

DYNAMICAL PROPERTIES OF DOUBLY 0-DIMENSIONAL MAPS

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1. INTRODUCTION

In this article, we will consider the following problems of one-sided dynamical systems:

Problem 1.1. *Is it possible to study arbitrary one-sided dynamical systems by use of appropriate 0-dimensional dynamical systems?*

Problem 1.2. *Is it possible to reconstruct arbitrary one-sided dynamical systems by use of appropriate time series analysis?, i.e., is it possible to extend Takens' reconstruction theorem to one-sided dynamical systems?*

In this article, we show that the above problems 1.1 and 1.2 have near-positive answers by using doubly 0-dimensional maps.

For a space X , $\dim X$ means the topological (covering) dimension of X (e.g. see [Eng95], [HW41] and [Nag65]). Let X be compact metric space and Y a space with a complete metric d_Y . Let $C(X, Y)$ denote the space consisting of all maps $f : X \rightarrow Y$. We equip $C(X, Y)$ with the metric d defined by

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Recall that $C(X, Y)$ is a complete metric space and hence Baire's category theorem holds in $C(X, Y)$. A map $g : X \rightarrow Y$ of separable metric spaces is n -dimensional ($n = 0, 1, 2, \dots$) if $\dim g^{-1}(y) \leq n$ for each $y \in Y$. Note that a closed map $g : X \rightarrow Y$ is 0-dimensional if and only if for any 0-dimensional subset D of Y , $\dim g^{-1}(D) \leq 0$ (see [Eng95, Hurewic's theorem (1.12.4)]). A map $T : X \rightarrow X$ is *doubly 0-dimensional* if for each closed set $A \subset X$ of dimension 0, one has $\dim T^{-1}(A) \leq 0$ and $\dim T(A) = 0$.

We have the following theorem ([Kat21] and [KOU16]) which is the key fact in this article.

Theorem 1.3. *Suppose that X is one of the following spaces: compact PL-manifolds, compact PL-manifolds with branched structures, Menger manifolds, Sierpinski carpet, Sierpinski gasket, dendrites. Then the followings hold.*

(1) *The set of all doubly 0-dimensional maps on X is dense in the space*

$C(X, X)$ (see [Kat21]).

(2) The set of maps T with $\dim P(T) \leq 0$ contains a G_δ -dense subset of the set $C(X, X)$ (see [KOU16]), where $P(T)$ denotes the set of periodic points of T .

So if we could study dynamical properties of doubly 0-dimensional maps, we can obtain approximate properties of any dynamical systems (X, T) and hence it is important to study the dynamical properties of “doubly 0-dimensional maps”. In this article, we show that the above problems 1.1 and 1.2 have near-positive answers through Theorem 1.3.

2. FINITE-TO-ONE 0-DIMENSIONAL COVERS OF DOUBLY 0-DIMENSIONAL MAPS

In this section, we consider the problem 1.1. Throughout this article, all spaces are separable metric spaces and maps are continuous functions. Let \mathbb{N} be the set of all nonnegative integers, i.e., $\mathbb{N} = \{0, 1, 2, \dots\}$ and let \mathbb{Z} be the set of all integers and \mathbb{R} the real line. A map $h : X \rightarrow Y$ is an *embedding* if $h : X \rightarrow h(X)$ is a homeomorphism. A pair (X, T) is called a *one-sided dynamical system* (abbreviated as *dynamical system*) if X is a separable metric space and $T : X \rightarrow X$ is any map. Moreover, if $T : X \rightarrow X$ is a homeomorphism, i.e., invertible, then (X, T) is called a *two-sided dynamical system*. Also if $T : X \rightarrow X$ is not a homeomorphism, (X, T) called a *non-invertible dynamical system*.

A dynamical system (Z, \tilde{T}) covers (X, T) via a map $p : Z \rightarrow X$ provided that p is an onto map and $p\tilde{T} = Tp$. We call the map $p : Z \rightarrow X$ a *factor mapping*. If Z is 0-dimensional, then we say that the dynamical system (Z, \tilde{T}) is a *0-dimensional cover* of (X, T) . Moreover, if the factor mapping is a finite-to-one map, then we say that the dynamical system (Z, \tilde{T}) is a *finite-to-one 0-dimensional cover* of (X, T) .

