# PROBLEMS ON ALGEBRAIC DYMANICS AND RELATED TOPICS

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## 1. Setting and Main interest

Let $X$ be a smooth projective variety of dimension $m$ over a field $k$. Unless stated otherwise, $k$ is assumed to be an algebraically closed field of characteristic zero (mostly $k = \mathbb{C}$), a point of $X$ means a closed, hence $k$-rational, point of $X$ and we identify $X = X(k)$ as varieties. We choose and fix an ample divisor $H := H_X$ on $X$.

We denote by $\mathrm{Rat}(X)$ the semi-group consisting of dominant rational self-maps of $X$, i.e., dominant rational maps from $X$ to $X$. Semi-group structure is given by the composition. We denote by $\mathrm{ord}\, f$ the order of $f$ in the semi-group $\mathrm{Rat}(X)$. $\mathrm{End}(X)$ is the sub-semi-group of $\mathrm{Rat}(X)$ consisting of surjective self-morphisms of $X$, $\mathrm{Bir}(X)$ is the subgroup of $\mathrm{Rat}(X)$ consisting of birational selfmaps of $X$, $\mathrm{PsAut}(X)$ is the subgroup consisting of birational selfmaps of $X$ being isomorphic in codimension one, and $\mathrm{Aut}(X)$ is the subgroup consisting of the biregular automorphisms of $X$. Clearly

$$\mathrm{End}(X) \subset \mathrm{Rat}(X) \supset \mathrm{Bir}(X) \supset \mathrm{PsAut}(X) \supset \mathrm{Aut}(X).$$

$\mathrm{Bir}(X) = \mathrm{Aut}(X)$ if $X$ has no rational curve (e.g. if $A$ is an abelian variety or its étale quotient) by Hironaka's resolution of indeterminacy of $f$ ([Hi64]), $\mathrm{Bir}(X) = \mathrm{PsAut}(X)$ if $K_X$ is nef by the negativity lemma (see e.g. [KM98]). Moreover, if $K_X$ is nef, then any element of $\mathrm{Bir}(X) = \mathrm{PsAut}(X)$ is written as a finite composition of flops and one isomorphism at the end by the fundamental theorem due to Kawamata [Ka08].

Let $f \in \mathrm{Rat}(X)$. We set $I(f) := X \setminus U(f)$. Here $U(f)$ is the largest Zariski open subset of $X$ on which $f$ is defined as a morphism to $X$. We call the Zariski closed subset $I(f)$ the indeterminacy set of $f$. $I(f)$ is of codimension at least 2 in $X$, as $X$ is normal and projective. We then define

$$X_f := \{x \in X \,|\, f^n(x) \notin I(f)\, \forall n \in \mathbb{Z} \cap [0, \infty)\},$$

$$\mathrm{Orb}_f(x) := \{f^n(x) \,|\, n \in \mathbb{Z} \cap [0, \infty)\} \text{ for } x \in X_f.$$

We call the set $\mathrm{Orb}_f(x)$ the (forward) orbit of $x$ under $f$.

Main interest in complex dynamics, algebraic dynamics and arithmetic dynamics is to understand the behaviour of orbits. For instance:

(i) Zariski closure and density in Zariski topology, also, closure and density in the Euclidean topology when $k = \mathbb{C}$;

(ii) description of the set of fixed points, i.e.,

$$\{x \in X_f \,|\, f(x) = x\}$$

and the distribution of the set of periodic points and preperiodic points

$$\{x \in X_f \,|\, f^n(x) = x\ \exists n \in \mathbb{Z} \cap [1, \infty)\}\ ,\ \{x \in X_f \,|\, |\mathrm{Orb}_f(x)| < \infty\}.$$

On the other hand, from birational algebraic geometry, we are more interested in:

(iii) structures, properties, descriptions of these 5 semi-groups, groups and their interesting subgroups.

Needless to say, (i), (ii), (iii) are closely related.

The following is a fundamental problem in any mathematics:

(iv) find/study/apply good invariants (e.g., for $(X, f)$ in our case).

## 2. Open Problems

In this section, I would like to pose some open problems related to (i), (iii), (iv). I indicate difficulty of each problem by the number of stars as requested by Professor Takayama. However, my evaluation might be wrong. From my experience, a ”good” answer for most problems posed here will deserve to (try to, at least) publish from an excellent journal such as JAG, JDG or higher. Here, assuming correctness, ”good” means that not only the result is striking and complete, but also the proof is innovative, ingenious and elegant.

