algorithms cannot be constructed globally in a non-contractible space. However, we may have local motion planning algorithms, that is, sections of the path fibration π on a subset of $X \times X$. If we have motion planning algorithms $s_i\colon\ U_i \to X^I$, $i = 0, \cdots, n$ such that $U_0 \cup U_1 \cup \cdots \cup U_n = X \times X$, a robot can move from x to y following algorithm s_i with $(x, y) \in U_i$.

Farber introduced a numerical invariant $\mathrm{TC}(X)$, which is one less than the minimum number of local motion planning algorithms for robotic motion design in X [Far03]. However, $\mathrm{TC}(X)$ is not easy to compute. Indeed, the topological complexity of a Klein bottle has been computed in recent years [CV17, Dra17, IST19].

In this paper, we introduce a discrete method for calculating the topological complexity for finite simplicial complexes using finite T_0-spaces or posets. Because a finite space has only a finite number of open sets, the topological complexity of a finite space can be theoretically computed in finite steps of discrete operations.

2. Topological complexity

Throughout this paper, we deal only with path-connected spaces. Let $\pi\colon X^I \to X \times X$ be the path fibration defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$. A *motion planning algorithm* on a subset U is a local section of π, i.e., a continuous map $s\colon U \to X^I$ such that $\pi \circ s$ agrees with the inclusion $U \hookrightarrow X \times X$.

Definition 2.1. For a space X, the *topological complexity* $\mathrm{TC}(X)$ is defined as the minimum number n such that we have $n+1$ open sets $U_0, \cdots, U_n$ covering $X \times X$, where each U_i admits a local motion planning algorithm. If no such number exists, we set $\mathrm{TC}(X) = \infty$.

The above definition adopts one less than the minimal size of open sets covering the product space with motion planning algorithms. As another option, we can consider arbitrarily subsets covering or separating the product space instead of open sets.

Definition 2.2. For a space X, the *generalized topological complexity* $\mathrm{TC}_g(X)$ is defined as the minimum number n such that we have $n+1$ subsets $U_0, \cdots, U_n$ such that

$$X \times X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n, \quad U_i \cap U_j = \emptyset, i \neq j,$$

where each U_i admits a local motion planning algorithm. If no such number exists, we set $\mathrm{TC}_g(X) = \infty$.

If $\mathrm{TC}(X) = n$ with open sets $U_0, \cdots, U_n$ covering $X \times X$ and motion planning algorithms $s_i\colon U_i \to X^I$, we have subsets

$$V_i = U_i - (U_0 \cup U_1 \cup \cdots \cup U_{i-1})$$

in $X \times X$. The product space $X \times X$ is decomposed by V_i, and $s_i|_{V_i}$ provides a motion planning algorithm on V_i. Hence, the inequality $\mathrm{TC}_g(X) \leq \mathrm{TC}(X)$ always holds. The converse inequality also holds for CW complexes.

Theorem 2.3 ([Gar19])**.** *For a CW complex X, we have* $\mathrm{TC}(X) = \mathrm{TC}_g(X)$.

We can consider the topological complexity for maps.

Definition 2.4. Let $f\colon Y \to X \times X$ be a continuous map. The topological complexity $\mathrm{TC}(f)$ of f is defined as the minimum number n such that we have $n+1$ open sets $U_0, \cdots, U_n$ covering Y, where each U_i admits a continuous map $s\colon U_i \to X^I$ with $\pi \circ s = f|_{U_i}$. If no such number exists, we set $\mathrm{TC}(f) = \infty$.

For the identity map $\mathrm{id}_{X\times X}\colon X \times X \to X \times X$, the topological complexity $\mathrm{TC}(\mathrm{id}_{X\times X})$ agrees with $\mathrm{TC}(X)$.

Proposition 2.5 ([Far03])**.** *The topological complexity for spaces has the following properties:*

(1) TC *is a homotopy invariant, i.e.,* $X \simeq Y$ *implies* $\mathrm{TC}(X) = \mathrm{TC}(Y)$.
(2) $\mathrm{TC}(X) = 0$ *if and only if* X *is contractible.*
(3) $\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X)$*, where* cat *denotes the LS-category.*
(4) $\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y)$ *for ANR spaces* X, Y.

