Topologically Representation of Cantor Cube Model for Geometric Patterns

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Abstract
The characteristic geometric structures found in disordered matters are discussed from the viewpoint of general topology. Cantor cube which is a topological space consisting of the infinite product space of 0 and 1 provides specific decomposition spaces representing topologically geometric patterns of matters such as graphs, clusterized structures, dendrites.

1 Introduction: Cantor cube model

Geometrically patterns of matters in solid and liquid states have been hugely studied from the viewpoint of disordered physics[1, 2]. In particular, fascinating characteristic topology structures of matters (the topology structures of matters is abbreviated as TSM hereafter), such as the graphic structure of polymers[3], the clusterized structure of molecular liquids[4, 5], or the dendritic structure in solidifications[6], have been found. To characterize such TSM, several mathematical methods based on topological concept have been studied. For instance, persistent homology is the method available for classification of geometric structures of amorphous materials[7, 8]. Note that this persistent homological analysis is mathematically based on a technique of algebraic topology.

The mathematical approach by using general topology to TSM has been successfully studied[9, 10, 11, 12]. In these studies, TSM are investigated based on the mathematical formation of a set of equivalence classes for a specific topological space X, independently of the detailed properties of each matter. That is, geometric patterns for TSM are discussed by connecting them with the decomposition spaces of X (For the details of this approach, see [12]). In the present article, we focus on characterizations of TSM by means of the Cantor cube model; in this model the specific topological space X is taken as a Cantor cube $(\{0,1\}^{\Lambda}, \tau_0^{\Lambda})$ and geometric patterns are discussed through the decomposition spaces of $(\{0,1\}^{\Lambda}, \tau_0^{\Lambda})$, where $(\{0,1\}^{\Lambda}, \tau_0^{\Lambda})$ is the $\Lambda-$product space of $(\{0,1\}, \tau_0)$ with an index set Λ of Card $\Lambda \geq \aleph_0$ and τ_0 is a discrete topology for $\{0,1\}$. To begin with, we will show a practical construction of a decomposition space of $(\{0,1\}^{\Lambda}, \tau_0^{\Lambda})$ homeomorphic to a compact metric space.

It is very known that any compact metric space is represented homeomorphically as a quotient space of fibers [13] of 0-dim, perfect, compact Hausdorff-space[14]. Indeed, the representation of a compact metric space can be obtained systematically as follows. Let X denote a Cantor cube $(\{0,1\}^{\Lambda}, \tau_0^{\Lambda})$ with Card $\Lambda \geq \aleph_0$, and let Y be a compact metric space. Then, there exists a closed cover $\{Y_1, \ldots, Y_n\}$ $(n < \infty)$ of Y, each diameter of which is less than $1/2$. To this cover there corresponds a partition $\{X_1, \ldots, X_n\}$ of $\{0,1\}^{\Lambda}$ such that

$$\begin{cases} X_1 = \{0\}_{\lambda_1} \times \{0,1\}^{\Lambda - \{\lambda_1\}}, \\ X_i = \{1\}_{\lambda_1} \times \cdots \times \{1\}_{\lambda_{i-1}} \times \{0\}_{\lambda_i} \times \{0,1\}^{\Lambda - \{\lambda_1, \ldots, \lambda_i\}} \; (i = 2, 3, \ldots, n-1), \\ X_n = \{1\}_{\lambda_1} \times \cdots \times \{1\}_{\lambda_{n-2}} \times \{1\}_{\lambda_{n-1}} \times \{0,1\}^{\Lambda - \{\lambda_1, \ldots, \lambda_{n-1}\}}, \end{cases} \tag{1}$$

where λ_i is arbitrarily element of Λ, $(i = 1, \ldots, n-1)$ and each $\{k_1\}_{\lambda_1} \times \cdots \times \{k_i\}_{\lambda_i} \times \{0,1\}^{\Lambda - \{\lambda_1, \ldots, \lambda_i\}} = \{x : \Lambda \to \{0,1\}, x(\lambda_l) = k_l \in \{0,1\}, l = 1, \ldots, i\}$ stands for a cone. Let $s_1 : X \to \Im(Y) - \{\emptyset\}$ be a map defined by $s_1(x) = Y_i$ if $x \in X_i$ for each i, where $\Im(Y)$ is the collection of closed sets of Y. Note that $Y = \cup_{x \in X} s_1(x)$. Since Y_i is a compact metric space, for each i, we have a closed cover $\{Y_{i_1}, \ldots, Y_{i_{n_i}}\}$ of Y_i, each diameter of Y_{i_j} being less than $1/2^2$. Also, X_i has a partition $\{X_{i_1}, \ldots, X_{i_{n_i}}\}$ composed of cones such that