The following theorem implies that the problem 1.1 has a near-positive answer (see [KM20]).

Theorem 2.1. *Suppose that $T : X \rightarrow X$ is a doubly 0-dimensional map of a compactum X with $\dim X = n < \infty$. If $\dim P(T) \leq 0$, then there exist a dense G_δ -set H of X and a zero-dimensional cover (Z, \tilde{T}) of (X, T) via an at most 2^n -to-one onto map $p : Z \rightarrow X$ such that $P(T) \subset H$ and $|p^{-1}(x)| = 1$ for $x \in H$. Moreover, if X is perfect, then Z can be chosen as a Cantor set. In particular, $h(T) = h(\tilde{T})$, where $h(T)$ denotes the topological entropy of T .*

For the special case of positively expansive maps, we have

Theorem 2.2. *Let $T : X \rightarrow X$ be a positively expansive map of a compactum X with $\dim X = n < \infty$. Then there exist $k \geq 1$ and a closed σ -invariant set Σ of the shift map $\sigma : \{1, 2, \dots, k\}^{\mathbb{N}} \rightarrow \{1, 2, \dots, k\}^{\mathbb{N}}$ such that (Σ, σ) is a zero-dimensional cover (= symbolic extension) of (X, T) via an at most 2^n -to-one map $p : \Sigma \rightarrow X$.*

An indexed family $(C_s)_{s \in S}$ of subsets of a set X will by abuse of notation also be denoted by $\{C_s\}_{s \in S}$ or $\{C_s : s \in S\}$. Hence if $\mathcal{C} = \{C_s\}_{s \in S}$ is such a family then its members C_s and C_t will be considered as different whenever $s \neq t$. We then put

$$\text{ord}(\mathcal{C}) = \sup\{\text{ord}_x(\mathcal{C}) : x \in X\}, \text{ where } \text{ord}_x(\mathcal{C}) = |\{s \in S \mid x \in C_s\}|.$$

Note that $\text{ord}(\mathcal{C})$ so defined is by 1 larger than it would be according to the usual definition, as e.g. in [Eng95, (1.6.6) Definition].

To prove the above theorems, we need the following lemma which is the key result to study the dynamical properties of doubly 0-dimensional maps (see [KM20]).

Lemma 2.3. *Suppose that $T : X \rightarrow X$ is a doubly 0-dimensional map of a compactum X such that $\dim X = n < \infty$ and $\dim P(T) \leq 0$. Let F be an F_σ -set of X with $\dim F \leq 0$. Then, for each $j \in \mathbb{N}$, there is a finite open cover $\mathcal{C}(j) = \{C(j)_i \mid 1 \leq i \leq m_j\}$ of X such that*

- (1) $\text{mesh}(\mathcal{C}(j)) < 1/j$ ($j \geq 1$),
- (2) $\text{ord}(\mathcal{G}) \leq n$, where $\mathcal{G} = \{T^{-p}(\text{bd}(C(j)_i)) \mid 1 \leq i \leq m_j, j \in \mathbb{N} \text{ and } p \in \mathbb{N}\}$, and
- (3) $F \cap L = \emptyset$, where $L = \bigcup\{\text{bd}(C(j)_i) \mid 1 \leq i \leq m_j, j \in \mathbb{N}\}$.

3. DYNAMICAL DECOMPOSITION THEOREM OF DOUBLY 0-DIMENSIONAL MAPS

In dimension theory, the following decomposition theorem is well-known.

