PROBLEMS IN TEICHMÜLLER THEORY

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1. Introduction

This paper collects problems in the Teichmuüller theory which the author concerns. The reference list at the last of the paper is possibly incomplete because it is given only from the author's knowledge. Though the author gives problems carefully, he approgizes if some problems given here are already solved or meaningless.

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2. Notation : Teichmüller spaces

2.1. **Notation.** Let D be a hyperbolizable domain in the Riemann sphere $\hat{\mathbb{C}}$, and G be a subgroup of the holomorphic automorphism group of D. Denote by $L^\infty(D,G)$ the complex Banach space of bounded measurable functions μ on D satisfying $\mu\circ g(z)\overline{g'(z)}/g(z) = \mu(z)$ for all $g \in G$ and $z \in D$ with the essential supremum norm $\|\mu\|_\infty = \mathrm{ess.sup}_{z\in D}|\mu(z)|$. Let $M(D,G)$ be the open unit ball in $L^\infty(D,G)$. Let $A^2(D,G)$ be the complex Banach space of holomorphic automorphic forms φ on D of weight -4 with the supremum norm $\|\varphi\|_\infty = \sup_{z\in D}\lambda_D(z)^{-2}|\varphi(z)|$ where $\lambda_D = \lambda_D(z)|dz|$ is the hyperbolic metric on D.

2.2. **Quasiconformal Teichmüller spaces of Fuchsian groups.** Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$ in $\mathbb{C}$. For $\mu \in M(\mathbb{D},\Gamma)$, we define a quasiconformal mapping W^μ on $\hat{\mathbb{C}}$ satisfying $\overline{\partial}W^\mu = \mu\,\partial W^\mu$ on $\mathbb{D}$, $\overline{\partial}W^\mu = 0$ on $\hat{\mathbb{C}}\setminus\mathbb{D}$, and $W^\mu(z) = z + o(1)$ as $z \to \infty$. For μ_1 and $\mu_2 \in M(\mathbb{D},\Gamma)$, we say that μ_1 and μ_2 are *(Teichmüller) equivalent* if $W^{\mu_1} = W^{\mu_2}$ on $\mathbb{D}^* = \hat{\mathbb{C}}\setminus\overline{\mathbb{D}}$. The *quasiconformal Teichmüller space* $\mathcal{T}_{qc}(\Gamma)$ of Γ is the quotient space of $M(\mathbb{D},\Gamma)$ by the Teichmüller equivalence relation (e.g. [26] and [44]). The projection $M(\mathbb{D},\Gamma) \ni \mu \to [\mu] \in \mathcal{T}_{qc}(\Gamma)$ is called the *Bers projection.* The image of the mapping

$$\beta_\Gamma :\ : \mathcal{T}_{qc}(\Gamma) \ni [\mu] \mapsto \mathrm{Sch}(W^\mu|_{\mathbb{D}^*}) \in A^2(\mathbb{D}^*,\Gamma)$$

is known to be an bounded open set containing the origin in $A^2(\mathbb{D}^*,\Gamma)$, where $\mathrm{Sch}(W)$ is the Schwarzian derivative of W. The mapping is called the *Bers embedding.* After

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identifying $\mathcal{T}_{qc}(\Gamma)$ with the image via the Bers embedding, $\mathcal{T}_{qc}(\Gamma)$ is thought of as a complex Banach manifold modeled on $A^2(\mathbb{D}^*,\Gamma)$.

2.3. **Reduced quasiconformal Teichmüller spaces of Fuchsian groups.** For references, see [15] and [16]. For $\mu \in M(\mathbb{D},\Gamma)$, we define a quasiconformal mapping w^μ on $\mathbb{D}$ with $\overline{\partial} w^\mu = \mu\,\partial w^\mu$ on $\mathbb{D}$ and $w^\mu(1) - 1 = w^\mu(i) - i = w^\mu(-i) + i = 0$. For μ_1 and $\mu_2 \in M(\mathbb{D},\Gamma)$, we say that μ_1 and μ_2 are *reduced Teichmüller equivalent* if $w^{\mu_1} = w^{\mu_2}$ on the limit set Λ_Γ of Γ. The set $\mathcal{T}^{\#}_{qc}(\Gamma)$ of reduced Teichmüller equivalence classes is called the *reduced quasiconformal Teichmüller space* of Γ. Let Ω_Γ be the component of the complement of Λ_Γ containing $\mathbb{D}^*$. Let $A^2_{\#}(\Omega_\Gamma,\Gamma)$ be a real subspace of $A^2(\Omega_\Gamma,\Gamma)$ consisting of $\varphi \in A^2(\Omega_\Gamma,\Gamma)$ which takes real along any component of $\partial\mathbb{D} \setminus \Lambda_\Gamma$ (as quadratic differentials).

Let $\pi_\Gamma\colon \mathbb{D} \to \Omega_\Gamma$ be the universal covering space such that π maps the imaginaly axis in $\mathbb{D}$ to a component of $\partial\mathbb{D} \setminus \Lambda_\Gamma$, and Γ^π be a subgroup of the automorphism group $\mathrm{Aut}(\mathbb{D})$ consisting of $g \in \mathrm{Aut}(\mathbb{D})$ with $\gamma \circ \pi = \pi \circ g$ for some $\gamma \in \Gamma$. Let $A^2_{\#}(\mathbb{D},\Gamma^\pi)$ be a real subspace of $A^2(\mathbb{D},\Gamma^\pi)$ consisting of $\varphi \in A^2(\mathbb{D},\Gamma)$ with $\varphi(-\overline{z}) = \overline{\varphi(z)}$ for $z \in \mathbb{D}$. Then $\pi^*_\Gamma(\varphi)(z) = \overline{\varphi(\pi(j(z)))\pi'(j(z))j_{\overline{z}}(z)^2}$ gives a real isometric isomorphism $\pi^*_\Gamma\colon A^2_{\#}(\Omega_\Gamma,\Gamma) \to A^2_{\#}(\mathbb{D}^*,\Gamma^\pi)$, where $j(z) = 1/\overline{z}$. Let $M_{\#}(\mathbb{D},\Gamma^\pi)$ be a subspace of $M(\mathbb{D},\Gamma^\pi)$ consisting of $\mu \in M(\mathbb{D},\Gamma^\pi)$ with $\mu(-\overline{z}) = \overline{\mu(z)}$.