**Question 2.1.** ((*) to (***)) Compute $d_k(f)$ for meaningful $(X, f)$.

Here $d_k(f)$ is the $k$-th dynamical degree of $f \in \mathrm{Rat}(X)$. $d_k(f)$ is (well-)defined, by using either operator norms or intersection numbers, as:

$$d_k(f) := \lim_{n\to\infty} ||(f^n)^*|_{N^k(X)_{\mathbb{R}}}||^{\frac{1}{n}} = \lim_{n\to\infty} ((f^n)^*(H^k).H^{d-k})^{\frac{1}{n}} \geq 1$$

$$(= \lim_{n\to\infty} ||(f^n)^*|_{H^{k,k}(X,\mathbb{R})}||^{\frac{1}{n}} = \lim_{n\to\infty} ((f^n)^*(\eta^k).\eta^{d-k})^{\frac{1}{n}} \geq 1$$

for any Kähler class $\eta$ when $k = \mathbb{C}$). Here $(f^n)^*$ is defined via Hironaka's resolution of the indeterminacy $I(f^n)$ and in general $(f^n)^* \neq (f^*)^n$; this is the case already for the standard Cremona involution of $\mathbb{P}^2$. See [DS05], [Tr15] for details and basic properties. The dynamical degrees $d_k(f)$ were first introduced and intensively studied by Dinh and Sibony ([DS05]) are now the most fundamental invariants for $(X, f)$ in complex dynamics and algebraic dynamics. The first dynamical degree $d_1(f)$ also plays a fundamental role in arithmetic dynamics. Among many important properties of $d_k(f)$, the two most fundamental properties are birational invariance of $d_k(f)$ and the exact formula for the topological entropy $h_{\mathrm{top}}(f)$:

$$h_{\mathrm{top}}(f) = \log \mathrm{Max}_{0\leq k\leq m} d_k(f),$$

when $f \in \mathrm{End}(X)$ and $k = \mathbb{C}$. However, it is in general very difficult to compute $d_k(f)$ even when $f \in \mathrm{Aut}(X)$.

### 2.1. **Salem number, Pisot number and $d_1(f)$ for surfaces.**

A Salem number is a real algebraic integer $a > 1$ whose Galois conjugates consist of $a$, $1/a$ and some numbers *on* the unit circle $S^1 \subset \mathbb{C}$. A Pisot number is a real algebraic integer $b > 1$ whose Galois conjugates consist of $b$ and some numbers *inside* the unit circle $S^1 \subset \mathbb{C}$.

Let $S$ be a smooth projective surface and $f \in \mathrm{Bir}(S)$ such that $d_1(f) > 1$. Then $d_1(f)$ is either a Salem number or a Pisot number and $d_1(f)$ becomes a Pisot number exactly when $f \in \mathrm{Bir}(\mathbb{P}^2)$ and it is not regularizable under any birational map $S \dashrightarrow \mathbb{P}^2$ ([DF01], [BC16]). The minimum Pisot number is $\lambda_P = 1.38...$ whose minimal polynomial is $x^3 - x - 1$.

The following question (going back to Lehmer around 1930s) is the most major open question for Salem numbers:

**Question 2.2.** (***) The absolute minimum of all Salem numbers exits. Moreover it is the Lehmer number $\lambda_{\rm Leh} = 1.176280...$ whose minimum polynomial is

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

McMullen [Mc07] discovered the following very interesting relation to algebraic geometry ([Mc07]):

**Theorem 2.3.** *Let $S$ be a smooth projective surface and $f \in \mathrm{Bir}(S)$ with $d_1(f) > 1$. Then $d_1(f)$ is a Salem number and satisfies $d_1(f) \geq \lambda_{\rm Leh}$ if $f$ is birationally conjugate to a biregular automorphism of a smooth projective surface.*

**Question 2.4.** (**) Find the minimum Pisot number realized as $d_1(f)$ for $f \in \mathrm{Bir}(\mathbb{P}^2)$.

See [DF01] [Xi15], [BC16] for interesting relevant results. It would be very exciting if the answer will *not* be $\lambda_P$. The minimum Salem number realized as $d_1(f)$ for $f \in \mathrm{Bir}(S)$ is completely determined by McMullen (cases where Rational surfaces, birational to a K3 surface and birational to an abelian surface) in [Mc07], [Mc11],[Mc16] and by Yu and me (the last case where birational to an Enriques surface) in [OY20]. They are the Lehmer number exactly when $S$ is a rational surface and birational to a K3 surface but not the Lehmer number otherwise. See also [Ca99] for surfaces with automorphism $f$ with $d_1(f) > 1$ and [Ueh16] for an elegant complete characterization of $d_1(f)$ of rational surface biregular automorphisms $f$.

2.2. $d_k(f)$ **for higher dimensional varieties.** There are very few cases where all $d_k(f)$ are computed for $k \in \mathbb{Z} \cap [1, m-1]$ especially when $m := \dim X \geq 4$ ([Lin12], [FW12], [Og09]). The following problem is doable but it is undoubtely meaningful and valuable to give a complete answer:

**Question 2.5.** (*) Let $A$ be an abelian variety of $\dim A = m$. Compute or characterize $d_k(f)$ for $f \in \mathrm{Aut}(A) = \mathrm{Bir}(A)$ when $m$ is smaller, e.g. $m = 3$ and 4.

See [Re12] for $m = 2$ and [OT14] for some relevant work in $m = 3$.

