Problems related to Kobayashi Hyperbolicity A Proposal of Asymptotic Methods

Ryoichi Kobayashi (Nagoya University)

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Abstract

I collect some problems in complex geometry with asymptotic nature including questions related to Kobayashi Hyperbolicity, Kobayashi Non-Hyperbolicity, Geometric Quantization and YTD Conjecture, etc. A proposal on applications of measure concentration phenomenon to problems in complex geometry is presented.

Contents

1 Large Freedom in Complex Geometry

Problem 1*.** Suppose that X is an n-dimensional smooth projective variety and D an s.n.c. divisor on X.

(1) Suppose that D is very ample. Let $|mD|'$ denote the linear subsystem of $|mD|$ consisting of those elements $[\sigma] \in |mD|$ s.t. $\sigma|_D = 0$. Then By $\mathfrak{P}_m(D)$, we denote the totality of all n-dimensional linear subsystem μ of $|mD|$ whose support contains D. Then $\mathfrak{P}_m(D)$ is identified with $\mathbb{G}(n, |mD|')$ realized in $\mathbb{G}(n, |mD|)$ as a totally geodesic submanifold. Let us introduce the Haar probability measure on $\mathfrak{P}_m(D)$ and consider **random projections**

$$\{\mu : X \to \mu^*\}_{\mu \in \mathfrak{P}_m(D)}$$

and the associated **random ramification divisors**

$$\{R_\mu\}_{\mu \in \mathfrak{P}_m(D)} \ .$$

Pick a large number m and put

$$\overline{B}(\mathfrak{P}_m(D)) := \cup_{k=1}^m \cap_{\mu \in \mathfrak{P}_k} R_\mu \ .$$

Then $\{\overline{B}(\mathfrak{P}_m(D))\}_{m:\text{large}}$ is an invariant of (X, D) in the form of the series of proper algebraic subsets paramaterized by large numbers m.

• Question*** : How to compute the invariant $\overline{B}(\mathfrak{P}_m(D))$?

(2) Suppose that D is not necessarily very ample. We add free moving divisor $E \in |L|$ to D so that $D + E$ is very ample. How to compute the invariant

$$\cap_{E\in|L|}\overline{B}(\mathfrak{P}_m(D + E))$$

is the first step of defining the invariant for (X, D) where D is just a s.n.c. divisor.

• Question*** : How to compute the invariant $\cap_{E\in|L|}\overline{B}(\mathfrak{P}_m(D + E))$?

Hope : (2) in the case when X is of general type and D empty will give a proper algebraic subset of X which is the **obstruction to the Kobayashi hyperbolicity**. In general case, this invariant is the obstruction for the Second Main Theorem on the approximation to D of entire holomorphic curves into X. Considering $\mathfrak{P}_m$ instead of $\mathbb{G}(n, |mD|)$ corresponds to considering the logarithmic version w.r.to D.

Note : (2) is an interesting question even in the case when X is a Calabi-Yau manifold (e.g., infinitely many $\mathbb{P}^1$'s in a K3 surface).

Background Questions behind Problem 1.

For simplicity, I restrict myself on the case where D is very ample.

(b1)* Establish the relationship between the series of the invariants $\overline{B}(\mathfrak{P}_m(D))$ and the Second Main (Griffiths-Lang) Conjecture in the Nevanlinna Theory.

Let $f : \mathbb{C} \to X$ be any holomorphic curve s.t. $f(0) \notin D$. Let f_μ be the composition

$$f_\mu := \mu \circ f : \mathbb{C} \to \mu^* = \mathbb{P}^n \ .$$

The Riemann-Hurwitz formula implies

$$\mu^* K_{\mathbb{P}^n}^{-1} = K_X^{-1} + R_\mu$$

if $\mu : X \to \mu^*$ is a finite map. Plugging this into the Cartan/Ahlfors Second Main Theorem for $f_\mu : \mathbb{C} \to \mu^* = \mathbb{P}^n$, we would get

$$m_{f,F}(r) + \{N_{W(f_\mu),S_0} - T_{f,R_\mu}(r)\}_{\mu\in\mathfrak{P}_m(D)} \leq T_{f,K_X^{-1}}(r) + S_f(r)/\!/$$

where $F \in |mD|$. We ask the following questions on this heuristic argument.

(b2)* Justify the use of the Cartan/Ahlfors Second Main Theorem to f_μ. This is a problem. Indeed, Cartan/Ahlfors Theorem is the Second Main Theorem for the case $X = \mathbb{P}^n$ and D a hyperplane arrangement in general position. In the present setting, $\mu(F)$ is set theoretically just a hyperplane in $\mathbb{P}^n$. However, it has multiplicity as the image of $\mu(F)$ ($F \in |mD|$), because $\deg \mu : X \to \mu^*$ is large if m is large. How to formulate this situation and justify the use of Cartan/Ahlfors Theory in this setting ? Can one formulate this situation more abstractly and formulate appropriate logic for working in such situation ?