The reduced quasiconformal Teichmüller space $\mathcal{T}^{\#}_{qc}(\Gamma)$ is naturally identified with the image of $M_{\#}(\mathbb{D},\Gamma^\pi)$ of the Bers projection $M(\mathbb{D},\Gamma^\pi) \to \mathcal{T}_{qc}(\Gamma^\pi)$. The Bers embedding $\beta_{\Gamma^\pi}\colon \mathcal{T}_{qc}(\Gamma^\pi) \to A^2(\mathbb{D}^*,\Gamma^\pi)$ induces an embedding $\mathcal{T}^{\#}_{qc}(\Gamma)$ onto a bounded open set in $A^2_{\#}(\Omega_\Gamma,\Gamma)$, and the identification induces a (canonical) real analytic Banach manifold structure on $\mathcal{T}^{\#}_{qc}(\Gamma)$. When, Γ is of the first kind, that is, $\Lambda_\Gamma = \partial\mathbb{D}$, $\mathcal{T}_{qc}(\Gamma) = \mathcal{T}^{\#}_{qc}(\Gamma)$ as sets, otherwise, $\mathcal{T}_{qc}(\Gamma) \neq \mathcal{T}^{\#}_{qc}(\Gamma)$.

Let $C \subset \partial\mathbb{D}$ be a closed set invariant under the action of Γ. The *Teichmüller (pseudo) distance* on δ^C_T on $M(\Gamma)$ with respect to C is defined by

$$\delta_T(\mu_1,\mu_2) = \frac{1}{2}\log\inf_h K(h)$$

where $h\colon \mathbb{D} \to \mathbb{D}$ is a quasiconformal mapping such that $h \circ \gamma \circ h^{-1} \in \mathrm{Aut}(\mathbb{D})$ for $\gamma \in w^{\mu_1}\Gamma(w^{\mu_1})^{-1}$ and $h \circ w^{\mu_1} = w^{\mu_2}$ on C. Then, $\delta^{\partial\mathbb{D}}_T$ and $\delta^{\Lambda_\Gamma}_T$ descends to distances d_T and $d^{\#}_T$ on $\mathcal{T}_{qc}(\Gamma)$ and $\mathcal{T}^{\#}_{qc}(\Gamma)$, respectively. These are called the *Teichmüller distances.*

2.4. **Teichmüller spaces and Reduced Teichmüller spaces of Riemann surfaces.** Let X be a hyperbolic Riemann surface and Γ be the Fuchsian group of X acting on $\mathbb{D}$. Let $\overline{X_0} = (\overline{\mathbb{D}} \setminus \Lambda_\Gamma)/\Gamma$. We call $\partial X = \overline{X} \setminus X$ is the *ideal boundary* of X. Any quasiconformal mapping on X extends on $\overline{X}$.

Fix a reference (hyperbolic) Riemann surface X_0. Two pairs (X_1, f_1) and (X_2, f_2) of quasiconformal mappings $f_i\colon X_0 \to X_i$ is said to be *Teichmüller equivalent* if there is a boholomorphism $c\colon X_1 \to X_2$ such that $f_2^{-1} \circ c \circ f_1$ is homotopic to the identity rel the ideal boundary ∂X_0. Two pairs (X_1, f_1) and (X_2, f_2) of quasiconformal mappings $f_i\colon X_0 \to X_i$ is said to be *reduced Teichmüller equivalent* if there is a boholomorphism $c\colon X_1 \to X_2$ such that $f_2^{-1} \circ c \circ f_1$ is homotopic to the identity. The set $\mathcal{T}_{qc}(X_0)$ of Teichmüller equivalence classes is called the *quasiconformal Teichmüller space* of X_0, and the set of reduced Teichmüller equivalence classes $\mathcal{T}^{\#}_{qc}(X_0)$ is called the *reduced quasiconformal Teichmüller space* of X_0.

Let Γ be the Fuchsian group of X_0 acting on $\mathbb{D}$. For $\mu \in M(\mathbb{D}, \Gamma)$, let $\Gamma^\mu = w^\mu \Gamma (w^\mu)^{-1}$ and $X_\mu \mathbb{D}/\Gamma^\mu$. The quasiconformal mapping w^μ descends to a quasiconformal mapping $f^\mu\colon X_0 \to X_\mu$. Then,

$$\mathcal{T}(\Gamma) \ni [\mu] \mapsto (X_\mu, f^\mu) \in \mathcal{T}_{qc}(X_0)$$
$$\mathcal{T}^{\#}(\Gamma) \ni [\mu] \mapsto (X_\mu, f^\mu) \in \mathcal{T}^{\#}_{qc}(X_0)$$

are bijection.

2.5. **Teichmüller space of a surface of type** (g, m)**.** Let $\Sigma_{g,m}$ be a closed orientable surface of genus g with m points removed. Assume that $2g - 2 + m > 0$. A *marked Riemann surface* of analytically finite type (g, m) is a pair (X, f) of a Riemann surface X of analytically finite type (g, m) and an orientation preserving homeomorphism $f\colon \Sigma_g \to X$. Two marked Riemann surfaces (X_1, f_1) and (X_2, f_2) are *Teichmüller equivalent* if there is a biholomorphism $h\colon X_1 \to X_2$ such that $h \circ f_1$ is homotopic to f_2. The *Teichmüller space* $\mathcal{T}_{g,m}$ of Riemann surfaces of analytically finite type (g, m) is the set of Teichmüller equivalence classes of marked Riemann surfaces of genus g. When $m = 0$, we abbreviate $\mathcal{T}_g$ to denote $\mathcal{T}_{g,0}$.

Let X_0 be a closed Riemann surface of analytically finite type (g, m) and Γ be a Fuchsian group of X_0 acting on $\mathbb{D}$. Then, there is a canonical identification

$$\mathcal{T}_{g,m} \cong \mathcal{T}_{qc}(X_0) \cong \mathcal{T}^{\#}_{qc}(X_0) \cong \mathcal{T}_{qc}(\Gamma) \cong \mathcal{T}^{\#}_{qc}(\Gamma).$$

3. Kerckhoff formula

In this section, we always assume that any Fuchsian group is not solvable.

3.1. **Nielsen core.** Let Γ be a torsion free Fuchsian group acting on $\mathbb{D}$, and $X = \mathbb{D}/\Gamma$. Let $\mathrm{CH}(\Gamma)$ be the convex hull of the limit set of Γ. The quotient $C(X) = \mathrm{CH}(\Gamma)/\Gamma$ is called the *convex core* of X. Let $\partial_0 C(X)$ be the union of all boundary components of $C(X)$ which are closed curves.