$$\begin{cases} X_{i_1} = \{1\}_{\lambda_1} \times \cdots \times \{1\}_{\lambda_{i-1}} \times \{0\}_{\lambda_i} \times \{0\}_{\mu_1} \times \{0,1\}^{\Lambda-(\{\lambda_1,\ldots,\lambda_i\}\cup\{\mu_1\})}, \\ X_{i_j} = \{1\}_{\lambda_1} \times \cdots \times \{1\}_{\lambda_{i-1}} \times \{0\}_{\lambda_i} \times \{1\}_{\mu_1} \times \cdots \times \{1\}_{\mu_{j-1}} \times \{0\}_{\mu_j} \times \{0,1\}^{\Lambda-(\{\lambda_1,\ldots,\lambda_i\})\cup(\{\mu_1,\ldots,\mu_j\})} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 2, 3, \ldots, n_i - 1), \\ X_{i_{n_i}} = \{1\}_{\lambda_1} \times \cdots \times \{1\}_{\lambda_{i-1}} \times \{0\}_{\lambda_i} \times \{1\}_{\mu_1} \times \cdots \times \{1\}_{\mu_{n_i-2}} \times \{1\}_{\mu_{n_i-1}} \times \{0,1\}^{\Lambda-(\{\lambda_1,\ldots,\lambda_i\})\cup(\{\mu_1,\ldots,\mu_{n_i-1}\})}, \end{cases} \tag{2}$$

where μ_i is arbitrarily element of $\Lambda - (\{\lambda_1, \ldots, \lambda_i\})$. Let $s_2 : X \to \Im(Y) - \{\emptyset\}$ be defined by $s_2(x) = Y_{i_j}$ for $x \in X_{i_j}$. Then, $Y = \cup_{x\in X} s_2(x)$ and $s_2(x) \subset s_1(x)$ for all x. Continuing the procedure we have a sequence of functions $\{s_n\}$ such that for each x and for each n, (i) s_n is upper semi-continuous, (ii) $s_{n+1}(x) \subset s_n(x)$, (iii) $Y = \cup_{x\in X} s_n(x)$, and (iv) $dia\ s_n(x) \to 0$ as $n \to \infty$, where dia stands for diameter of a set. Thus, we obtain a continuous map f from X onto Y, $x \mapsto \cap_n g_n(x)$ and the decomposition space $(\mathcal{D}_f, \tau(\mathcal{D}_f))$ of X relative to f homeomorphic to Y, where $\mathcal{D}_f = \{f^{-1}(y); y \in Y\}$ and $\tau(\mathcal{D}_f) = \{\mathcal{U} \subset \mathcal{D}_f; \bigcup \mathcal{U} \in \tau_0^\Lambda\}$ is a decomposition topology. Through the homeomorphism, each point y of Y can be associated with an unique point $f^{-1}(y)$ of $\mathcal{D}_f$. For instance, the decomposition space representing $[0,1]$ is obtained practically as the following two cases; letting $M \equiv \{l/2^n; n = 1, 2, \ldots$ and $l = 1, \ldots, 2^n - 1\}$, then (i) for $y = \Sigma_{i=1}^\infty a_i/2^i \notin M$

$$f^{-1}(y) = \{a_1\}_{\lambda_1} \times \{a_2\}_{\lambda_2} \times \cdots \times \{0,1\}^{\Lambda-\{a_1,a_2,\cdots\}}, \tag{3}$$

and (ii) for $y = l/2^n \in M$

$$\begin{aligned} f^{-1}(y) = &\Big[\{a_1\}_{\lambda_1} \times \{a_2\}_{\lambda_2} \times \cdots \times \{a_{n-1}\}_{\lambda_{n-1}} \times \{0\}_{\lambda_n} \times \{1\}_{\lambda_{n+1}} \times \{1\}_{\lambda_{n+2}} \times \cdots \times \{0,1\}^{\Lambda-\{\lambda_1,\lambda_2,\cdots\}}\Big] \\ &\cup\Big[\{a_1\}_{\lambda_1} \times \{a_2\}_{\lambda_2} \times \cdots \times \{a_{n-1}\}_{\lambda_{n-1}} \times \{1\}_{\lambda_n} \times \{0\}_{\lambda_{n+1}} \times \{0\}_{\lambda_{n+2}} \times \cdots \times \{0,1\}^{\Lambda-\{\lambda_1,\lambda_2,\cdots\}}\Big] \end{aligned} \tag{4}$$