(b3)* Study the behavior of the random variable

$$\{N_{W(f_\mu),S_0}(r) - T_{f,R_\mu}(r)\}_{\mu\in\mathfrak{P}_m(D)}$$

when m is very large. Find non random variable description at places where f hits D. I think, solving this kind of problem is absolutely necessary before solving the Second Main Conjecture

$$T_{f,D}(r) \leq N^{(1)}_{f,D}(r) + T_{f,K_X^{-1}}(r) + \varepsilon T_{f,E}(r) \ .$$

Cf. Yamanoi's problem session.

Background : The base locus

$$\overline{B}(\mathfrak{P}_m(D)) = \cup_{k=1}^{m} \cap_{\mu \in \mathfrak{P}_m(D)} R_\mu$$

arises as the **obstruction** to the use of **measure concentration** on $\mathfrak{P}_m(D)$. Indeed, in order to apply measure concentration to get useful information on the typical behavior of the random variable

$$\{N_{W(f_\mu),S_0}(r) - T_{f,R_\mu}(r)\}_{\mu \in \mathfrak{P}_m(D)} \ ,$$

we absolutely need the situation where the μ-pull back of affine coordinate system of $\mu^* = \mathbb{P}^n$ constitutes a local coordinate system on X at places where we are working. Therefore, if the image of f is contained in the above base locus, we are not allowed to use the measure concentration.

The idea of measure concentration will appear again in Problem 6, where we explain it more. Cf. [KL].

(b4)* Find examples of **non s.n.c.** divisor D such that the situation where the base locus

$$\overline{B}(\mathfrak{P}_m(D)) = \cup_{k=1}^{m} \cap_{\mu \in \mathfrak{P}_m(D)} R_\mu$$

covers topologically dense of Zariski dense subset of X.

(b5)*** What is the arithmetic version of Ahlfors-Yamanoi type LLD ? What is measure concentration in geometry of numbers ? (adèle version of measure concentration ?) Here LLD is the abbreviation of "Lemma on Logarithmic Derivative". Cf. [A, 5.9-10], [Y], [V].

[A] L. Ahlfors, "The theory of meromorphic curves", Acta Soc. Sci. Fenn. Nova. (1941) Ser. A 3 no.4, 1-31.

[KL] R. Kobayashi and P. Lin, "Riemann-Hurwitz Approach to Nevanlinna Theory and Measure Concentration Phenomenon". preprint.

[V] P. Vojta,"Diophantine Approximation and Nevanlinna Theory" (Vojta's new book).

[Y] K. Yamanoi, "Algebro-geometric version of Nevanlinna's lemma on logarithmic derivative and applications", Nagoya Math. J. 173 (2004) 23-63.

Problem 2*****.** This is a classical problem which motivates Problems 1 (and its background),3,4.

Let X be an n-dimensional smooth complex projective variety of **general type**. The classical Kobayashi-Ochiai Theorem states that any holomorphic map $\mathbb{C}^n \to X$ degenerates, i.e., its Jacobian is identically zero.

• Question*** : Is the Zariski closure $\overline{f(\mathbb{C}^n)}^{\mathrm{Zar}}$ a proper algebraic subset of X ?

• Related question** : Can one generalize Brody's reparameterization method to construct non-degenerate holomorphic maps $\mathbb{C}^k \to X^n$ s.t. $T_{f,E}(r) = O(r^2)$ $(k \geq 2)$ where $E \to X$ is an ample line bundle ?

I think if Problem 1 (and background questions as well) and 4 are settled, then Problem 2 will also be settled. Cf. [K].

[K] S. Kobayashi, "Hyperbolic Complex Spaces", 1998.

[Kod] K. Kodaira, "Nevanlinna Theory" (translated by T. Ohsawa), Springer (2017).

Problem 3***. As in Problem 1, we consider random projections

$$\{\mu : X \to \mu^* \cong \mathbb{P}^n\}_{\mu\in\mathbb{G}} \; .$$

This time, we replace $\mathfrak{P}_m(D)$ by the whole Grassmannian $\mathbb{G} = \mathbb{G}(n, |mD|)$. Problem 3 is concerned with another possible application of the large freedom in the random projection $\{\mu\}_{\mu\in\mathbb{G}(n,|mD|)}$. The idea is that we try to "asymptotically cancel" (as $m \to \infty$) $\mu^*[$ linear anticanonical divisor of $\mathbb{P}^n$ $]$ and the ramification divisor R_μ in X (at least when X is Calabi-Yau) and consider the limit of the "inverse $\mu^{-1} : (\mathbb{C}^*)^n \to X$".