A useful cohomological lower bound for TC is well-known. A *zero-divisor* of the cup product is an element in the kernel of the cup-product $\cup\colon H^*(X) \otimes H^*(X) \to H^*(X)$. The zero-divisor-cup-length $\mathrm{zcl}(X)$ is the maximal number n of zero-divisors $\alpha_1, \cdots, \alpha_n$ such that $\prod \alpha_i \neq 0$ in $H^*(X) \otimes H^*(X)$.

Theorem 2.6 ([Far03])**.** $\mathrm{zcl}(X) \leq \mathrm{TC}(X)$.

Example 2.7. The following are fundamental examples of topological complexity.

(1) $\mathrm{TC}(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even,} \end{cases}$ for an n-sphere S^n ([Far03]).
(2) $\mathrm{TC}(T^n) = n$ for the product $T^n = \prod_n S^1$ of circles ([Far03]).
(3) $\mathrm{TC}(B_k) = 2$ for the wedge $B_k = \vee_k S^1$ of circles when $k \geq 2$ ([Far04]).
(4) $\mathrm{TC}(\mathbb{C}\mathrm{P}^n) = 2n$ for an n-dimension complex projective space $\mathbb{C}\mathrm{P}^n$ ([FTY03]).
(5) $\mathrm{TC}(\Sigma_g) = \begin{cases} 2 & \text{if } g \leq 1, \\ 4 & \text{if } g \geq 2, \end{cases}$ for a compact orientable surface Σ_g with genus g ([Far03]).
(6) $\mathrm{TC}(K) = 4$ for a Klein bottle K ([CV17, Dra17, IST19]).

The calculations of topological complexity tend to be difficult. For example, although $\mathrm{TC}(\mathbb{R}\mathrm{P}^n) = n$ for a real projective space $\mathbb{R}\mathrm{P}^n$ with $n = 1, 3, 7$, it is difficult to find out a general formula for $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$. For $n \neq 1, 3, 7$, the topological complexity $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ is equal to the immersion dimension of $\mathbb{R}\mathrm{P}^n$ ([FTY03]).

3. Simplicial complexes and finite spaces

In this section, we study combinatorial homotopy theories on simplicial complexes and finite spaces.

3.1. **Simplicial complexes.** A *simplicial complex* K consists of the set $V(K)$ of vertices and the set $\Sigma(K)$ of simplices as a subset of the power set $2^{V(K)}$ satisfying the following face relation:

(1) The singleton $\{v\}$ is contained in $\Sigma(K)$ for any $v \in V(K)$.
(2) If $\tau \subset \sigma$ and $\sigma \in \Sigma(K)$, then $\tau \in \Sigma(K)$.

This study deals only with connected finite simplicial complexes (where $V(K)$ is finite).

For a simplicial complex K, let $|K|$ denote the geometric realization of K. This space is constructed by gluing the simplices along their boundaries. The geometric realization $|K|$ of a finite simplicial space K with $n+1$ vertices can be realized as a subcomplex of the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$: For $V(K) = \{v_0, v_1, \cdots, v_n\}$, we identify v_i with the i-th vertex of Δ^n. The space $|K|$ is constructed by taking the convex hull $|\sigma|$ of $\sigma = \{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$ if σ is a simplex in K.

For two simplicial complexes K and L, a *simplicial map* $f\colon K \to L$ is a map on vertices $f\colon V(K) \to V(L)$ sending a simplex σ of K to a simplex $f(\sigma)$ of L. A simplicial map $f\colon K \to L$ induces a continuous map $|f|\colon |K| \to |L|$ defined by $|f|(\sum_i t_i v_i) = \sum t_i f(v_i)$ for $t_i \in I$ and $v_i \in V(K)$.