3.2. **Curve family.** Let Γ be a torsion free Fuchsian group acting on $\mathbb{D}$, and $X_0 = \mathrm{CH}_0(\Gamma)/\Gamma$. A *curve system* on X_0 is a disjoint union of homotopically non-trivial properly embedded simple arcs and closed curves. Let $\mathcal{C}(X_0, \partial X_0)$ be the set of homotopy classes of curve systems under homotopies that keep the endpoits on the ideal boundary. Let $\mathcal{C}(X_0) \subset \mathcal{C}(X_0, \partial X_0)$ be the sef of homotopy classes of simple closed curves on X_0.

3.3. **Extremal length.** For $\mu \in M(\mathbb{D}, \Gamma)$, For any conformal metric $\sigma = \sigma(z)|dz|$ on X_μ and $\gamma \in \mathcal{C}(X_0, \partial X_0)$, we denote by $\ell_\sigma(\gamma)$ the infimum of the σ-length of curve systems in the homotopy class $f^\mu(\gamma)$. The *extremal length* of γ on a marked Riemann surface X_μ is defined by

$$\lambda_\sigma(\mu, \gamma) = \sup_\sigma \frac{\ell_\sigma(\gamma)^2}{A_\sigma}$$

where A_σ is the σ-area on X_μ.

We denote by X_μ^d the double of X_μ along ∂X_μ.

3.4. **Kerckhoff pseudo-distance.** We define the *Kerckhoff pseudo-distance*

$$d_{Ker}([\mu_1], [\mu_2]) = \frac{1}{2} \log \sup_{\gamma \in \mathcal{C}(X_0, \partial X_0)} \frac{\lambda_\sigma(\mu_1, \gamma)}{\lambda_\sigma(\mu_2, \gamma)}$$

for $[\mu_1]$,$[\mu_2] \in \mathcal{T}_{qc}^{\#}(\Gamma) \cong \mathcal{T}_{qc}^{\#}(X_0)$ (cf. [29]). Since the extremal length has quasiconformally invariant (cf. [1]),

$$d_{Ker}([\mu_1], [\mu_2]) \leq d_T^{\#}([\mu_1], [\mu_2])$$

for $[\mu_1]$,$[\mu_2] \in \mathcal{T}_{qc}^{\#}(\Gamma) \cong \mathcal{T}_{qc}^{\#}(X_0)$. Since

$$\begin{aligned} |d_{Ker}([\mu_1], [\mu_2]) - d_{Ker}([\mu_1'], [\mu_2'])| &\leq d_{Ker}([\mu_1], [\mu_1']) + d_{Ker}([\mu_2], [\mu_2']) \\ &\leq d_T^{\#}([\mu_1], [\mu_1']) + d_T^{\#}([\mu_2], [\mu_2']), \end{aligned}$$

the Kerckhoff pseudo-distance function

$$\mathcal{T}_{qc}^{\#}(X_0) \times \mathcal{T}_{qc}^{\#}(X_0) \ni ([\mu_1], [\mu_2]) \mapsto d_{Ker}([\mu_1], [\mu_2])$$

is continuous in terms of the topology defined by the Teichmüller distance.

When Γ is finitely generated, it is known that the *Kerckhoff formula*

$$d_T^{\#}([\mu_1], [\mu_2]) = d_{Ker}([\mu_1], [\mu_2]) \tag{3.1}$$

holds for $[\mu_1]$, $[\mu_2] \in \mathcal{T}_{qc}^{\#}(X_0) \cong \mathcal{T}_{qc}^{\#}(\Gamma)$ (cf. [29, Theorem 4] and [41, Theorem 2.1]).

Problem 1 (**). *Does the Kerckhoff formula* (3.1) *hold for all torsion free Fuchsian group?*

To the author's knowledge, there is less known on the Kerckhoff pseudo-distance. For instance, the following weaker problem is thought to be open.

Problem 2 (* or **). *Let X be a Riemann surface. Is the Kerckhoff pseudo distance d_{Ker} a distance on $\mathcal{T}_{qc}^{\#}(X)$? If so, is d_{Ker} complete?*

In the case of the topologically finite type, key facts for proving (3.1) are that the (weighted) curve systems are dense in the space of measured foliations (laminations), and that any measured foliation (lamination) is realized as the vertical foliation of a quadratic differential. From these facts, the ratio of the extremal lengths is presented as a "stretch factor" of the vertical foliation of a quadratic differential along the Teichmüller geodesic defined by the quadratic differential. For the case of the infinite type, there is less information on the geometric of vertical foliations of integrable quadratic differential.

Problem 3 (* or **)**.** *Study the vertical foliations of integrable quadratic differentials on Riemann surfaces X. For instance,*

(**) *If so, is the set of integrable quadratic differentials with such "rational foliations" dense in the space of integrable holomorphic quadratic differentials when X is in the class $\mathcal{O}_G$?*

(*) *Let φ be an integrable holomorphic quadratic differential on X. Suppose that the vertical foliation of φ is a weighted curve system on X. Does the Kerckhoff formula* (3.1) *hold along the Teichmüller ray defined by φ?*

In [36], Marden and Strebel discuss the approximation of quadratic differentials by "simple quadratic differentials" under the topology of the local uniformly convergence for Riemann surfaces of class $\mathcal{O}_G$.

Problems 1 to 3 are not trivial even for particular Riemann surfaces. For instance, a hyperbolic surface is called a *flute surface* if it is a sequence of pairs of pants glued in succession along common length boundaries. A flute surface is *tight* if all the pants holes that have not been glued along are in fact cusps.

Problem 4 (* or **)**.** *Study Problems 1 to 3 for a particular surface. For instance, do for (tight) flute surfaces or more precisely, for $X = \mathbb{C} - \mathbb{Z}$.*

4. Fenchel-Nielsen Teichmüller spaces

For reference, see [2].

4.1. **Nielsen convex hyperbolic structure.** Let S be an orientable hyperbolisable surface. A *hyperbolic structure* H on S is a local chart $\{(U_\alpha, z_\alpha)\}_{\alpha\in A}$ on S such that $z_\alpha(U_\alpha) \subset \mathbb{H}$ and for any $\alpha, \beta \in A$, $z_\beta \circ z_\alpha^{-1}$ is the restriction of a conformal automorphism on $\mathbb{H}$ to $z_\alpha(U_\alpha \cap U_\beta)$. A pair (S, H) is called a *hyperbolic surface* of the underlying surface S. A hyperbolic surface is a Riemann surface. For the simplicity, we abbreviate by omitting S when the underlying surface S is understood.