We call a topologically simply-connected smooth projective variety $X$ of dimension $m$ a Calabi-Yau $m$-fold in the strict sense (resp. a hyperkähler $m$-fold) if $X$ has no non-zero global $k$-form for all $k \in \mathbb{Z} \cap [1, m-1]$ but admits a nowhere vanishing global $m$-form (resp. $X$ admits everywhere non-degenerate global 2-form, say $\eta$, and satisfies $H^0(\Omega_X^2) = k\eta$).

The following problem is very difficult even to guess an answer (at least easy arguments in [Og09] for $f \in \mathrm{Aut}(X)$ can not be applied for $f \in \mathrm{Bir}(X) \setminus \mathrm{Aut}(X)$):

**Question 2.6.** (***) Let $X$ be a hyperkähler manifold of dimension $m$ and $f \in \mathrm{Bir}(X) \setminus \mathrm{Aut}(X)$. Compute or characterize $d_k(f)$ of $f \in \mathrm{Bir} X$ for $k \in \mathbb{Z} \cap [2, m-2]$.

In my opinion, a complete answer to the following question already deserve to publish from an excellent journal if an answer is "good":

**Question 2.7.** (**) Let $X = \mathrm{Hilb}^2(S)$ be the Hilbert scheme of two points on a K3 surface $S$. ($X$ is then a hyperkähler 4-fold.) Compute or characterize $d_2(f)$ of $f \in \mathrm{Bir}(X) \setminus \mathrm{Aut}(X)$ (or generalization to an arbitrary hyperkähler 4-fold).

Similar problems for some Calabi-Yau manifolds with rich $\mathrm{Aut}(X)$ or $\mathrm{Bir}(X)$ are equally interesting and meaningful:

**Question 2.8.** (**) Let $X$ be a general hypersurface of multi-degree $(2,2,\dots,2)$ in $(\mathbb{P}^1)^{m+1}$ with $m \ge 4$. Then $X$ is a Calabi-Yau $m$-fold in the strict sense. (See [CO15] also for the explicit description of $\mathrm{Bir}(X)$). Compute $d_k(f)$ for $f \in \mathrm{Bir}(X)$ and for $k \in \mathbb{Z} \cap [2, m-2]$. It is very interesting already for $m = 4$ and $k = 2$.

**Question 2.9.** (**) Let $S$ be an Enriques surface and let $X$ be the universal cover of $\mathrm{Hilb}^l(S)$ with $l \ge 2$. Then $X$ is a Calabi-Yau $2l$-fold in the strict sense ([OS11]). Compute $\mathrm{Bir}(X)$, $\mathrm{Aut}(X)$ and $d_k(f)$ for $f \in \mathrm{Bir}(X)$ and for $k \in \mathbb{Z} \cap [2, 2l-2]$. Again, it is very interesting already for $l = 2$ and $k = 2$.

2.3. **Distribution and algebraicity of $d_k(f)$.** The following result ([Ure18]) is very remarkable even though the proof based on the field of definition is quite simple:

**Theorem 2.10.** *The set $\{d_k(f)\,|\,f \in \mathrm{Rat}\,(X)\}$ is countable when $X$ runs through all smooth projective varieties and $f$ runs through all elements of* $\mathrm{Rat}\,(X)$.

One defines $d_k(f)$ for the compact Kähler manifold and its dominant self-bimeromorphic map $f$ (by using $H^{k,k}(X,\mathbb{R})$ instead of $N^k(X)$ and Kähler class $\omega$ instead of ample class $H$ in the definition). We denote by $\mathrm{Merom}\,(X)$ the set of all dominant meromorphic self maps of $X$. In order to answer the following natural question, one needs to find a new method completely different from [Ure18]:

**Question 2.11.** (***) Is the set $\{d_k(f)\,|\,f \in \mathrm{Merom}\,(X)\}$ is countable when $X$ runs through all compact Kähler manifolds and $f$ runs through all elements of $\mathrm{Merom}\,(X)$?

The following two results ([BDJ20], [BDJK21]) are the very striking also in the view of Theorem 2.10:

**Theorem 2.12.** (1) *There is $f \in \mathrm{Rat}(\mathbb{P}^2)$ such that $d_1(f)$ is transcendental. (See* [BDJ20] *for a very explicit form of $f$.)*
(2) *For each $m \ge 3$, there is $f \in \mathrm{Rat}(\mathbb{P}^m)$ such that $d_1(f)$ is transcendental. (See* [BDJK21] *for a very explicit form of $f$ in $m = 3$ and a relatively explicit form for $m \ge 4$.)*

*In both cases, there is such $f$ that is defined over $\mathbb{Q}$ in each dimension.*

Theorems 2.10 and 2.12 lead the following natural:

**Question 2.13.** ((*)-(***) for each, according to the depth of the study) Study the following sets:

$$\begin{aligned}
\mathrm{BFD}_{\mathbb{P}^m} &:= \{d_1(f)\,|\,f \in \mathrm{Bir}(\mathbb{P}^m)\}\ (m \ge 3),\\
\mathrm{RFD}_{\mathbb{P}^m} &:= \{d_1(f)\,|\,f \in \mathrm{Rat}(\mathbb{P}^m)\}\ (m \ge 2),\\
\mathrm{BFD}_m &:= \{d_1(f)\,|\,f \in \mathrm{Bir}(X)\}\ (m \ge 3),\\
\mathrm{RFD}_m &:= \{d_1(f)\,|\,f \in \mathrm{Rat}(X)\}\ (m \ge 2).
\end{aligned}$$

In $\mathrm{BFD}_m$ and $\mathrm{RFD}_m$, $X$ runs through all smooth projective varieties of dimension $m$. For instance, how are transcendental numbers distributed in each set?