Theorem 3.1. *A separable metric space X is $\dim X \leq n$ ($n \in \mathbb{N}$) if and only if X can be represented as the union of $n + 1$ subspaces Z_0, Z_1, \dots, Z_n of X such that $\dim Z_i \leq 0$ for each $i = 0, 1, \dots, n$.*

In this section, we study “dynamical decomposition theorem” of doubly 0-dimensional maps. Let $T : X \rightarrow X$ be a map. A subset Z of X is a *bright space* of T except n times ($n \in \mathbb{N}$) if for any $x \in X$,

$$|\{p \in \mathbb{N} \mid T^p(x) \notin Z\}| \leq n.$$

Note that for any $x \in X$, the positive orbit $O(x)$ appears in Z except n times. Also we say that $L = X - Z$ is a *dark space* of T except n times. Note that for any $x \in X$, the positive orbit $O(x)$ disappears from L except n times. Bright spaces Z would be expected to be very large spaces. But we can choose very “small” bright spaces.

Theorem 3.2. [KM20] *Suppose that $T : X \rightarrow X$ is a doubly 0-dimensional map of a compactum X with $\dim X = n < \infty$. Then $\dim P(T) \leq 0$ if and only if there is a zero-dimensional bright space Z of T except n times such that Z is a dense G_δ -set of X and the dark space $L = X - Z$ of T is an $(n - 1)$ -dimensional F_σ -set of X .*

This theorem implies that the bright space Z is very small like "small dots" and the dark space is very large like "dark matters" in physics. Such Z as in Theorem 3.2 satisfies the dynamical decomposition theorem of doubly 0-dimensional maps (see [KM20]).

Theorem 3.3. *Suppose that X is a compactum with $\dim X = n$ ($< \infty$) and $T : X \rightarrow X$ is a doubly 0-dimensional onto map. Then $\dim P(T) \leq 0$ if and only if there exists a zero-dimensional G_δ -dense set Z of X such that for any $n + 1$ integers $k_0 < k_1 < \dots < k_n$ ($k_i \in \mathbb{Z}$),*

$$X = T^{k_0}(Z) \cup T^{k_1}(Z) \cup \dots \cup T^{k_n}(Z).$$

4. TAKENS-TYPE RECONSTRUCTION THEOREM OF ONE-SIDED DYNAMICAL SYSTEMS

In this section, we consider the problem 1.2. Reconstruction of dynamical systems from a scalar time series is a topic that has been extensively studied. The theoretical basis for methods of recovering dynamical systems on compact manifolds from one-dimensional data was studied by Takens [Tak81, Tak02]. In 1981, Takens [Tak81], by use of Whitney's embedding theorem, proved that under some conditions of (two-sided) diffeomorphisms on a manifold, the dynamical system can be reconstructed from the observations made with generic functions.

Theorem 4.1. (Takens' reconstruction theorem for diffeomorphisms [Tak81] and [Noa91]) *Suppose that M is a compact smooth manifold of dimension d . Let $D^r(M)$ be the space of all C^r -diffeomorphisms on M and $C^r(M, \mathbb{R})$ the set of all C^r -functions ($r \geq 1$) to \mathbb{R} . If E is the set of all pairs $(T, f) \in D^r(M) \times C^r(M, \mathbb{R})$ such that the delay observation map $I_{T,f}^{(0,1,2,\dots,2d)} : M \rightarrow \mathbb{R}^{2d+1}$ defined by*

$$x \mapsto (fT^j(x))_{j=0}^{2d}$$

is an embedding, then E is open and dense in $D^r(M) \times C^r(M, \mathbb{R})$.

Moreover, in 2002 Takens [Tak02], extended his theorem for endomorphisms on compact smooth manifolds as follows.

Theorem 4.2. (Takens' reconstruction theorem for endomorphisms [Tak02]) *Suppose that M is a compact smooth manifold of dimension d . Then there is an open dense subset $\mathcal{U} \subset \text{End}^1(M) \times C^1(M, \mathbb{R})$, where $\text{End}^1(M)$ denotes the space of all C^1 -endomorphisms on M , such that, whenever $(T, f) \in \mathcal{U}$, there is a map $\pi : I_{T,f}^{(0,1,\dots,2d)}(M) \rightarrow M$ with $\pi \cdot I_{T,f}^{(0,1,\dots,2d)} = T^{2d}$.*

For a space K , we consider the (one-sided) shift $\sigma : K^{\mathbb{N}} \rightarrow K^{\mathbb{N}}$ which is defined by

$$\sigma(x_0, x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots), \quad x_i \in K.$$

Let (X, T) and (X', T') be dynamical systems. If a map $h : X \rightarrow X'$ satisfies the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \downarrow T & & \downarrow T' \\ X & \xrightarrow{h} & X' \end{array}$$

then we say that $h : (X, T) \rightarrow (X', T')$ is a *morphism* of dynamical systems.