A hyperbolic surface $H = (S, H)$ is called *Nielsen convex* if every point of H is contained in a geodesic arc with endpoints contained in simple closed geodesics in H. A geometric pair of pants decomposition $\mathcal{C} = \{C_i\}_i$ on H is a pair of pants decomposition such that every curve C_i in the decomposition is a simple closed geodesic, and every connected component of $S \setminus \cup_i C_i$ is isometric to the interior of a generalized hyperbolic pair of pants, where a *generalized hyperbolic pair of pants* is a pair of pants equipped with a convex hyperbolic metric with geodesic boundary, and possibly with cusps. In [2, Theorem 4.5], it is proved that when $\pi_1(H)$ is non-abelian and H is not a thrice punctured sphere, the following three conditions are equivalent: (1) H can be constructed by gluing some generalized hyperbolic pairs of pants along their boundary components; (2) H is Nielsen convex; (3) Every topological pair of pants decomposition of H by a system of simple closed curves is isotopic to a geometric pair of pants decomposition.

4.2. **Fenchel-Nielsen coordinates.** Henceforth, any hyperbolic structure in this section is assumed to be Nielsen convex. Fix a (topological) pants decomposition $\mathcal{P}$ defind by a collection of simple closed curves $\mathcal{C} = \{C_i\}_i$ on S.

Fix a hyperbolic structure H_0 on S. A *marked hyperbolic structure* is a pair $x = (f, H)$ of an orientation preserving homeomorphism $f\colon H_0 \to H$ and a hyperbolic surface H with base surface S. Let $x = (f, H)$ be a marked hyperbolic surface with base surface S. Let $\ell_x(C_i)$ be the hyperbolic length of the geodesic representative of $f(C_i)$ in terms of the hyperbolic structure H. The twist parameter $\theta_x(C_i)$ along C_i is defined as the same way as that in the case of Riemann surfaces of analytically finite type, in such a way that a complete positive Dehn twist along the curve C_i changes the twist parameter by addition of 2π. The *Fenchel-Nielsen parameters* of x is the collection of pairs $\{(\ell_x(C_i), \theta_x(C_i)\}_{C_i \in \mathcal{C}}$, where it is understood that if C_i is homotopic to a boundary component, then there is no associated twist parameter, and instead of a pair $(\ell_x(C_i), \theta_x(C_i))$, we have a single parameter $\ell_x(C_i)$. Given two marked hyperbolic metrics x and y on S, following [2], we define their *Fenchel-Nielsen distance* with respect to $\mathcal{P}$ by

$$d_{FN}(x, y) = \sup_{C_i} \max \left\{ \left| \log \frac{\ell_x(C_i)}{\ell_y(C_i)} \right|, |\ell_x(C_i)\theta_x(C_i) - \ell_y(C_i)\theta_y(C_i)| \right\},$$

again with the convention that if C_i is the homotopy class of a boundary component of S, then there is no twist parameter to be considered. Two marked hyperbolic structures x and y are said to be *Fenchel-Nielsen bounded* relative to $\mathcal{P}$ if $d_{FN}(x, y)$ is finite. Two marked hyperbolic structures $x_1 = (f_1, H_1)$ and $x_2 = (f_2, H_2)$ are said to be *Teichmüller equivalent* if there is an isometry $h\colon H_1 \to H_2$ such that $h \circ f_1$ is homotopic to f_2. The *Fenchel-Nielsen Teichmüller space* with respect to $\mathcal{P}$ and H_0, denoted by $\mathcal{T}_{FN}(H_0) = \mathcal{T}_{FN,\mathcal{P}}(H_0)$, is the space of Teichmüller equivalence classes of Fenchel-Nielsen bounded marked hyperbolic structures. The Fenchel-Nielsen distance is a distance on $\mathcal{T}_{FN}(H_0)$.

Problem 5 (* or **)**.** *Is the Fenchel-Nielsen distance a Finsler distance?*

When, S is topologically finite, $\mathcal{T}_{FN,\mathcal{P}}(H_0)$, $\mathcal{T}_{FN,\mathcal{P}'}(H_0)$ and $\mathcal{T}^{\#}_{qc}(H_0)$ are naturally homeomorphic. However, from [2, Proposition 6.2], there are a topologically infinite surface S, a hyperbolic structure H on S, and two pairs of pants decompositions $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{T}_{FN,\mathcal{P}}(H) \neq \mathcal{T}_{FN,\mathcal{P}}(H)$. This means that there is a marked hyperbolic structure $x = (f, H) \in \mathcal{T}_{FN,\mathcal{P}}(H)$ such that $x \notin \mathcal{T}_{FN,\mathcal{P}}(H)$.

Problem 6 (* or **)**.** *Let S be a surface. When $\mathcal{T}_{FN,\mathcal{P}}(H) = \mathcal{T}_{FN,\mathcal{P}'}(H)$ for any pants decomposition $\mathcal{P}$ and $\mathcal{P}'$ on S, is S topologically finite?*

A marked hyperbolic structure $x = (f, H)$ is said to satisfy the *upper bound condition* with respect to $\mathcal{P}$ if there is an $M > 0$ such that $\ell_x(C_i) \le M$ for all i. From [2, Theorem 8.5], when H_0 satisfies the upper bound condition with respect to $\mathcal{P}$, the identity map

$$\mathcal{T}^{\#}_{qc}(H_0) \ni (f, H) \mapsto (f, H) \in \mathcal{T}_{FN,\mathcal{P}}(H_0)$$

is locally bi-Lipschitz homeomorphism. In particular, when H_0 satisfies the upper bound condition with respec to two pairs of pants decompositions, $\mathcal{T}_{FN,\mathcal{P}}(H_0) = \mathcal{T}_{FN,\mathcal{P}'}(H_0)$.

Versions of the Fenchel-Nielsen distances. The Fenchel-Nielsen distance becomes a distance on $\mathcal{T}_{FN}(H_0)$, and

$$\mathcal{T}_{FN}(H_0) \ni x \mapsto (\log \ell_x(C_i) - \log \ell_{x_0}(C_i), \ell_x(C_i)\theta_x(C_i)) \in \ell^\infty \tag{4.1}$$

is an isometric bijection, where $x_0 = (id, H_0)$. Therefore, $(\mathcal{T}_{FN}(H_0), d_{FN}(H_0))$ is complete.

For $p > 0$, it is natural to consider the *Fenchel-Nielsen p-distance* $d_{FN,p}$ on the space of marked hyperbolic structures on S by

$$d_{FN,p}(x, y) = d_{FN,p,\mathcal{P}}(x, y) = \left\{ \sum_i \left| \log \frac{\ell_x(C_i)}{\ell_y(C_i)} \right|^p + |\ell_x(C_i)\theta_x(C_i) - \ell_y(C_i)\theta_y(C_i)|^p \right\}^{1/p}.$$

Hence, we can define the p-Fenchel-Nielsen Teichmüller space $\mathcal{T}_{FN,p}(H_0) = \mathcal{T}_{FN,p,\mathcal{P}}(H_0)$ in the similar way such that

$$(\mathcal{T}_{FN,p}(H_0), d_{FN,p}) \ni x \mapsto (\log \ell_x(C_i) - \log \ell_{x_0}(C_i), \ell_x(C_i)\theta_x(C_i)) \in \ell^p$$

is an isometric bijection.