See [BC16] for interesting properties of $\mathrm{BFD}_{\mathbb{P}^2}$ and $\mathrm{BFD}_2$. However both $\mathrm{BFD}_{\mathbb{P}^2}$ and $\mathrm{BFD}_2$ consist of algebraic integers only ([DF01]).

Transcendental numbers $d_1(f)$ in [BDJ20] and [BDJK21] are not so explicit even though $f$ is farily explicit. It is already very interesting to answer:

**Question 2.14.** (***) Find $f \in \mathrm{Rat}(\mathbb{P}^m)$ such that $d_1(f)$ is an "explicit" transcendental number. For instance, is there $f$ such that $d_1(f) = \pi$, $d_1(f) = e$, $d_1(f) = \log 3$ etc?

Here, for the choice of 3 (not 2), we note that $\log 2 < 1 < \log 3$ by $2 < e < 3$. The answer will be clearly negative for $\pi$ (resp. $e$, $\log 3$) if $\pi^{-1}$ (resp. $e^{-1}$, $(\log 3)^{-1}$) can not be the radius of convergence of some power series with positive integer coefficients (as observed in the subsection "Generating functions"). But I do not know the answer to this power series problem.

It is also natural to ask the following question (as asked by [BDJK21]):

**Question 2.15.** (***) Let $m \in \mathbb{Z} \cap [4, \infty)$. Is there $f \in \mathrm{Bir}(\mathbb{P}^m)$ such that $d_k(f)$ is transcendental for all $k \in \mathbb{Z} \cap [1, m-1]$?

This is false if $m = 2$ ([DF01]) and true if $m = 3$ ([BDJK21]).

2.4. **Kawaguchi-Silverman Conjecture.** Throughout this subsection, we choose the base field $k = \overline{\mathbb{Q}}$. Kawaguchi-Silverman Conjecture (KSC for short) asserts that *the arithmetic degree $a_f(x)$ exists, $a_f(x)$ is an algebraic integer and satisfies $a_f(x) = d_1(f)$, for all $x \in X_f$ with Zariski dense* $\mathrm{Orb}_f(x)$. KSC asks more but, in this note, we restrict ourselves to the above-mentioned part only.

See [KS16] and [Ma20] for the definition of $a_f(x)$, its basic properties and a full statement of KSC. See also [HS00], [MKI17] (in Japanese) for height functions and their basic properties needed, and more recent papers, e.g., [LS21], [Ma20a], for the current status.

**Question 2.16.** ((***) for each)

(1) Is KSC true for $f \in \mathrm{Bir}(\mathbb{P}^2)$?
(2) Is there $x \in \mathbb{P}^2_f$ with Zariski dense $\mathrm{Orb}_f(x)$ for $f \in \mathrm{Rat}(\mathbb{P}^2)$ with transcendental $d_1(f)$? If so, which part of KSC is false for this $f$?
(3) Let $m \geq 3$. Is there $x \in \mathbb{P}^m_f$ with Zariski dense $\mathrm{Orb}_f(x)$ for $f \in \mathrm{Bir}(\mathbb{P}^m)$ with transcendental $d_1(f)$? If so, which part of KSC is false for this $f$?

KSC for $f \in \mathrm{End}(X)$ is now relatively well studied (see e.g. [LS21], [Ma20a] and references therein for the current status) but almost nothing is known when $f \notin \mathrm{End}(X)$ (except special cases such as monomial maps). Each (1), (2), (3) needs really new idea and any solution (positive, negative) will be a major breakthrough for KSC for $f \notin \mathrm{End}(X)$.

To answer to some of the following question will also provide a major breakthrough for KSC for $f \notin \mathrm{End}(X)$:

**Question 2.17.** ((***) for each) Set $m = \dim X$.

(1) Is KSC true for $f \in \mathrm{Bir}(X) \setminus \mathrm{Aut}(X)$ for a hyperkähler 4-fold $X$? (See [LS21] when $f \in \mathrm{Aut}(X)$ in any $\dim X$.)
(2) Is KSC is true for $f \in \mathrm{Bir}(X)$ where $X$ is a Calabi-Yau manifold in Question 2.8 or in Question 2.9? Already the cases $m = 3$, 4 are highly non-trivial.
(3) Is KSC is true for $f \in \mathrm{Bir}(X) \setminus \mathrm{Aut}(X)$ when $X$ is a Calabi-Yau manifold of Picard number 2? Again, already any answer to any concrete $(X, f)$ with $m = 3$ is highly non-trivial. (See e.g. [Og14a] for some handy concrete examples.)