3.2. **Homotopy theory of simplicial complexes.**

Definition 3.1 (Simplicial approximation)**.** Let $f\colon |K| \to |L|$ be a continuous map. A simplicial map $\varphi\colon K \to L$ is called a *simplicial approximation* to f if $f(x) \in |\sigma|$ implies $|\varphi|(x) \in |\sigma|$ for any $x \in |K|$ and $\sigma \in \Sigma(L)$.

The realization $|\varphi|$ of a simplicial approximation to f is homotopic to f because we have a homotopy $H(x,t) = t|\varphi|(x) + (1-t)f(x)$.

Definition 3.2. Let K be a simplicial complex. The *barycentric subdivision* $\mathrm{sd}(K)$ consists of barycenters of (realized) simplices of K as vertices. A simplex of $\mathrm{sd}(K)$ consists of barycenters $\{b_{\sigma_0}, \cdots, b_{\sigma_n}\}$ of simplices $\sigma_0, \cdots, \sigma_n$ satisfying $\sigma_0 \subset \cdots \subset \sigma_n$. For $r \geq 1$, the r-iterated barycentric subdivision $\mathrm{sd}^r(K)$ is defined inductively by $\mathrm{sd}(\mathrm{sd}^{r-1}(K))$, where $\mathrm{sd}^0(K) = K$.

It should be noted that $|\mathrm{sd}^r(K)| \cong |K|$ for any $r \geq 0$, which we identify. From this viewpoint, we can choose a simplicial approximation $\lambda\colon \mathrm{sd}^r(K) \to K$ to the identity on $|K|$.

Theorem 3.3 (Simplicial approximation theorem ([Spa95]))**.** *Let $f\colon |K| \to |L|$ be a continuous map. There exist sufficiently large $r \geq 0$ such that we have a simplicial approximation $\varphi\colon \mathrm{sd}^r(K) \to L$ to f.*

Definition 3.4. Two simplicial maps $f, g\colon K \to L$ are called *contiguous* and are denoted by $f \sim g$ if $f(\sigma) \cup g(\sigma)$ constitutes a simplex of L for each simplex σ of K. The contiguous relation on simplicial maps from K to L is reflexive and symmetric, but not transitive. The equivalence relation generated from $\sim$ is denoted by $\approx$, i.e., $f \approx g$ if we have a finite number of simplicial maps $h_1, \cdots, h_n\colon K \to L$ such that $h_1 = f$ and $h_n = g$ and $h_i \sim h_{i+1}$ for each i. In this case, we say that f and g are in the same contiguity class.

Theorem 3.5 ([Spa95])**.** *Let $f, g\colon |K| \to |L|$ be homotopic maps. There exist sufficiently large $r \geq 0$ such that we have simplicial approximations $\varphi, \psi\colon \mathrm{sd}^r(K) \to L$ to f and g, respectively, in the same contiguity class.*

3.3. **Finite spaces.** In topology, spaces consisting of finite points are often regarded as pathological examples because such spaces are discrete under usual situations.

Proposition 3.6. *Any finite T_1-space must be discrete.*

For example, finite subspaces in a Hausdorff space must be discrete. However, finite T_0-spaces have fascinating combinatorial structures. For a point x in a finite T_0-space X, we have the minimal open neighborhood

$$U_x = \bigcap_{x \in U} U$$

defined as the intersection of all open sets $x \in U$. A partial order $x \leq y$ on X is defined as $U_x \subset U_y$.

By contrast, a poset (partially ordered set) P is equipped with a topology called the Alexandroff topology. A subset $Q \subset P$ is an open set in P if Q is an ideal (a down-set) closed under the lower order.

From this perspective, finite T_0-spaces can be regarded as finite posets. Throughout this paper, finite T_0-spaces are simply called finite spaces.

3.4. **Homotopy theory of finite spaces.** The homotopy theory of finite spaces was developed by Strong [Sto66] and Barmak-Minian [BM12]. A map $f\colon P \to Q$ between finite spaces P and Q is continuous if and only if f is an order-preserving map. For two continuous maps $f, g\colon P \to Q$, a partial order $f \leq g$ on the mapping space Q^P is defined by $f(x) \leq g(x)$ in Q for any $x \in P$.

