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Function spaces associated with the Dirichlet Laplacian

By

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Abstract

This is a survey of recent results on function spaces associated with the Dirichlet Laplacian. We study the well-definedness of the Besov spaces, properties of semigroup generated by the fractional Laplacian, and bilinear estimates.

§ 1. Introduction

Let Ω be an arbitrary open domain of $\mathbb{R}^d$ with $d \geq 1$. We consider the Dirichlet Laplacian A on $L^2(\Omega)$, namely,

$$A = -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

with the domain

$$D(A) := \{ f \in H_0^1(\Omega) \,|\, \Delta f \in L^2(\Omega) \},$$

where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with the H^1 norm $\|f\|_{H^1} := \{\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2\}^{1/2}$. We study several properties of function spaces such as Sobolev spaces and Besov spaces for applications to nonlinear partial differential equations on domains. As fundamental properties, we are interested in the well-definedness of those spaces, completeness, duality, lifting and so on. In this paper, we shall discuss semigroup generated by the Dirichlet Laplacian of fractional order and the bilinear estimates in those spaces.

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In the whole space $\mathbb{R}^d$, such results are nowadays well known and there are a lot of applications to partial differential equations. To the best of our knowledge for general domain case, such theory has not been well-established and the purpose of this paper is to give one of possibilities to approach the problems.

We start with defining spaces of test function spaces and Besov spaces, following the argument in [8]. The important property to introduce them is the Gaussian upper bound of the semigroup generated by the Dirichlet Laplacian, which allows us to obtain crucial spectral multiplier estimates. Let us remark here that it would be possible to apply this argument to more general operators such that the Gaussian upper bounds hold. Then we should define test function spaces on general domains appropriately, especially treating the low spectral component, which can give a theory of function spaces. We take $\phi_0(\cdot) \in C_0^\infty(\mathbb{R})$ a non-negative function on $\mathbb{R}$ such that

$$\operatorname{supp} \phi_0 \subset \{\, \lambda \in \mathbb{R} \,|\, 2^{-1} \le \lambda \le 2 \,\}, \quad \sum_{j\in\mathbb{Z}} \phi_0(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0, \tag{1.1}$$

and $\{\phi_j\}_{j\in\mathbb{Z}}$ is defined by letting

$$\phi_j(\lambda) := \phi_0(2^{-j}\lambda) \quad \text{for } \lambda \in \mathbb{R}. \tag{1.2}$$

Definition. (Spaces of test functions and distributions) (i) (*Inhomogeneous type*) *$\mathcal{X}(A)$ is defined by*

$$\mathcal{X}(A) := \Big\{ f \in L^1(\mathbb{R}^n_+) \cap \mathcal{D}(A) \,\Big|\, A^M f \in L^1(\mathbb{R}^n_+) \cap \mathcal{D}(A) \text{ for all } M \in \mathbb{N} \Big\}$$

equipped with the family of semi-norms $\{p_{A,M}(\cdot)\}_{M=1}^\infty$ given by

$$p_{A,M}(f) := \|f\|_{L^1(\mathbb{R}^n_+)} + \sup_{j\in\mathbb{N}} 2^{Mj} \|\phi_j(\sqrt{A})f\|_{L^1(\mathbb{R}^n_+)},$$

and $\mathcal{X}'(A)$ denotes the topological dual of $\mathcal{X}(A)$.
(ii) (*Homogeneous type*) *$\mathcal{Z}(A)$ is defined by*

$$\mathcal{Z}(A) := \Big\{ f \in \mathcal{X}(A) \,\Big|\, \sup_{j\le 0} 2^{-Mj} \big\|\phi_j\big(\sqrt{A}\big)f\big\|_{L^1(\mathbb{R}^n_+)} < \infty \text{ for all } M \in \mathbb{N} \Big\}$$

equipped with the family of semi-norms $\{q_{A,M}(\cdot)\}_{M=1}^\infty$ given by

$$q_{A,M}(f) := \|f\|_{L^1(\mathbb{R}^n_+)} + \sup_{j\in\mathbb{Z}} 2^{M|j|} \|\phi_j(\sqrt{A})f\|_{L^1(\mathbb{R}^n_+)},$$

and $\mathcal{Z}'(A)$ denotes the topological dual of $\mathcal{Z}(A)$.