Better aspect of Calabi-Yau and hyperkähler (compared with rational) is that $\mathrm{Bir}(X) = \mathrm{PsAut}(X)$. For instance $d_1(f)$ is the spectral radious $f^*|_{N^1(X)}$ and it is an algebraic integer. However, at least in my little trial, the lack of functoriality of the height functions under birational maps is one of the major obstructions to solve Question 2.17 (See [HS00] for an explicit counterexample).

2.5. **Generating functions.** Recall that

$$d_1(f) := \lim_{n\to\infty} \delta_{H,n}^{\frac{1}{n}} \text{ where } \delta_{H,n} := ((f^n)^*(H).H^{m-1}) \in \mathbb{Z} \cap [1,\infty).$$

Hence $d_1(f)^{-1}$ is the radious of convergence of the power series

$$F_{f,H}(x) = \sum_{n\geq 0} \delta_{H,n} x^n \; (= \sum_{n\geq 0} (\delta_{H,n}^{\frac{1}{n}} x)^n).$$

Clearly $F_{f,H}(x)$ encodes more information than $d_1(f)$ (good aspect) but $F_{f,H}(x)$ depends on the choice of $H$ (bad aspect). It is easy to see that $F_{f,H}(x)$ is a raional function and $F_{f,H}(d_1(f)^{-1}) = \infty$ when $f \in \mathrm{End}(X)$. It is natural to ask:

**Question 2.18.** ((*)-(***) for each) Find and study

(1) roles of $F_{f,H}(x)$ in complex dynamics and algebraic dynamics. For instance, is there a useful criterion for $d_1(f) \notin \overline{\mathbb{Q}}$ in terms of $F_{f,H}(x)$? Is there any applicable one for $f$ in Theorem 2.12?
(2) properties of $F_{f,H}(x)$ which are independent of the choice of $H$. For instance, is rationality of $F_{f,H}$ independent of $H$?
(3) (1) and (2) for the power series arizing from $d_k(f)$ for $k \geq 2$.

[BDJ20] and [BDJK21] also consider power series to study transcendence of $d_1(f)$ but they consider different power series from ours.

2.6. **Zariski density of orbits and primitivity.** We recall the following fundamental result by Amerik and Campana [AC08]:

**Theorem 2.19.** *Let $k = \mathbb{C}$ and $f \in \mathrm{Rat}(X)$. Then, there is a unique dominant rational map $\pi : X \dashrightarrow C$ up to birational equivalence such that $\pi^{-1}(\pi(x))$ is the Zariski closure of $\mathrm{Orb}_f(x)$ for all very general $x \in X$. In particular, there is a Zariski dense orbit if and only if $C$ is a point.*

Theorem 2.19 is formulated and proved for dominant meromorphic self-maps of a compact Kähler manifold. The uncountability of $k = \mathbb{C}$ is crucial, even for projective $X$, in their use of Chow varieties ([AC08]). It is then natural to ask:

**Question 2.20.** (***) Is it possible to make Amerik-Campana theorem over $k = \overline{\mathbb{Q}}$?

An affirmative answer will provide several applications in arithmetic dynamics. See [Am11] for some partial but important useful answer.

De-Qi Zhang [Zh09] introduced the following very interesting notion:

**Definition 2.21.** $f \in \mathrm{Bir}(X)$ is not primitive if there are a dominant rational map $\pi : X \dashrightarrow B$ and $f_B \in \mathrm{Bir}(B)$ such that $0 < \dim B < \dim X$ and $\pi \circ f = f_B \circ \pi$. For instance if $1 \leq \kappa(X) \leq \dim X - 1$, then any $f \in \mathrm{Bir}(X)$ is not primitive (by the pluri-canonical map). Also if $1 \leq \dim C \leq \dim X - 1$ in Theorem 2.19, then $f$ is not primitive.

The following natural problem is posed in my ICM2014 report [Og14] but yet largely open especially when $\kappa(X) = -\infty$ and $f$ is an automorphism:

**Question 2.22.** (**) Construct (many) explicit examples of primitive $(X, f)$ such that $f \in \mathrm{Aut}(X)$ with $d_1(f) > 1$. Is there a smooth projective rationally connected variety $X$ of odd dimension $m \geq 7$ with primitive $f \in \mathrm{Aut}(X)$ (more preferable if in addition $d_1(f) > 1$)? How about rational manifolds? How about over $\overline{\mathbb{Q}}$?

See [Og19] for the current status on this question.

2.7. **Density of orbits in Euclidean topology.** We can not expect the existence of dense orbit in Euclidean topology even if $d_1(f) > 1$ in dimension $m \geq 3$, as a product example shows. The same question is however non-trivial in dimension $m = 2$.