for some $a_1, \ldots, a_{n-1}$. Here, $f^{-1}(0) = \{0\}_{\lambda_1} \times \{0\}_{\lambda_2} \times \cdots \times \{0,1\}^{\Lambda-\{\lambda_1,\lambda_2,\cdots\}}$ and $f^{-1}(1) = \{1\}_{\lambda_1} \times \{1\}_{\lambda_2} \times \cdots \times \{0,1\}^{\Lambda-\{\lambda_1,\lambda_2,\cdots\}}$. Note that the decomposition space constructed in the above process is not unique.

2 Topologically representation for TSM

Here, we focus on several decomposition spaces of the Cantor cube $X = (\{0,1\}^\Lambda, \tau_0^\Lambda)$ which represent geometric models for TSM.

First let us consider two network patterns Y_1 and Y_g shown in (a) and (b) of Fig. 1; Y_1 is a figure composed of three nodes e_1, e_2, a and two bonds E_1 and E_2 connecting e_1 with a and e_2 with a, respectively. Y_g is a finite graph[15]. Since Y_1 is regarded as an arc, the construction of the decomposition space stated in the previous section for $Y = [0,1]$ can be directly applied to Y_1. Indeed, letting h be a homeomorphism from Y_1 onto $[0,1]$, each point x of Y_1 can be represented as the point of a decomposition space $\mathcal{D}_1$ of $\{0,1\}^\Lambda$ by the following two types; (i) if $h(x) \notin M (\equiv \{l/2^n; n = 1, 2, \ldots$ and $l = 1, \ldots, 2^n - 1\})$, then

$$x \doteq \{k_1\}_{\lambda_1} \times \{k_2\}_{\lambda_2} \times \cdots \times \{0,1\}^{\Lambda-\{\lambda_1,\lambda_2,\ldots\}}, \tag{5}$$

where $k_1, k_2, \ldots$ are points in $\{0,1\}$ satisfying $h(x) = \Sigma_{i=1}^\infty k_i/2^i$, and $\doteq$ is the sign of identification of x with a corresponding point $f^{-1}(x)$ of $\mathcal{D}_1$, and (ii) if $h(x) \in M$, then

$$\begin{aligned} x \doteq &\Big[\{k_1\}_{\lambda_1} \times \{k_2\}_{\lambda_2} \times \cdots \times \{k_m\}_{\lambda_m} \times \{0\}_{\lambda_{m+1}} \times \{1\}_{\lambda_{m+2}} \times \{1\}_{\lambda_{m+3}} \times \cdots \times \{0,1\}^{\Lambda-\{\lambda_1,\lambda_2,\cdots\}}\Big] \\ &\cup\Big[\{k_1\}_{\lambda_1} \times \{k_2\}_{\lambda_2} \times \cdots \times \{k_m\}_{\lambda_m} \times \{1\}_{\lambda_{m+1}} \times \{0\}_{\lambda_{m+2}} \times \{0\}_{\lambda_{m+3}} \times \cdots \times \{0,1\}^{\Lambda-\{\lambda_1,\lambda_2,\cdots\}}\Big] \end{aligned} \tag{6}$$

for some m, where $k_1, \ldots, k_m$ are points in $\{0,1\}$ giving $h(x) \in M$. If we introduce a sign S_x defined by

$$S_x \equiv \begin{cases} (5), & h(x) \notin M \\ (6), & h(x) \in M, \end{cases} \tag{7}$$

then

$$x \doteq S_x \tag{8}$$

for $x \in Y_1$. Note that assuming $h(e_1) = 0$ and $h(e_2) = 1$, the end points e_1 and e_2 form

$$e_1 \doteq \{0\}_{\lambda_1} \times \{0\}_{\lambda_2} \times \cdots \times \{0,1\}^{\Lambda - \{\lambda_1, \lambda_2, \cdots\}}, \; e_2 \doteq \{1\}_{\lambda_1} \times \{1\}_{\lambda_2} \times \cdots \times \{0,1\}^{\Lambda - \{\lambda_1, \lambda_2, \cdots\}}. \tag{9}$$