It is easy to see that $\mathcal{X}(A)$, $\mathcal{Z}(A)$ are independent of the choice of the partition of the unity $\{\phi_j\}_{j\in\mathbb{Z}}$ and that they are Fréchet spaces. We also remark that L^1-norm is considered in the definition, while A is initially defined on $L^2(\Omega)$. For this, we recall the boundedness of the spectral multiplier and its uniformity with respect to a scaling parameter.

Lemma. ([1,7]) Let $1 \le p \le \infty$. Then

$$\sup_{j\in\mathbb{Z}} \|\phi_j(\sqrt{A})\|_{L^p\to L^p} < \infty.$$

Furthermore, if $\alpha \in \mathbb{R}$ and $1 \le p \le q \le \infty$, then

$$\sup_{j\in\mathbb{Z}} \frac{\|A^{\alpha/2}\phi_j(\sqrt{A})\|_{L^p\to L^q}}{2^{\alpha j + d(\frac{1}{p}-\frac{1}{q})j}} < \infty. \tag{1.3}$$

We define Besov spaces on Ω in the following way. Let ψ be a non-negative function such that

$$\psi \in C_0^\infty(\mathbb{R}), \quad \psi(\lambda) + \sum_{j\in\mathbb{N}} \phi_j(\lambda) = 1 \quad \text{for any } \lambda \ge 0.$$

Definition. (Besov spaces) *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$.*
(i) $B^s_{p,q}(A)$ *is defined by*

$$B^s_{p,q}(A) := \{f \in \mathcal{X}'(A) \mid \|f\|_{B^s_{p,q}(A)} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}(A)} := \|\psi(\sqrt{A})f\|_{L^p} + \big\|\{2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p(\mathbb{R}^n_+)}\}_{j\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})}.$$

(ii) $\dot{B}^s_{p,q}(A)$ *is defined by*

$$\dot{B}^s_{p,q}(A) := \{f \in \mathcal{Z}'(A) \mid \|f\|_{\dot{B}^s_{p,q}(A)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(A)} := \big\|\{2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p(\mathbb{R}^n_+)}\}_{j\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})}.$$

It was proved that the fundamental properties of the Besov spaces holds, such as the completeness, the duality, the embedding of the Sobolev type, the lifting, and so on. More precisely we have the following.

Proposition. ([8]) *Let $s, \alpha \in \mathbb{R}$, $1 \le p, q, r \le \infty$. Then the following hold:*

(i) $\mathcal{X}(\Omega)$ *and* $\mathcal{Z}(\Omega)$ *are Fréchet spaces and* $\mathcal{X}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{X}'(\Omega)$, $\mathcal{Z}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{Z}'(\Omega)$ *in the sense of continuous embedding.*

(ii) $\dot{B}^s_{p,q}(A)$ *is a Banach space and* $\mathcal{Z}(\Omega) \hookrightarrow \dot{B}^s_{p,q}(A) \hookrightarrow \mathcal{Z}'(\Omega)$ *in the sense of continuous embedding.*

(iii) *If $p, q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$, the dual space of $\dot{B}^s_{p,q}(A)$ is $\dot{B}^{-s}_{p',q'}(A)$. More precisely, if f belongs to $\dot{B}^{-s}_{p',q'}$, then the bounded linear functional T_f given by*

$$\dot{B}^s_{p,q}(A) \ni g \mapsto \int_\Omega \Big(\phi_j(\sqrt{A})f\Big) \cdot \overline{\Phi_j(\sqrt{A})g}\, dx \in \mathbb{C}, \quad \Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1},$$

is well-defined and there exists $C > 0$ so that

$$\|T_f\|_{\dot{B}^s_{p,q}(A)\to\mathbb{C}} \le C\|f\|_{\dot{B}^{-s}_{p',q'}}.$$

Conversely, for every bounded linear functional T on $\dot{B}^s_{p,q}(A)$, there exists $f_T \in \dot{B}^{-s}_{p',q'}$ such that the functional is written as above and there exists $C > 0$ so that

$$\|f_T\|_{\dot{B}^{-s}_{p',q'}} \le C\|T_f\|_{\dot{B}^s_{p,q}(A)\to\mathbb{C}}.$$