Theorem 3.7 ([Sto66])**.** *Two continuous maps $f, g\colon P \to Q$ between finite spaces P and Q are homotopic if and only if we have a finite number of continuous maps $f = h_0, \cdots, h_n = g$ from P to Q such that $h_i \leq h_{i+1}$ or $h_i \geq h_{i+1}$ for each i.*

Definition 3.8. A point x in a finite space P is called an *up beat point* if there exists a unique maximal element in $P_{<x} = \{y \in P \mid y < x\}$. Conversely, x is called a *down beat point* if there exists a unique minimal element in $P_{>x} = \{y \in P \mid y > x\}$. A point x is simply called a *beat point* if it is either an up beat point or a down beat point.

For a beat point $x \in P$, we have a deformation retraction $r\colon P \to P \backslash \{x\}$ defined by $r(x) = y$, where y is the maximal (minimal) element in $P_{<x}$ ($P_{>x}$). An arbitrarily deformation retraction in finite spaces is described as removing beat points.

Definition 3.9. For a beat point x in a finite space P, we say that there is an *elementary strong collapse* from P to $P\backslash\{x\}$. For a subspace Q of P, if there exists a finite sequence of elementary strong collapses starting in P and ending in Q, we say that there is a *strong collapse* from P to Q, and use the notation $P \searrow\searrow Q$.

For two finite spaces P and Q, we say that P and Q *have the same strong equivalence type* if there exists a finite sequence of finite spaces

$$P = R_0, R_1, \ldots, R_n = Q$$

such that $R_i \searrow\searrow R_{i+1}$ or $R_{i+1} \searrow\searrow R_i$ for each i.

Proposition 3.10 ([Sto66, BM12])**.** *A subspace Q is a deformation retract of a finite space P if and only if $P \searrow\searrow Q$.*

Theorem 3.11 ([Sto66, BM12])**.** *Two finite spaces P and Q are homotopy equivalent if and only if they have the same strong equivalence type.*

Definition 3.12. A finite space is called *minimal* if it has no beat point.

Proposition 3.13 ([Sto66])**.** *Let $f\colon P \to P$ be a map homotopic to the identity* id_P*. If P is minimal, then $f = \mathrm{id}_P$.*

The above proposition implies that if two minimal finite spaces P and Q are homotopy equivalent, then they are homeomorphic.

Definition 3.14. A subspace Q of a finite space P is called a *core* if it satisfies the following two conditions:

(1) Q is minimal.
(2) Q is a deformation retract of P.

The homotopy type of finite spaces is completely classified by cores. The core of a finite space is uniquely determined up to homeomorphism.

Theorem 3.15 ([Sto66, BM12])**.** *Two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.*

In particular, a finite space is contractible if and only if it has a single-point core.

3.5. **Classifying space and barycentric subdivision.**

Definition 3.16. Let P be a finite space. The *order complex* $\mathcal{K}(P)$ is a simplicial complex defined as follows: The set of vertices $V(\mathcal{K}(P)) = P$, and the set of simplices $\Sigma(\mathcal{K}(P))$ consists of totally ordered subsets in P:

$$p_0 < p_1 < \cdots < p_n.$$

The geometric realization $|\mathcal{K}(P)|$ is called the *classifying space* of P.

Let K be a simplicial complex. The face poset $\mathcal{F}(K)$ consists of all simplices of K with the inclusion order.

Definition 3.17. For a finite space P, the *barycentric subdivision* $\mathrm{sd}(P) = \mathcal{F}(\mathcal{K}(P))$ is defined as the face poset of the classifying space of P. That is, $\mathrm{sd}(P)$ consists of totally ordered subsets of P with the inclusion order. We inductively define the k-iterated barycentric subdivision as follows:

$$\mathrm{sd}^k(P) = \mathrm{sd}(\mathrm{sd}^{k-1}(P)).$$

For convenience, we set $\mathrm{sd}^0(P) = P$.