The relation (8) shows that the geometric feature of Y_1 is completely characterized in the decomposition space $\mathcal{D}_1$ of X. For a finite graph Y_g, we denote the arcs composing of Y_g by $E_1, \ldots, E_r (r < \infty)$. There exists a partition $\{X^1, \ldots, X^r\}$ of X corresponding to these arcs where each X^i is defined as well as that in (1) with indexes $\mu_1, \ldots, \mu_{r-1} \in \Lambda$. It is confirmed that a decomposition space $\mathcal{D}_g$ of X represents Y_g; the representations for a node x with bonds $E_{t_1}, \ldots, E_{t_q}$ and a point y in a bond E_i are obtained as

$$x \doteq \cup_{j=1}^{q} (X^{t_j} \cap S_x^{t_j}), \; y \doteq X^i \cap S_y^i, \tag{10}$$

respectively. As a practical example of materials with the graphic structure we can consider a tree of a dendritic crystal[16]. A tree is a graph that does not contains a space homeomorphic to a unit sphere shown in (c) of Fig. 1. In this case, the representation for a tree by a decomposition space $\mathcal{D}_t$ is the same as the relation (10).

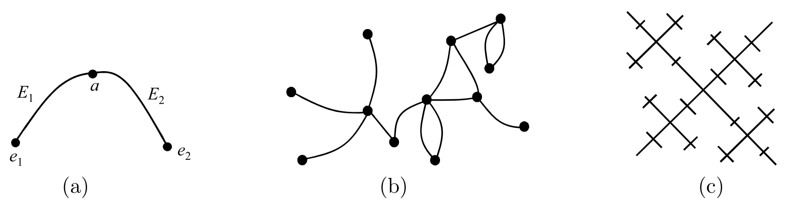


Figure 1: Schematic explanation of three types of geometric patterns with network configuration. (a) geometric model Y_1; two nodes e_1 and e_2 are connected by edges E_1 and E_2 thorough a node a. (b) a finite graph Y_g. (c) a tree Y_t.

Next, we focus on a cluster pattern Y_c for which each cluster is a finite graph, shown in Fig. 2 (a). Then, Y_c may be defined to be a topological space $(\bigoplus_{i=1}^{s} C_i, \bigoplus_{i=1}^{s} \tau_i)$ where $(\bigoplus_{i=1}^{s} C_i, \bigoplus_{i=1}^{s} \tau_i)$ is a disjoint union of a collection of finite graphs $\{(C_i, \tau_i), i = 1, \ldots, s\}$. To disjoint clusters $C_1, \ldots, C_s$, there corresponds a partition $\{J_1, \ldots, J_s\}$ of X using new elements $\xi_1, \ldots, \xi_{s-1} \in \Lambda$ such that

$$\begin{cases} J_1 &= \{0\}_{\xi_1} \times \{0,1\}^{\Lambda - \{\xi_1\}}, \\ J_j &= \{1\}_{\xi_1} \times \cdots \times \{1\}_{\xi_{j-1}} \times \{0\}_{\xi_j} \times \{0,1\}^{\Lambda - \{\xi_1, \cdots \xi_{j-1}\}} \; (j = 2, \ldots, s-1), \\ J_s &= \{1\}_{\xi_1} \times \cdots \times \{1\}_{\xi_{s-2}} \times \{1\}_{\xi_{s-1}} \times \{0,1\}^{\Lambda - \{\xi_1, \cdots \xi_{s-1}\}}. \end{cases} \tag{11}$$

By applying the relation (10) to each finite graph C_i for corresponding cone $J_i, i = 1, \ldots, s$, the representation of whole space Y_c by a decomposition space $\mathcal{D}_c$ of X can be obtained as follows; for $x \in Y_c$ contained in a cluster C_{i_0},

$$x \doteq J_{i_0} \cap \begin{cases} \cup_{j=1}^{q} (X^{t_j} \cap S_x^{t_j}), \\ X^i \cap S_x^i. \end{cases} \tag{12}$$

The relation (12) shows that x is either a node of C_{i_0} with bonds $E^{i_0}_{t_1}, \ldots, E^{i_0}_{t_q}$ or a point in a bond $E^{i_0}_i$ where $E^{i_0}_1, \ldots, E^{i_0}_{r(i_0)}$ $(t_q \leq r(i_0))$ are arcs composing of a finite graph C_{i_0}. Namely, the term J_{i_0} and the successive terms designate a point x belonging to a graphic cluster C_{i_0} and a location of x in the graph C_{i_0}, respectively .