(iv) *If $r \le p$, $\dot{B}^{s+d(\frac{1}{r}-\frac{1}{p})}_{r,q}(A)$ is embedded to $\dot{B}^s_{p,q}(A)$.*

(v) *For any $f \in \dot{B}^{s+\alpha}_{p,q}(A)$, $A^{\frac{\alpha}{2}} f \in \dot{B}^s_{p,q}(A)$.*

In this paper, let us overview results about semigroup generated by the fractional Laplacian in section 2, and bilinear estimates in section 3.

§ 2. Semigroup generated by the fractional Dirichlet Laplacian

We introduce the fractional Laplacian and the semigroup along the paper [5]. First of all, the boudedness of the spectral multipliers gives the following boundedness

$$\|A^{\frac{\alpha}{2}}\phi_j(\sqrt{A})\|_{L^1(\Omega)\to L^1(\Omega)} \le C2^{\alpha j}, \quad \|e^{-tA^{\frac{\alpha}{2}}}\phi_j(\sqrt{A})\|_{L^1(\Omega)\to L^1(\Omega)} \le Ce^{-ct2^{\alpha j}}$$

for all $j \in \mathbb{Z}$ with some positive constants $C, c > 0$. This allows us to define the operators $A^{\frac{\alpha}{2}}$ and $e^{-tA^{\frac{\alpha}{2}}}$ on the test function space $\mathcal{Z}(A)$. Thus they are also defined on the dual space $\mathcal{Z}'(A)$ with the resolution of identity (see Lemma 4.5 in [8]). In fact, for any $f \in \mathcal{Z}'(A)$, we define $A^{\frac{\alpha}{2}} f, e^{-tA^{\frac{\alpha}{2}}} f$ as elements of $\mathcal{Z}'(A)$ such that

$$A^{\frac{\alpha}{2}} f = \sum_{j\in\mathbb{Z}} A^{\frac{\alpha}{2}}\phi_j(\sqrt{A})f \quad \text{in } \mathcal{Z}'(A),$$

$$e^{-tA^{\frac{\alpha}{2}}} f = \sum_{j\in\mathbb{Z}} e^{-tA^{\frac{\alpha}{2}}}\phi_j(\sqrt{A})f \quad \text{in } \mathcal{Z}'(A).$$

The following is the result for the semigroup generated by the fractional Laplacian.

Theorem 2.1. ([5]) *Let $\alpha > 0$, $t > 0$, $s, s_1, s_2 \in \mathbb{R}$, $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$.*
(i) *There exists a positive constant C so that for every $f \in \dot{B}^s_{p,q}(A)$*

$$e^{-tA^{\frac{\alpha}{2}}} f \in \dot{B}^s_{p,q}(A) \quad \text{and} \quad \big\|e^{-tA^{\frac{\alpha}{2}}} f\big\|_{\dot{B}^s_{p,q}(A)} \le C\|f\|_{\dot{B}^s_{p,q}(A)}.$$

(ii) *If $s_2 \ge s_1$, $p_1 \le p_2$ and*

$$d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) + s_2 - s_1 > 0,$$

then a constant $C > 0$ exists such that

$$\big\|e^{-tA^{\frac{\alpha}{2}}} f\big\|_{\dot{B}^{s_2}_{p_2,q_2}(A)} \le Ct^{-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s_2-s_1}{\alpha}} \|f\|_{\dot{B}^{s_1}_{p_1,q_1}(A)}$$

for all $f \in \dot{B}^{s_1}_{p_1,q_1}(A)$.
(iii) *Assume that $q < \infty$ and $f \in \dot{B}^s_{p,q}(A)$. Then*

$$\lim_{t\to 0} \big\|e^{-tA^{\frac{\alpha}{2}}} f - f\big\|_{\dot{B}^s_{p,q}(A)} = 0.$$