Problem 7 (* or **)**.** *For $p \neq q$, study the relation between $(\mathcal{T}_{FN,p}(H_0), d_{FN,p})$ and $(\mathcal{T}_{FN,q}(H_0), d_{FN,q})$. For instance, are there a surface S, a pants decomposition $\mathcal{P}$ on S and a hyperbolic structure H_0 on S such that $\mathcal{T}_{FN,p}(H_0) \neq \mathcal{T}_{FN,q}(H_0)$ for any (or some) distinct p and q?*

Problem 8 (**)**.** *When $p = 2$, does $\mathcal{T}_{FN,p}(H_0)$ have a "nice" Hilbert manifold structure?*

Problem 9 (* or **)**.**
(1) *(*) For $p > 0$, is $(\mathcal{T}_{FN,p}(H_0), d_{FN,p})$ naturally embedded into the Teichmüller space of asymptotically conformal mappings? Namely, for $x_1 = (f_1, H_1)$, $x_2 = (f_2, H_2) \in \mathcal{T}_{FN,p}(H_0)$, is there an asymmtotically conformal mapping $h\colon H_1 \to H_2$ such that $h \circ f_1$ is homotopic to f_2?*
(2) *(* or **) If so, is the embedding locally (bi-)Lipschitz?*

See [19] for the Teichmüller space of asymptotically conformal mappings.

Problem 10 (*)**.** *For any $p > 0$, are there a topologically infinite surface S, a hyperbolic structure H on S, and two pairs of pants decompositions $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{T}_{FN,p,\mathcal{P}}(H) \neq \mathcal{T}_{FN,p,\mathcal{P}}(H)$?*

Problem 11 (*). *Fix $p > 0$. If a hyperbolic structure H on S satisfies the upper bound condition with respect to two pairs of pants decompositions $\mathcal{P}$ and $\mathcal{P}'$, does the identity mapping induce a locally bi-Lipschitz homeomorphism between $\mathcal{T}_{FN,p,\mathcal{P}}(H)$ and $\mathcal{T}_{FN,p,\mathcal{P}'}(H)$?*

In general, let L be a metrizable sequece space. We can also define the *L-Fenchel-Nielsen Teichmüller space* $\mathcal{T}_{FN,L}(H_0) = \mathcal{T}_{FN,L,\mathcal{P}}(H_0)$ and the *L-Fenchel-Nielsen distance* $d_{FN,L} = d_{FN,L,\mathcal{P}}$ on $\mathcal{T}_{FN,L}(H_0)$ such that

$$(\mathcal{T}_{FN,L}(H_0), d_{FN,p}) \ni x \mapsto (\log \ell_x(C_i) - \log \ell_{x_0}(C_i), \ell_x(C_i)\theta_x(C_i)) \in L$$

is an isometric bijection.

Problem 12 (* or ***?). *Study the L-*Fenchel-Nielsen Teichmüller space *with various sequence spaces L. Find a "nice" sequence space L such that the function theoretic properties of Riemann surfaces ($\mathcal{O}_G$, $\mathcal{O}_{HB}$, $\mathcal{O}_{AD}$ ...) are reflected.*

Complex Fenchel-Nielsen coordinates. We also have a complex Fenchel-Nielsen coordinates on the quasiconformal deformation spaces of Fuchsian groups (e.g. [30] and [49]). In these deformation spaces, we consider the complex translation length function instead of the length funtion $\ell_x(C)$ and the bending function instead of the twist parameter $\theta_x(C)$.

Problem 13 (**). *Let Γ_0 be a Fuchsian group of H_0. Suppose that Γ_0 is of the first kind. Let $\mathcal{R}(\Gamma_0)$ be the space of faithful discrete* $\mathrm{PSL}_2(\mathbb{C})$*-representations of Γ_0.*

(1) *Find (or characterize) a subspace $\mathcal{R}_0(\Gamma_0)$ of $\mathcal{R}(\Gamma_0)$ such that the embedding* (4.1) *extends to a well-defined holomorphic embedding*

$$\mathcal{R}_0(\Gamma_0) \ni x \mapsto (\log \ell_x(C_i) - \log \ell_{x_0}(C_i), \ell_x(C_i)\theta_x(C_i)) \in \ell^\infty_{\mathbb{C}}$$

by the "complexification".

(2) *When H_0 satisfies the upper bound condition, does the embedding* (4.1) *extend to a well-defined holomorphic embedding on the quasiconformal deformation space of Γ_0?*

(3) *If one of the previous problems is affirmatively solved, is the extension surjective? If not, study the boundary of the image. For instance, is the boundary locally-connected (cf.* [10] *and* [35]*)?*

5. Length spectrum Teichmüller spaces

We continue to use the notation defined in the previous section. In this section, we assume that H_0 has no ideal boundary. Namely, the Fuchsian group of H_0 is assumed to be of the first kind. For general H_0, see [33] for instance.

For two marked hyperbolic surfaces $x_1 = (f_1, H_1)$ and $x_2 = (f_2, H_2)$, we define the (symmetrized) *length spectrum distance*

$$d_{ls}(x_1, x_2) = \frac{1}{2} \sup_C \left| \log \frac{\ell_{x_1}(C)}{\ell_{x_2}(C)} \right|,$$

where C runs all simple closed curves on S. Define

$$\mathcal{T}_{ls}(H_0) = \{x = (f, H) \mid d_{ls}(x_0, x) < \infty\}/(\text{Teichmüller equivalence}),$$

where $x_0 = (id, H_0)$. The space $\mathcal{T}_{ls}(H_0)$ is called the *length spectrum Teichmüller space* of H_0. There are various investigations on the length spectrum Teichmüller spaces (e.g. [34], [47], [48]).