McMullen found a very remarkable example of non-algebraic K3 surface automorphism with Siegel disk [Mc02]. This automorphism then admits no dense orbit in Euclidean topology (whereas almost all are Zariski dense). However, any projective K3 surface automorphism admits no Siegel disk. In the same paper [Mc02], McMullen asked:

**Question 2.23.** (***) Let $S$ be a projective K3 surface and $f \in \mathrm{Aut}(S)$ such that $d_1(f) > 1$. Does $f$ admit a dense orbit in Euclidean topology? Are there any affirmative examples? Are there any counterexamples?

In order to give an affirmative answer, not only projectivity but also the fact that $S$ is a K3 surface should be crucial, as there is a projective rational surface automorphism with Siegel disk ([Mc07]). PC search for an explicit example in [BE21] may be interesting or might give some hint.

2.8. **Slow dynamics.** It seems less popular to study $f$ with $d_1(f) = 1$ before Cantat proposed more intensive study of slow dynamics in his ICM2018 report [Ca18]:

**Question 2.24.** ((*)-(***) according to choice of problems) Study slow dynamics, i.e., study $(X, f)$ with $d_1(f) = 1$, hence necessarily $f \in \mathrm{Bir}(X)$.

One might guess that in this case no orbit is Zariski dense. However, this is not true, as already $f \in \mathrm{Aut}(\mathbb{P}^1)$ with $\mathrm{ord} f = \infty$ and translation of infinite order of an elliptic curve show. Note that in both cases $f \in \mathrm{Aut}^0(X)$, the identity component of $\mathrm{Aut}(X)$. It is natural to ask:

**Question 2.25.** (*) Classify pairs of a smooth projective surface $S$ and $f \in \mathrm{Bir}(S)$ (or $f \in \mathrm{Aut}(S)$ if you prefer) such that $d_1(f) = 1$ and $f$ has a Zariski dense orbit.

**Question 2.26.** (***) Are there a Calabi-Yau 3-fold or a hyperkähler 4-fold $X$ and $f \in \mathrm{Bir}(X)$ (or $f \in \mathrm{Aut}(X)$ if you prefer) with $d_1(f) = 1$ such that $f$ has a Zariski dense orbit?

Here are a few remarks on Question 2.26. In Question 2.26, one can show that there is a non-zero integral divisor $D \in \overline{\mathrm{Mov}}(X)$ such that $f^*D = D$ by replacing $f$ by $f^l$ for some positive integer $l$. If $\dim |nD| \geq 1$ for some $n > 0$, then we may get more information through the $f$-equivariant rational map given by $|nD|$. Unfortunately, to find $n$ with $\dim |nD| \geq 1$ is a notoriously difficult problem, even for $X$ is a Calabi-Yau 3-fold and $f \in \mathrm{Aut}(X)$ so that $D$ is chosen to be nef:

**Question 2.27.** (***) Let $X$ be a Calabi-Yau 3-fold and let $D$ be a non-zero nef divisor on $X$. Is there a positive integer $n > 0$ such that $|nD| \neq \emptyset$?

I proposed this question about 30 years ago in [Og93], but it still remains open. See e.g. [LOP18] and references therein for the current status.

On the other hand, Question 2.24 itself is quite new and there are many (more tractable) open problems related. See, for instance [Ca18], [CPR21], [OZ21], [DLOZ18], [LOZ21] and references therein for some progress and many interesting open problems in slow dynamics.

2.9. $\mathrm{Aut}(X)$ **and** $\mathrm{Bir}(X)$ **as groups and cone conjecture.** There are so many works and problems concerning $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$. Here I would like to discuss only those problems that are related to the cone conjecture for Calabi-Yau manifolds posed by Morrison [Mo93] and Kawamata [Ka97] (see also for [LOP18] for the current status):

**Question 2.28.** (***) Assume that $K_X \equiv 0$. Then:

(1) the action of $\mathrm{Aut}^*(X)$ on the effective nef cone $\mathrm{Nef}^{\mathrm{eff}}(X) := \mathrm{Nef}(X) \cap \mathrm{Eff}(X)$ has a rational polyhedral fundamental domain $\Delta := \Delta_{\mathrm{nef}}$ in the sense that

$$\mathrm{Nef}^{\mathrm{eff}}(X) = \cup_{g^* \in Aut^*(X)} g^* \Delta \text{ and,}$$

$$\mathrm{Int}\,(\Delta) \cap \mathrm{Int}\,(g^*(\Delta)) = \emptyset \text{ unless } g^* = \mathrm{id}_{N^1(X)_{\mathbb{R}}}.$$

(2) the action of $\mathrm{Bir}^*(X)$ on the movable effective cone $\overline{\mathrm{Mov}}^{\mathrm{eff}}(X) (\subset N^1(X)_{\mathbb{R}})$ has a rational polyhedral fundamental domein $\Delta_{\mathrm{mov}}$ in the same sense as above.