(iv) *Assume that $1 < p \le \infty$, $q = \infty$ and $f \in \dot{B}^s_{p,\infty}(A)$. Then $e^{-tA^{\frac{\alpha}{2}}} f$ converges to f in the dual weak sense as $t \to 0$.*
(v) *Let $s_0 > s/\alpha$. Then a constant $C > 0$ exists such that*

$$C^{-1}\|f\|_{\dot{B}^s_{p,q}(A)} \le \Big\{ \int_0^\infty \Big(t^{-\frac{s}{\alpha}} \|(tA^{\frac{\alpha}{2}})^{s_0} e^{-tA^{\frac{\alpha}{2}}} f\|_{L^p}\Big)^q \frac{dt}{t}\Big\}^{\frac{1}{q}} \le C\|f\|_{\dot{B}^s_{p,q}(A)}$$

for all $f \in \dot{B}^s_{p,q}(A)$.
(vi) *Assume that $u_0 \in \dot{B}^{s+\alpha-\frac{\alpha}{q}}_{p,q}(A)$, $f \in L^q(0,\infty; \dot{B}^s_{p,q}(A))$. Let u be given by*

$$u(t) = e^{-tA^{\frac{\alpha}{2}}} u_0 + \int_0^t e^{-(t-\tau)A^{\frac{\alpha}{2}}} f(\tau) d\tau.$$

Then there exists a constant $C > 0$ independent of u_0 and f such that

$$\|\partial_t u\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))} + \|A^{\frac{\alpha}{2}} u\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))} \le C\|u_0\|_{\dot{B}^{s+\alpha-\frac{\alpha}{q}}_{p,q}(A)} + C\|f\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))}.$$

It is well known that the assertions above hold in the case when $\Omega = \mathbb{R}^d$ and $\alpha = 1$ (see e.g. [10]). The proof is based on the spectral multiplier estimate (1.3). Furthermore, we can verify

$$C^{-1}2^{\alpha j} \le \|A^{\alpha/2}\phi_j(\sqrt{A})\|_{L^p\to L^p} \le C2^{\alpha j},$$

$$C^{-1}e^{-C2^{\alpha j}} \le \|e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})\|_{L^p\to L^p} \le Ce^{-C^{-1}2^{\alpha j}},$$

for all $j \in \mathbb{Z}$ with an absolute constant $C > 0$ (see Lemma 5.1 in [5]). These inequalities from above and below allow us to prove Theorem 2.1 by estimating them directly.

It is natural to expect corresponding results in function spaces of in inhomogeneous type, which will be the future work.

§ 3. Bilinear estimates

We consider the bilinear estimates of the form

$$\|fg\|_{\dot{H}^s_p} \le C(\|f\|_{\dot{H}^s_{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|g\|_{\dot{H}^s_{p_4}}),$$

where $s > 0$ and p, p_j $(j = 1, 2, 3, 4)$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. We focus on the domain, the half space $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d \,|\, x_d > 0\}$, for the sake of simplicity. The spaces we consider are the Sobolev spaces and the Besov spaces generated by the Dirichlet and the Neumann Laplacian. This kind of inequalities are well known in several domains, but because of the problem on how to introduce the fractional derivatives, the most of the results are for the whole space case, and general domains case required certain restriction, such as only small regularity number s, smoothness of the boundary of the domain. Classical proof of the bilinear estimates in the whole space case can be found in papers by Grafakos and Si [3], Tomita [12], and we refer a book by Runst and Sickel [11] on the detailed analysis of multi-linear estimates (see also [2, 4, 9]).

We shall reveal the optimal regularity s such that the above bilinear estimates holds for the Dirichlet and the Neumann boundary conditions. There will be a restriction on the regularity s only for the Dirichlet case, whose crucial point is how to handle behaviour of functions near the boundary, which will be discussed below theorems.

We introduce the Sobolev spaces generated by the Dirichlet and the Neumann Laplacian and state the results. Hereafter let us write the Dirichlet Laplacian A_D and the Neumann Laplacian A_N.