Since

$$d_{ls}(x, y) \leq d_T(x, y)$$

(cf. [53]), there is a natural Lipschitz embedding

$$\mathcal{T}_{qc}(H_0) \ni x \mapsto x \in \mathcal{T}_{ls}(H_0).$$

In [4], Basmajian and Saric showed the following: For a geodesically complete tight flute surface X_0 built by gluing pairs of pants with rapidly increasing cuff lengths $\{\ell_n\}_n$, where the *geodesically completeness* means that every geodesic can be extended infinitely far in both directions, and the *rapidly increasing sequence* is an increasing sequence $\{\ell_n\}_n$ such that $\ell_n \to \infty$, $\sum_{k=1}^{n} \ell_k = o(\ell_{n+1})$ $(n \to \infty)$. Then, the closure $\overline{\mathcal{T}_{qc}(X_0)}$ of $\mathcal{T}_{qc}(X_0)$ contains all surfaces with the Fenchel-Nielsen coordinates $\{(\ell_n, \theta_n)\}_n$, where $-C\ell_n \leq \theta_n \leq C\ell_n$, for $C > 0$, and the lengths $\{\ell_n\}$ correspond to a marked surface in $\mathcal{T}_{qc}(X_0)$.

Problem 14 (** or ***)**.** *For any hyperbolic surface H_0, characterize the closure $\overline{\mathcal{T}_{qc}^{\#}(H_0)}$ in $\mathcal{T}_{ls}(H_0)$.*

6. Mapping class group

6.1. **Isometries.** First we start with the closed surfaces with finite points removed. The *mapping class group* $\mathrm{MCG}_{g,m}$ is the group of homotopy classes of orientation preserving homeomorphisms on $\Sigma_{g,m}$. Any $[\omega] \in \mathrm{MCG}_{g,m}$ acts biholomorphically on $\mathcal{T}_{g,m}$ by

$$[\omega]_*(X, f) = (X, f \circ \omega^{-1}).$$

Then, we have a homomorphism

$$\mathrm{MCG}_{g,m} \ni [\omega] \mapsto [\omega]_* \in \mathrm{Aut}(\mathcal{T}_{g,m}).$$

The image $\mathrm{Mod}_{g,m}$ is called the *Teichmüller modular group*. Royden [45] shows

$$\mathrm{Aut}(\mathcal{T}_2) = \mathrm{Mod}_2 = \mathrm{MCG}_2/\mathbb{Z}_2 \quad (\mathbb{Z}_2 = \langle \text{Hyperelliptic involution} \rangle)$$
$$\mathrm{Aut}(\mathcal{T}_g) = \mathrm{Mod}_g = \mathrm{MCG}_g \quad (g \geq 3)$$

Let X_0 be an arbitrary Riemann surface. The biholomorphic automorphism group $\mathrm{Aut}(\mathcal{T}_{qc}(X_0))$ acts on isometrically on the quasiconformal Teichmüller space $\mathcal{T}_{qc}(X_0)$ (cf. [45] and). We consider the *quasiconformal mapping class group* $\mathrm{QC}(X_0)$, which is set of the homotopy classes of quasiconformal self-homeomorphisms on X_0, instead of the mapping class group, and the image $\mathrm{Mod}_{qc}(X_0)$ ($\subset \mathrm{Aut}(\mathcal{T}_{qc}(X_0))$ is called the *quasiconformal Teichmüller modular group* of X_0. Royden [45], Earle-Kra [17], Lakic [31], and (finally) Markovic [37] have shown that X_0 is either of infinite type or an analytically finite type with $2g + m > 5$,

$$\mathrm{Aut}(\mathcal{T}_{qc}(X_0)) = \mathrm{Mod}_{qc}(X_0) = \mathrm{QC}(X_0).$$

We can consider the similar problem for the length spectrum Teichmüller space. Namely, we pose the following problem.

Problem 15 (** or ***). *Characterize the isometry group of the length spectrum Teichmüller space. For instance,*

- *(**) is any isometry of the length Teichmüller space induced by a quasiconformal mapping?*
- *(* or **) Is there a Riemann surface X_0 (of infinite type) with the property that some isometry on the length spectrum Teichmüller space is not induced by any quasiconformal self-mapping on X_0. If yes, characterize the self-mappings on X_0 which induce isometries.*

It is also interesting to formulate the cataclysm coordinates for the length spectrum Teichmüller spaces of surfaces of infinite type. Saric and his collaborators give a series of investigations on the (bounded) measured laminations on hyperbolic (Riemann) surfaces of infinite type (e.g. [51], [8] and [52]).

Problem 16 (***). *Can we embed the length spectrum Teichmüller space into the space of (bounded or some) measured laminations, in the similar way as the cataclysm coordinates by Thurston?*

6.2. **Classifications.** When X_0 is of analytically finite type (g, m), the conjugacy classes of mapping classes are classified by Bers and Thurson (cf. [6] and [50]) as follows. For $[\omega]_* \in \mathrm{Mod}_{qc}(X_0)$, we define the *translation length* $a([\omega]_*)$ of $[\omega]_*$ by

$$a([\omega]_*) = \inf\{d_T(x, [\omega]_*(x)) \mid x \in \mathcal{T}_{qc}(X_0)\}.$$

We say $a([\omega]_*)$ is *attained* if there is $x \in \mathcal{T}_{g,m}$ such that $a([\omega]_*) = d_T(x, [\omega]_*(x))$.

The idea of Thurston's classification is to see the natural action of the mapping class on the set $\mathcal{S}(\Sigma_{g,m})$ of homotopy classes of non-trivial and non-peripheral simple closed curves on $\Sigma_{g,m}$. Namely, if the action of $[\omega]$ on $\mathcal{S}(\Sigma_{g,m})$ is of finite order, so is $[\omega]$. If $[\omega]$ admits a fixed point on $\mathcal{S}(\Sigma_{g,m})$, $[\omega]$ is reducible. Otherwise, $[\omega]$ is irreducible. When

$a([\omega]_*)$	Bers' Classification	Thruston's Classification
$= 0$, attained	elliptic	finite order
$= 0$, not attained	parabolic	reducible
> 0, attained	hyperbolic	irreducible (pseudo-Anosov)
> 0, not attained	pseudo-hyperbolic	reducible

Table 1. Bers-Thurston classification

$[\omega]$ is irreducible, the action of $[\omega]$ has a fixed point on the "completion" $\mathcal{PML}(\Sigma_{g,m})$ of $\mathcal{S}(\Sigma_{g,m})$ defined by the geometric intersection number. The completion $\mathcal{PML}(\Sigma_{g,m})$ is called the *space of projective measured laminations*, which is homeomorphic to $\mathbb{S}^{6g-7+2m}$. A surface homeomorphism is called *pseudo-Anosov* if it preserves a transverse pair of measured lamination, expanding one lamination uniformly by a factor $\lambda > 1$ and contracting the other by a factor $1/\lambda$. For the case of irreducible mapping class acting on $\Sigma_{g,m}$, the action on $\mathcal{PML}(\Sigma_{g,m})$ has exactly two fixed points and they are nothing but the invariant laminations.