• Question*. Perform the above process in the case of dimension 1, i.e., the case where X is a polarized elliptic curve.

• Question**. Perform the above process in the case of a family of dimension 1. For instance, the case where X is a K3 surface given in the Weierstrass form like an elliptic curve (e.g., a cubic curve in $\mathbb{P}^2$).

(1) Let X^n be an n-dimensional polarized Calabi-Yau manifold (ample $L \to X$ is given).

• Question*** : Can we use the freedom in $\{\mu\}_{\mu\in\mathbb{G}}$ to make the branch divisor B_μ to approach $\cup_{i=1}^{n+1} H_i$ (the anti-canonical divisor of $\mathbb{P}^n$ consisting of coordinate hyperplanes in $\mathbb{P}^n$) in a "consistent" way along an increasing sequence $\{m\}$ tending to ∞ so that, in the limit as $m \to \infty$, the "asymptotic cancellation" between $\mu^*(\cup_{i=1}^{n+1} H_i)$ and R_μ takes place so that the limiting holomorphic map $f : (\mathbb{C}^*)^n \to X$ (in the opposite direction) is defined and the relationship

$$f^*(\Omega_X) = \Omega_{(\mathbb{C}^*)^n}$$

holds ? Here $\Omega_{(\mathbb{C}^*)^n} = \bigwedge_{i=1}^n \frac{dz^i}{z^i}$ and $\Omega_X$is a non-vanishing holomorphic n-form on X.

Suppose that $f^*(\Omega_X) = \Omega_{(\mathbb{C}^*)^n}$ holds. Let $\omega_X^{(1,1)}$ be a Ricci-flat Kähler metric on X. Then $(f^*\omega_X)^n = \Omega_{(\mathbb{C}^*)^n} \wedge \overline{\Omega_{(\mathbb{C}^*)^n}}$. The lift of $f^*\omega_X^{(1,1)}$ on $\mathbb{C}^n$ does not satisfy the condition of the following result of Chen-Wang.

Chen-Wang (2014) : Let $g_{i\bar{j}}$ be a Ricci-flat Kähler metric on $\mathbb{C}^n$ satisfying the condition $\exists C > 0$ s.t. $C^{-1}(\delta_{i\bar{j}}) < (g_{i\bar{j}}) < C(\delta_{i\bar{j}})$. Then $(g_{i\bar{j}})$ is the Euclidean metric.

• Question*** : on $\mathbb{C}^n$: $C^{-n} < \det(g_{i\bar{j}}) < C^n \overset{??}{\Rightarrow} C^{-1}(\delta_{i\bar{j}}) < (g_{i\bar{j}}) < C(\delta_{i\bar{j}}) \overset{\text{Chen–Wang}}{\Rightarrow} (g_{i\bar{j}}) = (\delta_{i\bar{j}})$.

$(1')$ The question in (1) seems to be related to the **SYZ conjecture** (cf. Hosono's problem session) (a conjectural geometric construction of the mirror CY manifold). Suppose that there exists a maximal rank limiting holomorphic map $f : (\mathbb{C}^*)^n \to X$ with the equation $f^*(\Omega_X) = \Omega_{(\mathbb{C}^*)^n}$. Then the image in X of compact tori in the LHS ($=(\mathbb{C}^*)^n$) constitutes a family of tori in X parameterized by $\mathbb{R}^n$ (the singular tori will appear by taking the closure in X).

• Question*** (same question as the first one in (1) but from different view point) : Can one show that the image of standard compact tori in $(\mathbb{C}^*)^n$ constitutes a family of special Lagrangian tori in X ?

(2) Let X^n be a Fano manifold. We ask the same question as in (1). In this setting, some part D of $\mu^*(\cup_{i=1}^{n+1} H_i)$ would persist after "asymptotic cancellation" with R_μ.

• Question*** : Does there exist an anti-canonical divisor D of X satisfying the condition that the limiting $f : (\mathbb{C}^*)^n \to X \setminus D$ (in the opposite direction) is defined and the relationship

$$f^*\Omega_{X\setminus D} = \Omega_{(\mathbb{C}^*)^n}$$

holds ? Here $\Omega_{X\setminus D}$ is a non-vanishing holomorphic n-form on $X \setminus D$ having log poles along D.

Problem 4*.** We would like to propose another approach to Problem 3 from the view point of Brody's reparameterization method.

Let X^n be a Calabi-Yau manifold or a Fano manifold. Let D be an anti-canonical divisor (the pole divisor of a meromorphic n-form) when X is Fano. Let Ω_X (resp. $\Omega_{X\setminus D}$) be a canonically defined holomorphic n-form on X (resp. $X \setminus D$).