Definition. *Let $A = A_D$ or A_N, $s \in \mathbb{R}$ and $1 \le p \le \infty$.*
(i) *$H^s_p(A)$ is defined by*

$$H^s_p(A) := \{f \in \mathcal{X}'(A) \,|\, \|f\|_{H^s_p(A)} := \|(1 + A)^{s/2} f\|_{L^p(\mathbb{R}^n_+)} < \infty\}.$$

(ii) *$\dot{H}^s_p(A)$ is defined by*

$$\dot{H}^s_p(A) := \{f \in \mathcal{Z}'(A) \,|\, \|f\|_{\dot{H}^s_p(A)} := \|A^{s/2} f\|_{L^p(\mathbb{R}^n_+)} < \infty\}.$$

Theorem 3.1. ([6]) *Suppose that p, p_1, p_2, p_3, p_4 satisfy*

$$1 < p, p_1, p_4 < \infty, \quad 1 < p_2, p_3 \le \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

(i) (*Dirichlet case*) *Let* $A = A, 0 < s < 2 + 1/p$. *There exists a positive constant* C *so that*

$$\|fg\|_{\dot H^s_p(A_D)} \le C(\|f\|_{\dot H^s_{p_1}(A_D)}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|g\|_{\dot H^s_{p_4}(A_D)}) \tag{3.1}$$

for all $f \in \dot H^s_{p_1}(A_D) \cap L^{p_3}(\mathbb{R}^n_+)$, $g \in L^{p_2}(\mathbb{R}^n_+) \cap \dot H^s_{p_4}(A_D)$.
(ii) (*Neumann case*) *Let* $A = A_N, s > 0$. *There exists a positive constant* C *so that*

$$\|fg\|_{\dot H^s_p(A_N)} \le C(\|f\|_{\dot H^s_{p_1}(A_N)}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|g\|_{\dot H^s_{p_4}(A_N)}) \tag{3.2}$$

for all $f \in \dot H^s_{p_1}(A_N) \cap L^{p_3}(\mathbb{R}^n_+)$, $g \in L^{p_2}(\mathbb{R}^n_+) \cap \dot H^s_{p_4}(A_N)$
(iii) *The corresponding assertions to* (i) *and* (ii) *in the inhomogeneous Sobolev spaces hold.*

Theorem 3.2. ([6]) *Suppose that* $s \ge 2 + 1/p$. *Then the bilinear estimate* (3.1) *of the Dirichlet case does not hold.*

The result in the Besov spaces also holds.

Theorem 3.3. ([6]) *Suppose that* p, p_1, p_2, p_3, p_4, q *satisfy*

$$1 \le p, p_1, p_2, p_3, p_4 \le \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Let s *be as in Theorem 3.1. Then the corresponding bilinear estimates in* $\dot B^s_{p,q}(A_D)$, $B^s_{p,q}(A_D)$, $\dot B^s_{p,q}(A_N)$, $B^s_{p,q}(A_N)$ *hold, respectively, by replacing the Sobolev spaces with the Besov spaces which have the interpolation index* q. *Furthermore, if* $s > 2 + 1/p$ *or* $s = 2 + 1/p$ *with* $1 \le q < \infty$, *the bilinear estimate does not hold for the Dirichlet case.*

Multi-linear case contains a complexity that some cases hold true but the others do not. This is a result for the trilinear estimates.

Corollary 3.4. ([6]) *Let* $s > 0$, p, p_j $(j = 1, 2, \cdots, 9)$ *be such that*

$$1 < p, p_j < \infty \text{ for } j = 1, 5, 9, \quad 1 < p_j \le \infty \text{ for } j = 2, 3, 4, 6, 7, 8,$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_4} + \frac{1}{p_5} + \frac{1}{p_6} = \frac{1}{p_7} + \frac{1}{p_8} + \frac{1}{p_9}.$$

Then there exists a positive constant C *so that*

$$\begin{aligned}&\|fgh\|_{\dot H^s_p(A_D)}\\ \le& C(\|f\|_{\dot H^s_{p_1}(A_D)}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}} + \|f\|_{L^{p_4}}\|g\|_{\dot H^s_{p_5}(A_D)}\|h\|_{L^{p_6}} + \|f\|_{L^{p_7}}\|g\|_{L^{p_8}}\|h\|_{\dot H^s_{p_9}(A_D)}).\end{aligned}$$

We expect the positive result for products of functions of the odd numbers for all regularity $s > 0$, and there should be restriction for the even number case. Let us discuss the difference between two cases and conjecture the optimal regularity number s for the Dirichlet case.