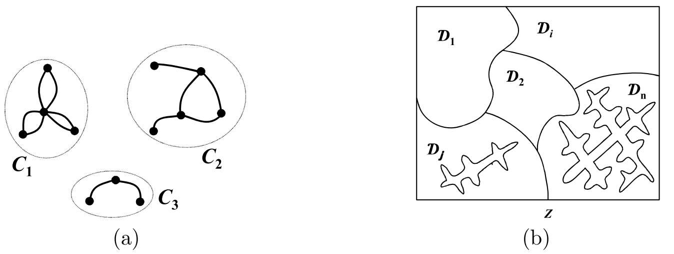
This topologically representation by a decomposition space $\mathcal{D}_c$ for a clusterized structure can be applied to the tiling issue in material science that a polycrystal can be filled with an arbitrary finite number of single crystals characterized by a specific geometric structure, i.e., dendritic, or self-similar structure (according to the mathematical setting and discussion for the issue based on general topology, see [11]). Figure 2 (b) shows the roughly sketch of situation for this issue. Here, we consider dendritic crystals as single crystals. Then, the situation can be identified with the clusterized geometrical pattern in which each cluster is dendritic. Actually, we regard each dendritic crystal composing of the polycrystal as a cluster and then the geometric structure of the polycrystal is described by a kind of clusterized structure. Based on the representation (12) of the clusterized structure, we can obtain the following decomposition space $\mathcal{D}_c$ of X representing the geometric structure of the polycrystal:

$$\mathcal{D}_c = \cup_{i=1}^{n} \mathcal{D}_i, \tag{13}$$

where

$$\mathcal{D}_i = \left\{ y \doteq J_i \cap \cup_{j=1}^{q} (X^{t_j} \cap S^{t_j}_y); y \in Y^i_t \right\} \cup \left\{ y \doteq J_i \cap X^j \cap S^j_y; y \in Y^i_t \right\}. \tag{14}$$

Note that $\mathcal{D}_i,\ i = 1, \cdots, i$ are mutually disjoint each other. (13) and (14) show the relationship between each single dendritic crystal $\mathcal{D}_i$ and a whole polycrystal $\mathcal{D}_c$ for the tiling issue. The representation of decomposition spaces for the clusterized structure stated in this section can be widely applicable to discuss geometric aggregation structures of matters such as noncrystalline and amorphous as well as this tiling issue for a polycrystal.



(a) (b)

Figure 2: Geometric models of (a) a clusterized structure Y_c where the number of clusters $s = 3$, and (b) schematic explanations of a polycrystal Z filled with dendritic decomposition spaces $\mathcal{D}_i$.

Finally, we comment that in this short article a Cantor cube is introduced as a conceptional model to obtain several topologically representations of TSM, e.g., the graphic and clusterized structures. Indeed, each character of these geometric structures can be connected with decomposition spaces of a Cantor cube. Therefore, by analyzing a mathematical property of a Cantor cube model even more, new universal properties of the geometric structures of matters might be revealed.

3 Conclusion

The mathematical method to characterize geometric patterns for TSM universally based on a Cantor cube $(\{0,1\}^\Lambda, \tau_0^\Lambda)$ has been shown. Typical geometric patterns such as graphic and clusterized structures are focused on and their representations by decomposition spaces of the Cantor cube are investigated. A practical form of a decomposition space of a polycrystal in the tiling issue that a polycrystal filled with an arbitrary finite number of dendritic crystals is shown by handling it as a special case of the decomposition representation for the clusterized structural geometric model. (More details are shown in [12].)

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[16] A dendrite is defined as a topological space which is a Peano continuum (namely, a connected, locally connected, compact, metric space) containing no simple closed curve. Mathematically, a tree is a dendrite but it is not necessarily that a dendrite implies a tree. However, the dendritic structure composed of infinitely many branches may not be observed in diffusion process such as solidification in practice. Therefore, we consider a dendrite as a tree here. According to the detailed discussions for the dendritic structures in solidification, see [6].