We say that a holomorphic map $f : \mathbb{C}^n \to X$ is **pseudo-toric** if the condition

$$f^*\Omega_X \wedge \overline{f^*\Omega_X} = (\mathrm{const})\, \beta_{\mathbb{C}^n} \wedge \overline{\beta_{\mathbb{C}^n}} \quad \text{or} \quad f^*\Omega_{X\setminus D} \wedge \overline{f^*\Omega_{X\setminus D}} = (\mathrm{const})\, \beta_{\mathbb{C}^n} \wedge \overline{\beta_{\mathbb{C}^n}}$$

holds (cf. Problem 3). We ask the following questions concerning the construction of a pseudo-toric holomorphic map.

Let ω_X be any Kähler form on X. The zero-locus of $f^*(\omega_X^n)$ stems from the Jacobian $|J(f)|^2$. We write it as $|\psi_f|^2$ where ψ_f is a holomorphic function on $\mathbb{C}^n$. Then ψ_f is a globally defined holomorphic function on $\mathbb{C}^n$.

(1) We choose d so that $d > n$ holds. We consider the function

$$\frac{(\mathrm{tr}_{\beta_{\mathbb{C}^n}} f^*\omega_X)^d}{\mathrm{tr}_{\beta^n_{\mathbb{C}^n}} |\psi_f|^2 (f^*\omega_X)^n} + \frac{(\mathrm{tr}_{\beta^n_{\mathbb{C}^n}} |\psi_f|^2 (f^*\omega_X)^n)^{\frac{d}{n}}}{(\mathrm{tr}_{\beta_{\mathbb{C}^n}} f^*\omega_X)^n} \ .$$

We apply Brody's reparameterization argument [K, Chapt. 3, 6] to this function, where we use the Bergman metric on $\mathbb{D}(r) = \{z \in \mathbb{C}^n \,|\, |z| < r\}$.

• Question*** : We ask if we can apply the normal family argument to Brody's procedure to conclude the convergence of the reparameterized sequence of holomorphic maps defined on concentric balls whose diameter diverges to ∞. More precisely we ask whether the reparamerterized functions on $\mathbb{D}(R)$ (R being any fixed number) remains uniformly bounded.

(2) We ask the following question.

• Question*** : Is the quantity $\sup_{\mathbb{D}(R)} |\psi_f|$ for reparameterized sequence bounded above uniformly in R ? If so, we would get a pseudo-toric map $f : \mathbb{C}^n \to X$ (or $f : \mathbb{C}^n \to X \setminus D$) as a limit of reparameterized sequence.

(3) Problem 3 & 4 seem to be related to the concept of numerical Kodaira dimension for pseudo-effective canonical divisor. Suppose that K_X is pseudo-effective and k the largest nonnegative integer s.t. K_X^k is not zero in $H^{2k}(X)$. It is natural to ask whether there exists a rank $n-k$ holomorphic map $(\mathbb{C}^*)^{n-k} \to X$ (or $\mathbb{C}^{n-k} \to X$) whose Zariski closure has dimension $n-k$ in X. This means that this f is something like $f : (\mathbb{C}^*)^n \to X^n$ with $f^*\Omega_X \wedge f^*\overline{\Omega_X} = \beta_{(\mathbb{C}^*)^n} \wedge \overline{\beta_{(\mathbb{C}^*)^n}}$ in (1) where X is Calabi-Yau. We ask whether the Zariaki closure of such $f : \mathbb{C}^k \to X$ is a k-dimensional Calabi-Yau manifold and serves as a fiber of the virtual Iitaka fibration. If so, this may become

a seed to apply Ohsawa-Takegoshi extension theorem to the non-vanishing of $H^0(X, \mathcal{O}(K_X^m))$ (for some positive integer m). Cf. Fujino's problem session.

• Question***. Setting is as above. Does there exist a rank $n-k$ holomorphic map $f : (\mathbb{C}^*)^k \to X$ or $f : \mathbb{C}^{n-k} \to X$ s.t. its Zariski closure is Calabi-Yau of dimension $n-k$?

[K] S. Kobayashi, "Hyperbolic Complex Spaces", 1998.

Problem 5.** (1) Let X be a Calabi-Yau or Fano.

• Question** : We ask for $\forall x \in X$, $\xi_x \in T_xX$,

$$? \ \exists f : \mathbb{C} \to X \quad \text{(a Brody curve) s.t.} \quad f(0) = x \text{ and } f'\left(\left(\frac{\partial}{\partial z}\right)_0\right) = \xi_x.$$

(2) The relationship with Royden's extension theorem is of interest.

• Question** : Let X be a Calabi-Yau or Fano. Is the set of all Brody curves in X open w.r.to the compact-open topology ? Can one extend Royden's Extension Theorem [K, Chapt 3, A] globally to a Brody curve so that extended curves remain Brody ?