The Neumann case can be handled by the even extension of the function on the half space to the whole space: For $f : \mathbb{R}^d_+ \to \mathbb{R}$, let us define the even extension $f_{\rm even}$ by

$$f_{\rm even}(x) = \begin{cases} f(x), & \text{if } x_d > 0, \\ f(-x), & \text{if } x_d < 0. \end{cases}$$

Then we deduce that

$$\|A_N^\alpha f\|_{L^p(\mathbb{R}^d_+)} \simeq \|(-\Delta)^\alpha f_{\rm even}\|_{L^p(\mathbb{R}^d)},$$

where $(-\Delta)^\alpha$ is defined by the Fourier multiplier $|\xi|^{2\alpha}$, and the formula

$$(fg)_{\rm even} = f_{\rm even}\, g_{\rm even}$$

and the bilinear estimates in the whole space $\mathbb{R}^d$ implies the one on the half space $\mathbb{R}^d_+$ in the Sobolev spaces associated with the Neumann Laplacian A_N. The important point is smoothness continued from $\{x_d < 0\}$ to $\{x_d < 0\}$ successfully accrossing the boundary $\partial\mathbb{R}^d_+$.

On the other hand, the Dirichlet case causes an different situation in this viewpoint. Let us define the odd extension: For $f : \mathbb{R}^d_+ \to \mathbb{R}$, we define

$$f_{\rm odd}(x) = \begin{cases} f(x), & \text{if } x_d > 0, \\ -f(-x), & \text{if } x_d < 0. \end{cases}$$

Then we deduce that

$$\|A_D^\alpha f\|_{L^p(\mathbb{R}^d_+)} \simeq \|(-\Delta)^\alpha f_{\rm odd}\|_{L^p(\mathbb{R}^d)},$$

however,

$$(fg)_{\rm odd} = ({\rm sign}\, x_d) f_{\rm odd}\, g_{\rm odd}$$

which would causes an restricted regularity for connection of continuity near the boundary. In fact, let us consider smooth f satisfying the Dirichlet boundary condition $f|_{\partial\Omega} = 0$. The Dirichlet condition for f, g gives again the Dirichlet condition for the product fg, however, higher order Dirichlet Laplacians can not act on fg, namely,

$$A_D(fg) = (A_D f)g - \nabla f \cdot \nabla g + f A_D g,$$

and $\nabla f \cdot \nabla g$ must be non-zero on the boundary in general. Hence $\nabla f \cdot \nabla g$ does not belong to the domain of the Dirichlet Laplcian, in particular, $(\partial_{x_d} f)(\partial_{x_d} g)$ does not necessarily satisfy the Dirichlet boundary condition. The threshold regularity of the bilinear estimate (3.1) is $s = 2 + 1/p$, whose second order derivative corresponds to A_D and the regularity less than $1/p$ is understood by the possibility of functions non-zero boundary value belonging to the Sobolev space associated with the Dirichlet Laplacian $H^{1/p-\varepsilon}(A_D)$. For $m \in \mathbb{N}$ and $f_j \in \mathcal{X}(A_D)$ $(j = 1, 2, \cdots, 2m)$, as the multi-linear case, it would be reasonable to conjecture that the threshold regularity of the multilinear estimates for $f_1 f_2 \cdots f_{2m}$ is $2m + 1/p$, since we see

$$A_D^m(f_1 f_2 \cdots f_{2m}) = (-1)^m (\nabla f_1 \cdot \nabla f_2)(\nabla f_3 \cdot \nabla f_4) \cdots (\nabla f_{2m-1} \cdot \nabla f_{2m}) + (\text{the other terms}),$$

the first term can have the non-zero boundary value in general and the other terms should satisfy the Dirichlet boundary condition.

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