We have a natural continuous map $\tau_P\colon\ \mathrm{sd}(P) \to P$ defined by $\tau(p_0 < \cdots < p_n) = p_n$. Moreover, $\tau_P^k\colon\ \mathrm{sd}^k(P) \to P$ is defined as follows:

$$\mathrm{sd}^k(P) \xrightarrow{\tau_{\mathrm{sd}^{k-1}(P)}} \mathrm{sd}^{k-1}(P) \longrightarrow \cdots \longrightarrow \mathrm{sd}(P) \xrightarrow{\tau_P} P.$$

Theorem 3.18 ([McC66])**.** *For any finite space P, the map $\tau\colon\ \mathrm{sd}(P) \to P$ is a weak homotopy equivalence.*

Theorem 3.19 ([BM12])**.** *Let K, L be simplicial complexes, and let P, Q be finite spaces.*

(1) If two simplicial maps $f, g\colon\ K \to L$ are in the same contiguity class, then the induced maps $\mathcal{F}(f), \mathcal{F}(g)\colon\ \mathcal{F}(K) \to \mathcal{F}(L)$ are homotopic.
(2) If two continuous maps $f, g\colon\ P \to Q$ are homotopic, then the induced simplicial maps $\mathcal{K}(f), \mathcal{K}(g)\colon\ \mathcal{K}(P) \to \mathcal{K}(Q)$ are in the same contiguity class.

Theorem 3.20 ([BM12])**.** *A finite space P is contractible if and only if* $\mathrm{sd}(P)$ *is contractible.*

4. Topological complexity of simplicial complexes and finite spaces

The topological complexity $\mathrm{TC}(X)$ is defined as one less than the minimal number of open sets covering $X \times X$ with motion planning algorithm. Here a motion planning algorithm is a continuous local section $s\colon\ U \to X^I$ of the path fibration. We notice that s provides a homotopy between the projections $\mathrm{pr}_1, \mathrm{pr}_2 \colon U \to X$. Hence, a subset $U \subset X \times X$ admits a motion planning algorithm if and only if the projections $\mathrm{pr}_1, \mathrm{pr}_2 \colon U \to X$ are homotopic on U.

From this perspective, González introduced a simplicial version of topological complexity for simplicial complexes [Gon18].

4.1. **Simplicial complexity.** The product of simplicial complexes is not naturally determined because the Cartesian product $\Delta^n \times \Delta^m$ of geometric simplices is no longer a simplex. To define the Cartesian product of simplicial complexes, we consider ordered simplicial complexes.

An *ordered simplicial complex* is a simplicial complex with a total order on the vertices in each simplex compatible with the face relation. For an ordered simplicial complex K, the Cartesian product $K \times K$ is a simplicial complex with the vertex set $V(K) \times V(K)$. A binary relation on $V(K) \times V(K)$ is defined by $(v, w) \leq (v', w')$ if and only if $v \leq v'$ and $w \leq w'$ in $V(K)$. A simplex of $K \times K$ is a totally ordered subset S of $V(K) \times V(K)$ with respect to the relation $\leq$ such that $\pi_i(S)$ constitutes a simplex in K, where π_i is the projection to the i-th coordinate for each $i = 1, 2$. In this setting, the projections $\pi_i \colon K \times K \to K$ become simplicial maps and induce a homeomorphism $|K \times K| \cong |K| \times |K|$. We note that for any finite simplicial complex K, we can always choose a total order on the vertices, and it makes K an ordered simplicial complex.

Definition 4.1. Let K be a (an ordered) simplicial complex K, and $r \geq 0$ be a nonnegative integer. We say that a subcomplex L of $\mathrm{sd}^r(K \times K)$ admits a *motion planning algorithm* if the two simplicial maps $\pi_1 \circ \lambda$ and $\pi_2 \circ \lambda \colon L \to K$ are in the same contiguity class for a simplicial approximation $\lambda \colon L \to K \times K$ to the inclusion $|L| \hookrightarrow |K| \times |K|$.

The *simplicial complexity* $\mathrm{SC}_r(K)$ is defined as one less than the smallest size of subcomplexes covering $\mathrm{sd}^r(K \times K)$, where each subcomplex admits a motion planning algorithm.