I think, if we can settle Problem 4, then we can settle Problem 5, too. However, I want more direct approach to (2).

[K] S. Kobayashi, "Hyperbolic Complex Spaces", 1998.

Problem 6*.** The Measure Concentration is a phenomenon one observes on large dimensional spheres, projective spaces, Grassmannians and so forth. An r-Lipschitz function (r is a fixed number) on these spaces looks like a constant function. For instance, randomly chosen k-vectors from a large dimensional Euclidean space are almost surely orthogonal to each other. So one can ask the following questions. Cf. [M].

(1) Concerning Problem 1, we ask the following :

• Question : What can one say on Problem 1 from the view point of the measure concentration ? Cf. [KL].

(2) Adiprasito-Sanyal [AS] proved a long standing conjecture on the log-concavity of Whitney numbers of arrangements by using the measure concentration on intrinsic volumes. Adiprasito's view point is there is a combinatorial background behind hard Lefschetz theorem. I think the relationship between the hard Lefschetz phenomenon and measure concentration (of intrinsic volumes) should be studied more systematically. For instance, by the Brunn-Minkowski inequality, the volume functional is log-concave w.r.to the Minkowski sum :

$$\mathrm{Vol}(\lambda K + (1-\lambda)L) \geq (\mathrm{Vol}(K))^{\lambda}(\mathrm{Vol}(L))^{1-\lambda} \ .$$

• Question : What is the origin (unified view point) for the log-concavity observed in different fields ?

• Question** : Y. Kazukawa established a convergence of complex projective spaces whose dimension going infinity to to the Hopf quotient Q of the infinite dimensional Gaussian space. The question is how to define the anti-canonical class (or rather the concept of Ricci curvature) on such a space. It would be very interesting to generalize Cartan/Ahlfors theory for holomorphic curves (resp. Calson-Griffiths theory) of holomorphic maps $f : \mathbb{C} \to Q$ (resp. $\mathbb{C}^\infty \to Q$) with target

space Q obtained by Kazukawa. Cf. D. Kazukawa, "Convergence of metric transformed spaces", arXiv:2008.04038 [math.MG].

[M] E. Meckes, "Concentration of Measure and the Compact Classical Matrix Groups", https://www.math.ias.edu/files/wam/Haar_notes-revised.pdf

[KL] R. Kobayashi and P. Lin, "Riemann-Hurwitz Approach to Nevanlinna Theory and Measure Concentration Phenomenon". preprint.

[AS] K. Adiprasito-R. Sanyal, "Whitney Numbers of Arrangements via Measure Concentration of Intrinsic Volumes", arXiv:1606.09412.

[S] G. Schechtman, "Concentration, results and applications"

[MS] V. Milman and G. Schechtman, "Asymptotic theory of finite dimensional normed spaces" (1986).

[GM] A. Giannopoulos and V. Milman, "Concentration property on probability spaces" (2000).

Problem 7***. It is believed that the quantization of pre-quantized compact symplectic manifold (M, L) is independent of polarization chosen. Andersen [An] disproved this folklore by showing that the Kähler quantization is not projectively flat but is only asymptotically projectively flat on the parameter space w.r.to the Hitchin connection ("asymptotically" means the tensor power of the pre-quantized line bundle becomes indefinitely large).

• Question : Is there a similar observation for real polarization, i.e., a Lagrangian fibration ?

Comment 1. The idea I have introduced in Problem 1 seems to be related to this problem. Let L be an ample line bundle with a fixed Hermitian metric whose Chern form is a Kähler form. Let $X \to |mL|^*$ be the Kodaira embedding ($m \geq m_0$ and m_0 sufficiently large) into the Fubini-Study space $|mL|^*$ (determined canonically from the given Hermitian metric of L). Using the collection of projections

$$\bigcup_{m \geq m_0} \{\mu_m : X \to \mu_m^*\}_{\mu_m \in \mathbb{G}(n,|mL|)} \ .$$

we would like to compare X and $\mathbb{P}^n$. In order to do so, we should consider the discrete analogue of $\mathbb{G}(n, |mL|)$ in the following way. We consider the moment map of the Fubini-Study space $|mL|^*$ w.r.to the action of the maximal torus of the maximal compact subgroup of $\mathrm{Aut}_{\mathcal{O}}(|mL|^*)$. Then we have $(\dim |mL| + 1)$ points corresponding to the vertices of the moment polytope. Instead of the full Grassmannian $\mathbb{G}(n, |mL|)$, we consider such $\mathbb{P}^n$'s in $|mL|^*$ spanned by $n+1$ points among these $(\dim |mL|+1)$ points. We then consider the pre-image of the unitary basis of $\mathcal{O}_{\mathbb{P}^n}(1)$ (corresponding to the fixed maximal torus) via the projection μ determined by this chosen $\mathbb{P}^n \in \mathbb{G}(n, |mL|)$. Then the pre-image of each point among these $(n + 1)$ points constitutes a "cluster" of $\deg(\mu)$ points in X as well as a section of mL having "peak" along this "cluster" of points. By considering the sequence of m's divisible by m_0 we can compare the images of X and $\mathbb{P}^n$'s chosen in $|m_0L|$ as above in the same ambient space $|mL|^*$ ($m = km_0$). A natural question is the Gromov-Hausdorff convergence of the "cluster" into a Lagrangian subspace of X as well as the asymptotic behavior of the sequence of sections having "peak" at the "cluster". Cf. [HY].