Here $\mathrm{Aut}^*(X)$ is the image of the natural map $\mathrm{Aut}^*(X) \to \mathrm{GL}\,(N^1(X))$ and similarly for $\mathrm{Bir}^*(X)$. Note that $\mathrm{Bir}^*(X)$ preserves $\overline{\mathrm{Mov}}^{\mathrm{eff}}(X)$, because $\mathrm{Bir}(X) = \mathrm{PsAut}(X)$ as $K_X$ is nef. There are variants in which the convex hull $\mathrm{Nef}^+(X)$ of $\mathrm{Nef}\,(X)(\mathbb{Q})$ and the convex hull $\overline{\mathrm{Mov}}^+(X)$ of $\overline{\mathrm{Mov}}\,(X)(\mathbb{Q})$ are used instead of $\mathrm{Nef}^{\mathrm{eff}}(X)$ and $\overline{\mathrm{Mov}}^{\mathrm{eff}}(X)$ (See e.g. [LOP18] for details and differences).

Question 2.28 is affirmative when $m = 2$, when $X$ is an abelian variety and when $X$ is a projective hyperkähler manifold (for + version), but almost nothing is known for Calabi-Yau manifolds in the strict sense. See [LOP18] and references therein for more precise informations. An affirmative answer to the cone conjecture also gives a strong consequence about the automorphism groups. For instance, $\mathrm{Aut}^*(X)$ (resp. $\mathrm{Bir}^*(X)$) is finitely generated if Question 2.28 (1) (resp. (2)) is affirmative as explained in [Lo14]. So, in particular, $\mathrm{Aut}(S)$ is finitely generated group when $S$ is a K3 surface.

In the statement of cone conjecture, it is interesting at least for me the following question, which to my best knowledge is unknown:

**Question 2.29.** ((*)?)

(1) Can one choose a finite rational fundamental domain $\Delta_{\mathrm{nef}}$ (resp. $\Delta_{\mathrm{mov}}$) so that $\{g^*\Delta\}_{g \in \mathrm{Aut}^*(X)}$ (resp. $\{g^*\Delta\}_{g \in \mathrm{Bir}^*(X)}$) forms a fan when Question 2.28 (1) (resp. (2)) is true for $X$?
(2) Can one choose so that $\Delta_{\mathrm{mov}} = \Delta_{\mathrm{nef}}$ if Question 2.28 is affirmative?

**Question 2.30.** ((**) or (***))

(1) Can one deduce Question 2.28 (2) from Question 2.28 (1)?
(2) Conversely, can one deduce Question 2.28 (1) from Question 2.28 (2)?

Any (non-trivial case) answer to the following question is very interesting:

**Question 2.31.** (**) Let $Z$ be a smooth Fano manifold of dimension $m+1 \geq 4$ and assume $X \in |-K_Z|$ is smooth (so that $Z$ is a Calabi-Yau manifold in the strict sense). Is the cone conjecture (2) true for $X$?

An affirmative answer to Question 2.30 (1) gives an affirmative answer to Question 2.31. Indeed, Kollár proved that $\mathrm{Nef}(X)$ is a finite rational polyhedral cone for $X$ in Question 2.31. From this, one can easily deduce that $\mathrm{Aut}(X)$ is also finite and Question 2.28 (1) holds for $X$ in Question 2.31. For instance, one can apply [Lo14].

The following question (for quotient type) is also worth studying:

**Question 2.32.** (**) Is the cone conjecture true for Calabi-Yau threefolds of Borcea-Voisin type, i.e., Calabi-Yau threefolds obtained as a crepant projective resolution of a finite Gorenstein quotient of either the product of an elliptic curve and a K3 surface or an abelian threefold?

2.10. **Finite generation of $\pi_0\mathrm{Aut}(X)$ and real forms.** $\mathrm{Aut}(X)$ is not only an abstract group but also a locally neotherian group scheme and one can speak of the identity component $\mathrm{Aut}^0(X)$ and consider the quotient group $\pi_0\mathrm{Aut}(X) := \mathrm{Aut}(X)/\mathrm{Aut}^0(X)$. The group $\pi_0\mathrm{Aut}(X)$ is discrete and countable so that it is natural to ask if $\pi_0\mathrm{Aut}(X)$ is finitely generated or not. The first negative answer is found by Lesieutre [Le18] when $\dim X = 6$. The following fairly satisfactory form is obtained by Dinh-Oguiso [DO19], Dinh-Oguiso-Yu [DOY21]:

**Theorem 2.33.** *Let $X$ be a smooth projective variety. We denote by $\kappa(X)$ the Kodaira dimension of $X$ (*[Ue75]*).*

(1) *For each $m \in \mathbb{Z} \cap [2, \infty)$ and $\kappa \in \{-\infty\} \cup (\mathbb{Z} \cap [0, m-2])$, there is $X$ such that $\dim X = m$, $\kappa(X) = \kappa$ and $\pi_0\mathrm{Aut}(X)$ is not finitely generated, except possibly $(m, \kappa) = (2, -\infty)$.*
(2) *Moreover, for each $m \in \mathbb{Z} \cap [3, \infty)$, there is a smooth rational variety such that $\pi_0\mathrm{Aut}(X)$ is not finite generated.*
(3) *If $\kappa(X) \geq \dim X - 1$, then $\pi_0\mathrm{Aut}(X)$ is finitely generated.*