It should be noted that $\mathrm{SC}_r(K)$ does not depend on the choice of ordering on K, and hence, the simplicial complexity is defined purely for simplicial complexes [Gon18, Remark 3.2]. The inequality $\mathrm{TC}(|K|) \leq \mathrm{SC}_r(K)$ always holds for any $r \geq 0$, and $\mathrm{SC}_r(K)$ decreases as r increases:

$$\mathrm{SC}_0(K) \geq \mathrm{SC}_1(K) \geq \cdots \geq \mathrm{SC}_r(K) \geq \cdots \geq 0.$$

González showed that the above monotone sequence converges to $\mathrm{TC}(|K|)$.

Theorem 4.2 ([Gon18])**.** *For any simplicial complex K, the equality* $\mathrm{SC}_r(K) = \mathrm{TC}(|K|)$ *holds for sufficiently large $r \geq 0$.*

4.2. **Topological complexity for finite spaces.** The reminder of this paper is an overview of [Tan18] regarding topological complexity for finite spaces and the classifying spaces.

The *finite interval* J_m of length m is a finite space consisting of $m + 1$ points $\{0, 1 \cdots, m\}$ with the zigzag order as follows:

$$0 < 1 > 2 < 3 > \cdots < (>)m.$$

For a finite space P, the mapping space P^{J_m} consists of continuous (order-preserving) maps $J_m \to P$. This is a finite space with the partial order $f \leq g$ given by $f(i) \leq g(i)$ for any $i \in J_m$.

An element in P^{J_m} can be regarded as a zigzag-ordered $(m + 1)$-tuple of elements in P:

$$p_0 \leq p_1 \geq p_2 \leq \cdots \leq (\geq) p_m.$$

We have a continuous map $\pi_m \colon P^{J_m} \to P \times P$ defined by $\pi_m(\gamma) = (\gamma(0), \gamma(m))$.

Definition 4.3. For a nonnegative integer $m \geq 0$ and a map $f \colon Q \to P \times P$ between finite spaces Q and P, $\mathrm{TC}_m(f)$ is defined as the minimum number n such that we have $(n + 1)$ open sets $U_0, \cdots, U_n$ covering Q, where each U_i admits a map $s \colon U_i \to P^{J_m}$ with $\pi_m \circ s = f|_{U_i}$. In particular, $\mathrm{TC}_m(\mathrm{id}_{P\times P})$ is denoted by $\mathrm{TC}_m(P)$.

We have a deformation retraction $J_{m+1} \to J_m$ sending $m + 1$ to m. This induces a map $P^{J_m} \to P^{J_{m+1}}$ preserving both ends. This implies that if a subset U of $P \times P$ admits an m-length motion planning algorithm, then U also admits an $(m + 1)$-length motion planning algorithm.

Proposition 4.4. *For a map $f \colon Q \to P \times P$, we have* $\mathrm{TC}_m(f) \geq \mathrm{TC}_{m+1}(f)$.

Theorem 4.5. *For a map $f \colon Q \to P \times P$, we have the following decreasing sequence:*

$$\mathrm{TC}_0(f) \geq \mathrm{TC}_1(f) \geq \cdots \geq 0$$

and

$$\lim_{m\to\infty} \mathrm{TC}_m(f) = \mathrm{TC}(f).$$

4.3. **Topological complexity for the classifying space.** We focus on the relationship between $\mathrm{TC}(|\mathcal{K}P|)$ and $\mathrm{TC}(P)$ for a finite space P. The following inequality always holds.

Proposition 4.6. *For a finite space P, we have* $\mathrm{TC}(P) \geq \mathrm{TC}(|\mathcal{K}P|)$.