Comment 2. I suspect its relationship to the measure concentration on large dimensional projective spaces $\mathbb{P}^N$ into which polarized manifolds are Kodaira embedded (the measure concentration on the relative position of a maximal torus of $\mathrm{Aut}_{\mathcal{O}}(\mathbb{P}^N)$ and embedded M).

[An] J. Andersen, "Geometric quantization of curved phase spaces via the theory of resurgence", in preparation.

[HY] K. Hattori and M. Yamashita, "Spectral Convergence in Geometric Quentization – The Case of Toric Symplectic Manifolds", arXiv : 2002.12495.

Problem 8*.** A complete Ricci-flat Kähler metric with Eucdlidean volume growth is interpreted as a positive Kähler-Einstein metric at infinity. From this view point, it is natural to ask the following question. Suppose that (X, D) is a pair of smooth projective variety X and a smooth hypersurface D s.t. there exists an ample line bundle $L \to X$ satisfying $c_1(L) = \alpha[D]$, $\mathbb{Q} \ni \alpha > 1$.

• Question : Is the uniform K-stability of $(D, L|_D)$ equivalent to the existence of a complete scalar-flat Kähler metric on $X \setminus D$?

I think, if the polarization is anti-canonical, by using the solution to the YTD conjecture (Chen-Donaldson-Sun) one can confirm this statement by interpreting the existence of a Ricci-flat Kähler metric as a limit of certain sequence of Calabi-Yau Theorems on a compact Kähler manifold. Recently, Aoi [Ao] was able to show (under certain technical condition) that a complete scalar-flat Kähler metric with Euclidean volume growth is interpreted as a positive cscK metric at infinity. His method is a combination of Monge-Ampère equations (!) and Arezzo-Pacard method.

I would like to ask the following questions on Aoi's method :

• Question : Can one understand Aoi's result [Ao] as a limit of some sequence of Calabi-Yau Theorems on a compact Kähler manifold ? How is Arzzo-Pacard method modified in this setting ?

• Question : More generally, even under the assumption that the divisor D at infinity is K-unstable, can one still make a sequence of Calabi-Yau Theorems ? Does there exist a pair $(\overline{X}, \overline{D}$ birational to (X, D) (not holomorphic only along D) polarized by $\overline{L}$ s.t. $c_1(\overline{L}) = \alpha\overline{D}$ $(\alpha > 1)$ and $\overline{D}$ admits a positive cscK metric ($(\overline{D}, \overline{L}|_{\overline{D}})$ is K-stable) ?

• Question : In the anti-canonical polarization, such sequence of Calabi-Yau Theorems are in fact constructed and provide a sequence of metrics on D (on which the balanced metric does not exist). The question to be solved is its convergence "virtually to some positive KE metric on D' birational to D".

The above questions combined with Problem 6 motivates the following Problem 9.

[Ao] T. Aoi, "Constant Scalar Curvature Kähler Metrics on Noncompact Complex Manifolds", Thesis, Osaka University.

Problem 9*.** The YTD conjecture asks the equivalence between the K-stability (formulated in algebraic geometry) and the existence of the minimum of certain functionals (K-energy, Ding) on the space of Kähler potentials.

Motivated by Problems 7 and 8, I would like to propose the following questions concerning the "measure concentration approach" to the YTD conjecture.

Let (X, L) be a polarized manifold. The parameter space of the Bergman metrics defined from the polarization mL is a non-compact type symmetric space

$$M_N := \mathbf{GL}(N)/\mathbf{U}(N) \ .$$

Its complexification $M_N^{\mathbb{C}}$ is interpreted as a family of $\mathbf{GL}(N)/\mathbf{U}(N)$ parameterized by the compact dual

$$M_N' = G_N/K_N$$

where $G_N$and K_N are compact. We compactify the complexification $M_N^{\mathbb{C}}$ and denote it by

$$\mathcal{M}_N = \overline{M_N^{\mathbb{C}}} \ .$$

The compactification $\mathcal{M}_N$ is not necessarily a rational homogeneous variety but something similar, because there are rational homogeneous varieties which arises this way.