The following question, which asks the major missing case in Theorem 2.33, is very difficult (at least for me) to solve:

**Question 2.34.** (***) Is there a smooth rational surface $S$ such that $\pi_0\mathrm{Aut}(S)$ is not finitely generated? More concretely, is $\pi_0\mathrm{Aut}(X_{\lambda,Q}) = \mathrm{Aut}(X_{\lambda,Q})$ finitely generated for some $\lambda$ and $Q$? Here $X_{\lambda,Q}$ is a rational surface studied by [DOY21b] (see also the next subsection for $X_{\lambda,Q}$).

It is also natural to ask the following:

**Question 2.35.** (**) Let $k$ be an algebraically closed field of characteristic $p \geq 2$. To what exent can one generalize Theorem 2.33?

See [Og20] for some attempt when $p \geq 3$. [Og20] is a rather easy paper but it contains a very interesting observation that the answer depend if the transcendental degree of $k$ over the prime field is zero or not. Question 2.35 is completely open for $p = 2$.

2.11. **From Real form problem.** The smooth complex projective rational surface $\mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^1$ is defined over $\mathbb{R}$ in a natural manner. Let $\lambda \in \mathbb{R}$ be a *generic* real number and $T_\lambda \to \mathbb{P}$ be the blow up of 16 real points $(a, b)$ of $\mathbb{P}$, where

$$a \in \{0, 1, 2, \infty\}\ ,\ b \in \{0, 1, \lambda, \infty\}.$$

Then $T$ is also defined over $\mathbb{R}$. Let $C \simeq \mathbb{P}^1$ be the exceptional curve on $T$ over $(\infty, \infty)$. Let $Q \in C(\mathbb{R}) = \mathbb{P}^1(\mathbb{R})$ be a *generic* real point and let

$$X := X_{\lambda,Q} \to T_\lambda$$

be the blow up of $T$ at $Q_T$. Then the surface $X$ is a smooth complex projective rational surface defined over $\mathbb{R}$. The following theorem [DOY21b] due to Dinh, Yu and me, gives the first definite answer to a long standing problem asked by Kharlamov [Kh02]:

**Theorem 2.36.** *If both $\lambda \in \mathbb{R}$ and $Q \in C(\mathbb{R})$ are generic, then the smooth complex projective rational surface $X = X_{\lambda,Q}$ has infinitely many mutually non-isomorphic real forms. That is, there are infinitely many $\mathbb{R}$-schemes $V_i$ such that $X \simeq V_i \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$ over* $\mathrm{Sepc}\,\mathbb{C}$ *but $V_i$ are not isomorphic over* $\mathrm{Sepc}\,\mathbb{R}$.

We know that $\mathrm{Aut}(T_\lambda)$ is (discrete and) finitely generated and $\mathrm{Aut}(X_{\lambda,Q})$ is also discrete but we do not know if $\mathrm{Aut}(X_{\lambda,Q})$ is finitely generated or not. To my best knowledge, $\pi_0\mathrm{Aut}V$ is not finitely generated for all *previously known* smooth complex projective varieties $V$ with infinitely many real forms ([Le18], [DO19], [DOY21]). Also, the number of the real forms of $X$ (if exists) is the cardinality of the Galois cohomology set $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(X))$. So the real form problem has a very close relation with $\mathrm{Aut}(X)$ both experimentally and theoretically. I think that it may not be so unreasonable to ask (as asked in [DOY21b]):

**Question 2.37.** (***) Is $\pi_0\mathrm{Aut}(X)$ not finitely generated if $X$ has infinitely many mutually non-isomorphic real forms (and $\mathrm{Aut}(X)$ is discrete if you prefer to assume)?

Obviously, an affirmative answer to Question 2.37 with Theorem 2.36 gives an affirmative answer to Question 2.34.

It is also interesting to describe explicit real forms of interesting series of varieties. For instance, the following question is worth solving:

**Question 2.38.** ((***) for $(m, d) = (2, 4)$ and (*) for other $(m, d)$) Let $F_m^d$ be the Fermat hypersurface of dimension $m$ and degree $d$:

$$F_m^d := (x_0^d + x_1^d + \ldots + x_{m+1}^d = 0) \subset \mathbb{P}^{m+1}.$$

Find all real forms of $F_m^d$ up to isomorphisms over $\mathrm{Spec}\,\mathbb{R}$.

Note that $F_2^4$ is a K3 surface so that $\mathrm{Aut}(F_2^4)$ is finitely generated as remarked before. However, $\mathrm{Aut}(F_2^4)$ is a notoriously difficult group which no one succeed to describe explicitly by now. For other $(m, d)$, we have a very explicit clean description of $\mathrm{Aut}(F_m^d)$ ([Sh88]). Mr. Sasaki, one of my master students, is now studying Question 2.38 and is getting some non-trivial results.

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