In this article, we need the following definition from [Kat20].

Definition 4.3. Let $T : X \rightarrow X$ be a map of a compact metric space X .

(a) Given a set $S \subset \mathbb{N}$ and a map $f : X \rightarrow \mathbb{R}$, the map $(fT^j)_{j \in S} : X \rightarrow \mathbb{R}^S$ will be denoted by $I_{T,f}^S$. We call this map the *delay observation map at times* $j \in S$. Note that $I_{T,f} := I_{T,f}^{\mathbb{N}} : (X, T) \rightarrow (\mathbb{R}^{\mathbb{N}}, \sigma)$ is a morphism of dynamical systems. We call $I_{T,f}$ the *infinite delay observation map for* (T, f) .

(b) We say that I_f^S is a *trajectory-embedding* if $I_f^S(x) \neq I_f^S(y)$ whenever $T^j(x) \neq T^j(y)$ for all $j \in S$.

Let (X, T) be a dynamical system of a compact metric space X . For $n \geq 1$, let $P_n(T)$ be the set of all periodic points of T with period $\leq n$ and $P(T)$ the set of all periodic points of T , i.e.

$$P_n(T) = \{x \in X \mid \text{there is an } i \text{ such that } 1 \leq i \leq n \text{ and } T^i(x) = x\}$$

$$\text{and } P(T) = \bigcup_{n \geq 1} P_n(T).$$

Two points x and y of X are *trajectory-separated* for T if $T^j(x) \neq T^j(y)$ for $j \in \mathbb{N}$. A morphism $h : (X, T) \rightarrow (X', T')$ is a *trajectory-monomorphism* if $h(x), h(y)$ are trajectory-separated for T' , whenever $x, y \in X$ are trajectory-separated for T .

For $x, y \in X$, let $o_T(x) = (T^i(x))_{i \in \mathbb{N}}$ and $o_T(y) = (T^i(y))_{i \in \mathbb{N}}$ be two orbits of T . We say that the orbit $o_T(x)$ is *eventually equivalent* to the orbit $o_T(y)$ if the orbits will be equal in the future, i.e., there exists an $n \in \mathbb{N}$ such that $T^i(x) = T^i(y)$ for each $i \geq n$. In this case, we write $o_T(x) \sim_e o_T(y)$. We see that this relation is an equivalence relation. So we have the equivalence class

$$[o_T(x)] = \{o_T(y) \mid o_T(x) \sim_e o_T(y)\}$$

containing $o_T(x)$ and we put

$$[O(T)] = \{[o_T(x)] \mid x \in X\}.$$

Note that if $T : X \rightarrow X$ is injective, the function $o : X \rightarrow [O(T)]$ defined by $x \mapsto [o_T(x)]$ is bijective, i.e., $o : X \cong [O(T)]$. Also, note that if $h : (X, T) \rightarrow (X', T')$ is a morphism of dynamical systems, then h induces the function $h : [O(T)] \rightarrow [O(T')]$ defined by $h([o_T(x)]) = [o_{T'}(h(x))]$ for $x \in X$. A morphism $h : (X, T) \rightarrow (X', T')$ of dynamical systems is a *trajectory-isomorphism* if h induces the bijection $h : [O(T)] \cong [O(T')]$.