• Question 1 : Does the measure concentration phenomenon take place on $\mathcal{M}_N$ for large N (i.e., for large m) ?
Basic logic : Bakry-Émery criterion (Ricci $\geq \frac{1}{c}g$) $\Rightarrow$ LSI (constant c) $\Rightarrow$ concentration (constant $1/c$). smaller c, stronger concentration.
Can we use Bakry-Émery criterion effectively ?

Note : The complexification $M_N^{\mathbb{C}}$ is group theoretically embedded in the Grassmannian $\mathbb{G}(\mathfrak{k}_N^{\mathbb{C}}, \mathfrak{g}_N^{\mathbb{C}})$. Indeed, for each $x \in M_N^{\mathbb{C}}$, one associates its isotropy subalgebra of $\mathfrak{g}_N^{\mathbb{C}}$ and therefore $M_N^{\mathbb{C}}$ is embedded in the Grassmannian $\mathbb{G}(\mathfrak{k}_N^{\mathbb{C}}, \mathfrak{g}_N^{\mathbb{C}})$. Then we take its Euclidean closure to get the compactification $\mathcal{M}_N$.
$\Rightarrow M_N^{\mathbb{C}}$ is Fano.
Necessary : computation of Ricci curvature of $\mathcal{M}_N$ as $N \to \infty$.

Fubini-Study metric of the Grassmannian $\mathbb{G}(\mathfrak{k}_N^{\mathbb{C}}, \mathfrak{g}_N^{\mathbb{C}})$ restricts to $\mathcal{M}_N$. The compact dual M_N' is totally geodesic submanifold w.r.to this Kähler metric. In N-dimensional standard examples of measure concentration (spheres, projective spaces and Grassmannians), an ε-neighborhood of a k-dimensional totally geodesic submanifold ($k = \nu N$, $0 < \nu < 1$ is fixed) is almost everything measure theoretically.

• Question 2 : Is an ε-neighborhood of the compact dual M_N' in $\mathcal{M}_N$ measure theoretically almost everything ?

Suppose the above two questions 1 & 2 are true. Then we can continue our heuristic argument in the following way.

Consider the K-energy (or the Ding functional extended to a general polarization). We choose such functional and write it as F.

Under the hypothesis of K-stability, we consider the minimum point of the function F on each $\mathbf{GL}(N)/\mathbf{U}(N)$ in the family parameterized by the compact dual $M_N' = G_N/K_N$.

In the present setting this coincides with $U(N)$ and therefore its tangent bundle is trivial. We can extend F to $\mathcal{M}_N$ by choosing a trivialization (for instance, w.r.to the left action of $U(N)$ to $M_N' = U(N)$) of the tangent bundle.

The totality of the parameterized minimum constitutes an M_N' embedded in $\mathcal{M}_N$. On the other hand, the reference Kähler metric constitutes a totally geodesic M_N' (after moving it by varying the parameter).

• Question 3 : Are the embedded M_N' and the totally geodesic M_N' indefinitely separated if $N \to \infty$.
Suppose that the embedded M_N' indefinitely escapes from the totally geodesic M_N' as $N \to \infty$.
Does this contradicts to the K-stability ?

I ask if one can formulate the relationship between the question (3) and the expected measure concentration property on $\mathcal{M}_N$ with the Fubini-Study metric induced from the natural realization in the Grassmannian $\mathbb{G}(\mathfrak{k}_N^{\mathbb{C}}, \mathfrak{g}_N^{\mathbb{C}})$. The embedded M_N' constructed by extending the minimum point of the functional F on M_N by the left action of $U(N)$ on M_N' (extended to $\mathcal{M}_N$) is a totally geodesic submanifold of $\mathcal{M}_N$ w.r.to the induced metric described above. If N is large, by combining the affirmative answers to (1) & (2), we infer that the ε-tubular neighborhood ($\varepsilon > 0$ is fixed) of this embedded M'_N is measure theoretically everything and this tendency will become stronger as N becomes large.

• Question 4 : Does this imply that there exists a minimum point of the functional F on the space of Kähler potentials.

2 Other Asymptotic Problems

Problem 10****.** This problem is concerned with the characterization of the period condition among "pseudo-algebraic" minimal surfaces.