Example 4.7. Let $\mathcal{B}_n$ be a finite space consisting of $(n+3)$-points $\{a, b, c_0, \cdots, c_n\}$ for $n \geq 1$. The partial order on $\mathcal{B}_n$ is given by $a < c_i$ and $b < c_i$ for each $0 \leq i \leq n$. The space $|\mathcal{K}(\mathcal{B}_n)|$ is a bouquet with n circles. Hence, $\mathrm{TC}(|\mathcal{K}(\mathcal{B}_n)|) = 1$ for $n = 1$ and $\mathrm{TC}(|\mathcal{K}(\mathcal{B}_n)|) = 2$ for $n \geq 2$, whereas $\mathrm{TC}(\mathcal{B}_n) = (n + 1)^2 - 1$.

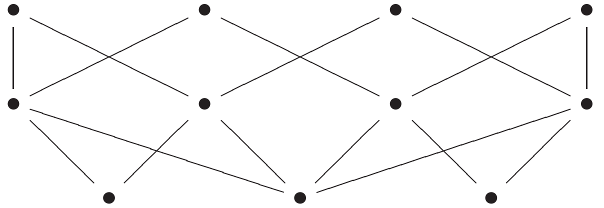
In the above example, the difference between TC(P) and TC($|\mathcal{K}P|$) can be significantly large, depending on n. This results from a small number of open sets of $P \times P$ compared with $|\mathcal{K}(P)| \times |\mathcal{K}(P)|$. One of the ideas to fill the gap between them is to consider the barycentric subdivisions.

Example 4.8. Consider $\mathcal{B}_1$ in Example 4.7. This is a finite-space model of a circle. The barycentric subdivision $\mathrm{sd}(\mathcal{B}_1)$ consists of eight points, and we have

$$\mathrm{TC}(\mathrm{sd}(\mathcal{B}_1)) \leq \mathrm{cat}(\mathrm{sd}(\mathcal{B}_1) \times \mathrm{sd}(\mathcal{B}_1)) = 2 < \mathrm{TC}(\mathcal{B}_1) = 3.$$

The above example provides a case of strict inequality $\mathrm{TC}(P) > \mathrm{TC}(\mathrm{sd}(P))$. We expect that $\mathrm{TC}(\mathrm{sd}^k(P))$ will decrease and converge to $\mathrm{TC}(|\mathcal{K}P|)$ as $k \to \infty$. However, this is not true in general.

Example 4.9. Let $\mathcal{W}$ be a finite space described as the following Hasse diagram.



$\mathcal{W}$ is not contractible because it has no beat point, whereas the classifying space $|\mathcal{K}(\mathcal{W})|$ is contractible. We have $\mathrm{TC}(|\mathcal{K}(\mathcal{W})|) = 0$, whereas $\mathrm{TC}(\mathrm{sd}^k(\mathcal{W})) > 0$ for any $k \geq 0$ by Theorem 3.20.

The topological complexity $\mathrm{TC}(\mathrm{sd}(P))$ is based on open sets in the product $\mathrm{sd}(P)\times\mathrm{sd}(P)$. However, the product $\mathrm{sd}(P)\times\mathrm{sd}(P)$ is not described as the face poset of a simplicial complex in general. This is a disadvantage in that we cannot use the combinatorial homotopy theory of simplicial complexes, including the simplicial approximation theorem.

We use $\mathrm{sd}(P\times P)$, which is the face poset of the order complex $\mathcal{K}(P\times P)$, instead of $\mathrm{sd}(P) \times \mathrm{sd}(P)$. Let us recall the natural map defined in Section 3.5:

$$\tau^k_{P\times P}\colon\ \mathrm{sd}(P \times P) \longrightarrow P \times P.$$

The following is our main result in [Tan18]. The proof is essentially based on the simplicial approximation theorem (Theorems 3.3 and 3.5).

Theorem 4.10. *Let P be a finite space. We have the following monotone decreasing sequence*

$$\mathrm{TC}(P) = \mathrm{TC}(\tau^0_{P\times P}) \geq \mathrm{TC}(\tau^1_{P\times P}) \geq \cdots \geq 0$$

and

$$\lim_{k\to\infty} \mathrm{TC}(\tau^k_{P\times P}) = \mathrm{SC}(\mathcal{K}P) = \mathrm{TC}(|\mathcal{K}P|).$$

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