Problem 17 (Shiga, ** or ***)**.** *Classify the isometry of the length spectrum Teichmüller space.*

Problem 18 (* or **)**.** *When an orientation preserving homeomorphism ω on X_0 is irreducible (in the sense of Thurston), study the geometric property of ω.*

There are several examples for pseudo-Anosov homeomorphisms on surfaces of infinite type:

- de Carvalho-Hall (cf. [14]. Train track, unimodal map, horseshoe)
- Chamanara (cf. [12]. Flat surface, Affine automorphism group)
- Hubert-Schmithüsen (cf. [25]. Flat surface, Infinite origami)
- Hooper (cf. [23]. Flat surface, Infinite Interval exchange)
- Morales-Valdez (cf. [43]. Flat surface, Hooper-Thurston-Veech construction))

Problem 19 (** or ***)**.** *Find topological conditions of mapping classes for which they are pseudo-Anosov.*

In the case where the surface is topologically finite, a mapping class is pseudo-Anosov if and only if it is irreducible of infinite order. In the case where the surface is of infinite type, these conditions are not enough. For instance, the translation $\omega(z) = z + 1$ is irreducible of infinite order on $\mathbb{C} - \mathbb{Z}$.

6.3. **A Classical problem.** The following is a famous unsolved problem (cf. [20, §5.4]).

Problem 20 (***)**.** *For any $h \geq 2$, does MCG_h contain a purely hyperbolic subgroup which is isomorphic to $\pi_1(\Sigma_g)$ for some $g \geq 2$?*

A subgroup of MCG_h is said to be *purely hyperbolic* if all element in the subgroup is pseudo-Anosov (except for the identity). Relating the problem, the following are known for instance.

- (Leininger-Reid [32]) For ever $g \geq 2$, there exist subgroups of MCG_g isomorphic to $\pi_1(\Sigma_{2g})$ for which all but one conjugacy class of elements (up to powers) is pseudo-Anosov.
- (Bowditch [9]) For any $h, g \geq 2$, there are only finitely many conjugacy classes of purely pseudo-Anosov surface subgroups in MCG_h which are isomorphic to $\pi_1(\Sigma_g)$.

In Kahn-Markovic [27] and Kahn-Wright [28], it was shown that for any cofinite volume Kleinian group Γ and $K > 1$, there is a subgroup $H < \Gamma$ that is K-quasiconfomally conjugate to a discrete cocompact Fuchsian group. It is natural to ask if the similar result holds for the mapping class group. Namely, we pose the following.

Problem 21 (***)**.** *For any $h > 2$ and $K > 1$, are there a cocompact Fuchisian group H_0 acting on the unit disk $\mathbb{D}$ and an equivariant K-Lipschitz immersion $f\colon \mathbb{D} \to \mathcal{T}_h$ such that the homomorphism $f_*\colon H_0 \to \mathrm{MCG}_h \cong \mathrm{Aut}(\mathcal{T}_h)$ induced by f is injective and the image $f_*(H_0)$ is purely hyperbolic?*

A holomorphically and isometrically embedded Poincaré disk (of curvature -4) in the Teichmüller space is called the *Teichmüller disk*. A stabilizer subgroup of a Teichmüller disk is called a *Veech group* (for the semi-translation surface defined from the quadratic differential associated to the Teichmüller disk). When a Veech group is cofinite (as acting on the Teichmüller disk, which is isomorphic to the Poincaré disk), the qutient surface is called the *Veech surface*. The Veech surface is isometrically embedded in the Moduli space (that is, it is the image of a 1-Lipschitz map), but unfortunately, it is never closed, that is, its Veech group contains parabolic elements, which correspond to reducible elements in the Teichmüller modular group (cf. [24]).

We pose the following problem which is motivated from Bowditch's result stated above:

Problem 22 (* or **)**.** *for any $h > 2$ and $g \geq 2$, is there $K_0 = K_0(g,h) > 1$ with the following property?: For a cocompact Fuchisian group H_0 of genus g acting on the unit disk $\mathbb{D}$, there is no equivariant K-Lipschitz immersion $f\colon \mathbb{D} \to \mathcal{T}_h$ with $1 \leq K < K_0$ such that the homomorphism $f_*\colon H_0 \to \mathrm{MCG}_h \cong \mathrm{Aut}(\mathcal{T}_h)$ induced by f is injective and the image $f_*(H_0)$ is purely hyperbolic.*

Therefore, if Probrem 21 and Probrem 22 are affirmatively solved, the genus of $\mathbb{D}/H_0$ (in Probrem 21) must diverge as $K \to 1$ when h is fixed.

6.4. **Holomorphic families.** Let $\hat{M}$ be a two-dimensional complex manifold and let C be a non- singular one dimensional analytic subset of $\hat{M}$ or empty. Let B be a Riemann surface. Assume that there exists a holomorphic mapping $\hat{\pi} : \hat{M} \to B$ satisfying the following two conditions;

(1) π is proper and of maximal rank at every point of $\hat{M}$, and
(2) setting $M = \hat{M} - C$ and $\pi = \hat{\pi} \mid_M$, the fiber $S_b = \pi^{-1}(b)$ of M over each b in B is an irreducible analytic subset of M and is of fixed finite type (g, n) as a Riemann surface.

We call such a triple (M, π, B) a *holomorphic family of Riemann surfaces of type* (g, n) *over* B. A holomorphic family is called *locally trivial* if for any $b \in B$, there is a neighborhood V of b in B such that the fiber space $\pi\colon \pi^{-1}(V) \to V$ is isomorphic to $S_b \times V \to V$ (the canonical projection on the second coordinate). For a holomorphic family (M, π, B) of Riemann surfaces of type (g, m) over a hyperbolic surface B and the univeral covering $\tilde{B} \to B$ with the Deck transformation group Γ_0, there are a holomorphic map $\Phi\colon \tilde{B} \to \mathcal{T}_{g,m}$, called the *representation*, and a homomorphism $\rho\colon \Gamma_0 \to \mathrm{Mod}_{g,m}$, called the *monodromy*, such that $\rho(g) \circ \Phi = \Phi \circ g$ for all $g \in \Gamma_0$. When $\tilde{B}$ is either $\hat{\mathbb{C}}$ or $\mathbb{C}$, the family is locally trivial. Hence, we assume that B is hyperbolic. A subgroup H of $\mathrm{Mod}_{g,m}$ is said to be *reducible* if the (natural) action of H on $\mathcal{S}(\Sigma_{g,m})$ has a fixed point. Otherwise H is said to be *irreducible*. Shiga [46] shows that when B is of analytically finite type, the monodromy of locally non-trivial holomorphic family over B is infinite and irreducible.