Let (g, ω) be a Weierstrass data defined on a finite Riemann surface M (i.e., a Riemann surface obtained by removing finitely many points from a compact Riemann surface). Suppose that (g, ω) satisfies the regularity condition and completeness and extends meromorphically across the punctured points (in short, "pseudo-algebraic"). Suppose moreover that the universal covering of M is the unit disk $\mathbb{D}$. We define a holomorphic function

$$H(z) = \int_{z_0}^{z} p(g)\omega$$

on $\mathbb{D}$, where

$$p(g) \in V_{\mathbb{R}} := \mathbb{R}\frac{1-g^2}{2} \bigoplus \mathbb{R}\frac{i(1+g^2)}{2} \bigoplus \mathbb{R}g$$

is a random element of the unit sphere of $V_{\mathbb{R}}$, the Euclidean metric is defined so that

$$\left\{\frac{1-g^2}{2}, \frac{i(1+g^2)}{2}, g\right\}$$

is an orthonormal basis. Then $\exists C > 0$ s.t.

$$C^{-1}\left(\log\frac{1}{1-r}\right)^2 < T_{e^H}(r) < C\left(\log\frac{1}{1-r}\right)^2$$

holds. The Weierstrass data satisfies the period condition (i.e., $e^{H(z)}$ is $\pi_1(M)$-invariant, i.e., the Enneper-Weierstrass representation is single-valued) if and only if $\exists C > 0$ s.t.

$$C^{-1}\log\frac{1}{1-r} < T_{e^H}(r) < C\log\frac{1}{1-r}$$

holds.

• Question : Can one take such C uniformly over all algebraic (or pseudo-algebraic) minimal surfaces ? Cf. [KM].

[KM] R. Kobayashi and R. Miyaoka, "Period Condition on Algebraic Minimal Surfaces and Nevanlinna Theory", preprint.

Problem 11****.** This is a real and non-compact analogue of Donaldson's dynamical system made from "Hilb" and "FS".

To each real division algebra F ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$) associates the projective spaces $\mathbb{P}^n(F)$ (for $F = \mathbb{O}$, $n = 1, 2$). We consider a pair (x, P_x) of a point $x \in \mathbb{P}^n(F)$ and its polar set P (i.e.,

$P_x = \{x \in \mathbb{P}^n(F) \,|\, \mathrm{dist}(x, P) = \mathrm{diameter}(\mathbb{P}^n(F))\})$. Consider a family of metric balls $\{\Omega\}$ centered at x contained in $B_0 := \{y \in \mathbb{P}^n(F) \,|\, \mathrm{dist}(y, x) = \frac{1}{2}\mathrm{diameter}(\mathbb{P}^n(F))\}$. We consider the following dynamical system of metrics on Ω.

1. The totality of harmonic extensions of δ-functions (Poisson kernels) on $\partial\Omega$ defines a family of probability measures on $\partial\Omega$ parameterized by Ω.

2. Let $\mathfrak{P}$ be the space of positive probability measures on $\partial\Omega$. We consider the embedding $P : \Omega \to \mathfrak{P}$ defined by the Poisson kernel. Then we replace the original Riemannian metric on Ω by the pull-back of the Fisher metric of $\mathfrak{P}$.

3. Return to Step 1 with the new metric obtained in Step 2.

4. Perform the Step 2 with the new Poisson kernel obtained in Step 3.

By iteration of the above procedure we get a sequence of metrics $\{g_i\}_{i=0}^{\infty}$ where g_0 is the initial metric induced from the standard metric from $\mathbb{P}^n(F)$.

• Question : Suppose that $\Omega \neq B_0$. Does the limit $\lim_{i\to\infty} g_i$ exist ? If so, does $\lim_{i\to\infty} g_i$ coincide with the $\mathbb{R}$-hyperbolic metric ?

Cf. [孫] : If the initial condition is $(\Omega, g) = (B^n, g_{\mathrm{euc}})$ (the Euclidean ball with Euclidean metric), then the first step transforms the Euclidean metric to a complete asymptotically $\mathbb{R}$-hyperbolic metric.

This phenomenon is very similar to the construction of the Bergman metric (= the $\mathbb{C}$-hyperbolic metric) on $\mathbb{B}^n_{\mathbb{C}}$ starting with L^2-holomorphic functions w.r.to the Lebesgue measure.

• Question : Suppose that $\Omega = B_0$. Does the limit $\lim_{i\to\infty} g_i$ exist ? If so, does $\lim_{i\to\infty} g_i$ coincide with the F-hyperbolic metric ?

• Question : Consider the totality of the δ-functions on $\partial\Omega$ and imagine that this constitutes a "contour" γ in "$\partial\mathfrak{P}$" and further imagine a solution to the Plateau problem on the existence of a "minimal surface" spanning γ (minimal, w.r.to the Fisher metric). The question is that whether the limit (if exists) of the sequence $\{g_i\}_{i=0}^{\infty}$ converges to this "minimal surface". Cf. [IS].

[孫] 孫旭, "ポアソン核によるフィッシャー計量の引き戻し", 修士論文 (名古屋大学 2021).

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Ryoichi Kobayashi,
Graduate School of Mathematics,
Nagoya University.
e-mail address : ryoichi@math.nagoya-u.ac.jp