Proposition 4.4. *Suppose that a morphism $h : (X, T) \rightarrow (X', T')$ is a trajectory-monomorphism and h is surjective, i.e., $h(X) = X'$. Then h is a trajectory-isomorphism:*

$$h : [O(T)] \cong [O(T')]$$

We need the definition of topological entropy and we give the definition by Bowen [Bow78]. Let $T : X \rightarrow X$ be any map of a compact metric space X . A subset E of X is (n, ϵ) -separated if for any $x, y \in E$ with $x \neq y$, there is an integer j such that $0 \leq j < n$ and $d(T^j(x), T^j(y)) \geq \epsilon$. If K is any nonempty closed subset of X , $s_n(\epsilon; K)$ denotes the largest cardinality of any set $E \subset K$ which is (n, ϵ) -separated. Also we define

$$s(\epsilon; K) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon; K),$$

$$h(T; K) = \lim_{\epsilon \rightarrow 0} s(\epsilon; K).$$

It is well known that the topological entropy $h(T)$ of T is equal to $h(T; X)$ (see [Bow78]).

Let (X, T) and (Y, S) be one-sided dynamical systems of compact metric spaces. The *inverse limit* of T is the space

$$\varprojlim (X, T) = \{(x_i)_{i=0}^{\infty} \mid T(x_{i+1}) = x_i \text{ for each } i \in \mathbb{N}\} \subset X^{\mathbb{N}}$$

which has the topology inherited as a subspace of the product space $X^{\mathbb{N}}$. If $h : (X, T) \rightarrow (Y, S)$ is a morphism of dynamical systems, then the map

$$\varprojlim h : \varprojlim (X, T) \rightarrow \varprojlim (Y, S)$$

is defined by $\varprojlim h((x_i)_i) = (h(x_i))_i$ for $(x_i)_i \in \varprojlim (X, T)$. Note that if T is a homeomorphism, then $X \cong \varprojlim (X, T)$.

By [Kat20, Theorem 3.1], we have the following result.

Theorem 4.5. *Let X be a compact metric space with $\dim X = d < \infty$ and let $T : X \rightarrow X$ be a doubly 0-dimensional map with $\dim P(T) \leq 0$. Then there is a dense G_{δ} -set D of $C(X, \mathbb{R})$ such that for all $f \in D$,*

$$I_{T,f} = T_{T,f}^{\mathbb{N}} : (X, T) \rightarrow (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions:

- (a) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$,
- (b) $\varprojlim I_{T,f} : \varprojlim (X, T) \rightarrow \varprojlim (I_{T,f}(X), \sigma_{T,f})$ is a homeomorphism,
- (c) $h(T) = h(\sigma_{T,f})$ and
- (d) if $x, y \in X$ are trajectory-separated for T , then

$$|\{i \in \mathbb{N} \mid I_{T,f}(x)_i = I_{T,f}(y)_i\}| \leq 2d.$$

Now, we will introduce the notion of *reconstruction space* of dynamical systems (see [Kat21]).

Definition 4.6. A compact metric space X is a reconstruction space of dynamical systems if there exists a G_δ -dense set E of $C(X, X) \times C(X, \mathbb{R})$ such that for $(T, f) \in E$, the infinite delay observation map

$$I_{T,f} := I_{T,f}^{\mathbb{N}} : (X, T) \rightarrow (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions (1) and (2):

- (1) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$, where $\sigma_{T,f} = \sigma|_{I_{T,f}(X)}$, and
 (2) $\varprojlim I_{T,f} : \varprojlim (X, T) \rightarrow \varprojlim (I_{T,f}(X), \sigma_{T,f})$ is a homeomorphism.

$$\begin{array}{ccc} X & \xrightarrow{I_{T,f}} & I_{T,f}(X) \subset \mathbb{R}^{\mathbb{N}} \\ \downarrow T & & \downarrow \sigma_{T,f} \quad \downarrow \sigma \\ X & \xrightarrow{I_{T,f}} & I_{T,f}(X) \subset \mathbb{R}^{\mathbb{N}} \end{array}$$

Finally we obtain Theorem 4.7 by use of Theorem 1.3 (see [Kat21]). Theorem 4.7 implies that the problem 1.2 has a near-positive answer.

Theorem 4.7. Let X be one of the following spaces: PL-manifold, PL-manifold with branch structures, Menger manifold, Sierpiński carpet, Sierpiński gasket and dendrite. Then X is a reconstruction space of dynamical systems.

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