Problem 23 (** or ***). *Characterize infinite irreducible subgroups of* $\mathrm{Mod}_{g,m}$ *which are the images of monodromies of holomorphic families of Riemann surfaces of type* (g, m).

McMullen [40] observes that the limit set of the action of the mapping class group (Teichmüller modular group) on the Bers slice is the whole Bers boundary. The following problem is motivated from McMullen's observation.

Problem 24 (** or ***). *Does the limit set of the action of the monodromy of a locally non-trivial holomorphic family of Riemann surfaces of type* (g, m) *over a Riemann surface of class* $\mathcal{O}_G$ *acting on the Bers slice coincide with the whole Bers boundary?*

7. Complex structure

7.1. The following is a kind of a classical question.

Problem 25 (** or ***). *Study the Teichmüller space* $\mathcal{T}_{g,m}$ *as a complex manifold. For instance, does the algebra of holomorphic functions have some special properties?*

Probrem 25 is motivated from the following Daskalopoulos-Mese's result [13]: Assume that MCG_g acts (as a discrete automorphism group) on a contractible Kähler manifold $\tilde{M}$ such that there is a finite index subgroup Γ' of MCG_g satisfying the properties:

(i) $M := \tilde{M}/\Gamma'$ is a smooth quasiprojective variety.

(ii) M admits a compactification M as an algebraic variety such that the codimension of $\overline{M} \setminus M$ is ≥ 3.

Then $\tilde{M}$ is equivariantly biholomorphic or conjugate biholomorphic to the Teichmüller space $\mathcal{T}_g$ where MCG_g acts on $\mathcal{T}_g$ as the mapping class group.

7.2. The Teichmüller space $\mathcal{T}_{g,m}$ is biholomorphic to a bounded domain in $\mathbb{C}^{3g-3+m}$ (cf. [5]). However, it is conjectured that the boundary is very wild, see discussion in [11, §10]. It is natural to ask if the Teichmüller space is realized in some "nice" domain. Namely, we pose the following problem:

Problem 26 (** or ***). *Is the Teichmüller space holomorphically and properly embedded into a "nice" pseudoconvex domain(e.g. a pseudoconvex domain with smooth boundary or a convex domain)?*

Probrem 26 is motivated from the result by Fornaess [18] who shows that any strongly pseudoconvex domain is holomorphically and properly embedded into a (higer dimensional) convex domain. It is known that the Teichmüller space $\mathcal{T}_{g,m}$ is not biholomorphic to a convex domain (cf. [38]). Moreover, $\mathcal{T}_{g,m}$ is not biholomorphically equivalent to a bounded domain in $\mathbb{C}^{3g-3+m}$ which is strictly locally convex at even one boundary point (cf. [22]).

The Teichmüller space $\mathcal{T}_{g,m}$ is Stein (cf. [7]). Hence, $\mathcal{T}_{g,m}$ is realized as a closed submanifold of $\mathbb{C}^N$ for some $N \leq 6g-4+2m$ (cf. [21, Chapter VII, C, 10 Theorem]).

Problem 27 (* or trivial?). *For g, m with $2g-2+m>0$, determine the minimum $N = N(g,m)$ such that $\mathcal{T}_{g,m}$ is realized as a closed submanifold in $\mathbb{C}^N$.*

Problem 28 (* or **). *Construct the embedding $\mathcal{T}_{g,m} \to \mathbb{C}^N$ geometrically. For instance, can $\mathcal{T}_{g,m}$ be realized as a closed submanifold in the complex Euclidean space by using a finite number of the trace functions defined from projective structures?*

Problem 29 (** or ***). *Can the embedding be taken to be equivariant under the actions of* $\mathrm{Aut}(\mathcal{T}_{g,m})$ *and* $\mathrm{Aut}(\mathbb{C}^N)$*? If yes, consider the previous two problems for equivariant embeddings.*

Relating Probrem 29, we pose

Problem 30 (** or ***). *Is there an injective homomorphism from* $\mathrm{Mod}_{g,m}$ *(or* $\mathrm{MCG}_{g,m}$*) into* $\mathrm{Aut}(\mathbb{C}^N)$ *for some N?*

Relating the discussion in §3, we pose

Problem 31 (* or **). *Is there a finite system $\{\alpha_i\}_{i=1}^N$ of simple closed curves on $\Sigma_{g,m}$ such that the extremal length functions of α_i's define a global coordinate of $\mathcal{T}_{g,m}$?*

7.3. Antonakoudis [3] shows the following: Let $\mathcal{B}$ be a bounded symmetric domain and $\mathcal{T}_{g,m}$ be a Teichmüller space with $\dim_{\mathbb{C}} \mathcal{B}$, $\dim_{\mathbb{C}} \mathcal{T}_{g,m} \geq 2$. There are no holomorphic isometric immersions $\mathcal{B} \xrightarrow{f} \mathcal{T}_{g,m}$ or $\mathcal{T}_{g,m} \xrightarrow{f} \mathcal{B}$ equivalently, there are no holomorphic maps f such that df is an isometry for the Kobayashi norms on tangent spaces.

Problem 32 (** or ***). *Is Antonakoudis' result true even when $\mathcal{B}$ is a bounded homogeneous domain?*

Problem 33 (* or **). *Characterize the period mapping from $\mathcal{T}_g$ to the Siegel upper half-space $\mathfrak{S}_g$. For instance, is any equivariant holomorphic map $\mathcal{T}_g \to \mathfrak{S}_g$ "essentially" the period mapping?*

7.4. The following problem is somewhat basic and classical.

Problem 34 (* or **). *Study the complex analytical properties of conformal invariants on marked Riemanns surfaces as functions of $\mathcal{T}_{g,m}$. For instance, calculate their first and second derivatives and the Levi forms.*

For instance, it is known that the reciprocal of either the hyperbolic length function or the extremal length function is plurisuperharmonic (cf. [54] and [42]). Furthermore, the minus of the reciprocal of either the extremal length function is maximal (cf. [42]).

Problem 35 (* or **). *Find a necessary and sufficient condition for a negative maximal plurisubharmonic funcion to be the minus of the reciprocal of either the extremal length function.*

7.5. Masur [39] discusses the random walk on the Teichmüller space $\mathcal{T}_g$. Masur's random walk satisfies the following property: If $f\colon \mathcal{T}_g \to \mathbb{R}$ is pluriharmonic with respect to the Ahlfors-Bers complex structure on $\mathcal{T}_g$, it is harmonic with respect to the random walk (cf. [39, Proposition 2.1]). We pose:

Problem 36 (* or **). *Study Masur's random walk in the complex analytical setting.*

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